

## DEFORMATION OF AN INFINITE DIELECTRIC MEDIUM WITH A HOLE IN THE SHAPE OF PASCAL LIMAÇON

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### ABSTRACT:

A two dimensional problem of electrostriction with a hole in the shape of Pascal's limaçon is solved by complex variable method. The distributions of stresses in an infinite dielectric plate when the hole is filled up by air and is subjected to an electric field uniform at infinity as well as it is acted on by applied two dimensional tractions at infinity. The hoop stress is calculated on the boundary of the hole.

### INTRODUCTION:

It is known that a dielectric medium is deformed when it is placed before an electric field and the accompanying strains are directly proportional to the even powers of the components of the electric intensity vector. A systematic development of this theory and its applications to different types of problems can be found in the literature [1,2]. Knops [3] developed the complex variable method for solving two dimensional problems of electrostriction. This method of Knops is largely expository in character and admits of wide application to different problems of electrostriction. Maikap and Sengupta [4] also applied the theory to some specific problems of electrostriction.

In the present paper, it is proposed to find out the distribution of stresses in an infinite dielectric plate containing a hole in the form of Limaçon filled up by air and is subjected to an electric field uniform at infinity. Moreover, at infinity there also acts tensions along the coordinate axes. The results obtained are found to be in good agreement in the absence of dielectric [5]. The hoop stress is obtained and is shown graphically.

### FUNDAMENTAL EQUATIONS:

Consider a homogeneous isotropic dielectric medium subjected to an electric field uniform at infinity. Then the deformation of the medium is called electrostrictive deformation or electrostriction. We also suppose that the geometry of the medium is such and the conditions of the problem are so imposed that it is in a state of either plain strain or plane stress.

Now the relation between the electric displacement vector  $\mathbf{D}$  and the electric intensity vector  $\mathbf{E}$  for a linear homogeneous isotropic dielectric medium  $\epsilon$  as its permeability is  $\mathbf{D} = \epsilon \mathbf{E}$ . As in classical plane theory of electrostatics, we define an analytic complex potential function  $W(z)$  of the complex variable  $z$  whose real part is single valued electrostatic potential so that

$$\kappa = \kappa_x + i\kappa_y = -\overline{W'(z)} \quad (1)$$

Following Knops [3] and using the complex potential function, we have the following relations to determine the components of stress and displacement:

$$\tau_{xx} + \tau_{yy} = 4KW'(z)\overline{W'(z)} + 4[\phi'(z) + \overline{\phi'(z)}], \quad (2)$$

$$\tau_{xx} - \tau_{yy} + 2i\tau_{xy} = 4KW''(z)\overline{W'(z)} + 4[\bar{z}\phi''(z) + \psi'(z)], \quad (3)$$

$$\mu(u + iv) = k\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} - KW'(z)\overline{W'(z)} + \frac{1}{2}a\overline{\Omega(z)}, \quad (4)$$

Where  $\mu$  is the material shear modulus,  $\phi(z)$  and  $\psi(z)$  are arbitrary functions of  $z$ , bar represents the conjugate variable and prime denotes differentiation with respect to its argument. Moreover the plane strain and plane stress  $K$  and  $k$  appearing in the relations (2), (3) and (4) have respectively the following pair of values:

$$\text{Plane strain} \begin{cases} k = \frac{1}{2} \frac{1-\nu}{1+\nu} (a + 2b), \\ k = 3 - 4\nu, \end{cases} \quad (5)$$

$$\text{Plane stress} \begin{cases} k = \frac{1}{8} \{2(1-2\nu)b + (1-\nu)a\}, \\ k = \frac{2-\nu}{1+\nu}, \end{cases} \quad (6)$$

$\nu$  being the Poisson's ratio and  $a, b$  are constants given in [4]. Also

$$\Omega'(z) = [W'(z)]_P^Q \quad (7)$$

Where the symbol  $[ ]_P^Q$  denotes the change of function inside the bracket as its argument moves from a point P to a point Q along a contour PQ. Then the complex resultant force and the moment on a contour PQ is given by

$$X + iY = -2i[kW(z)\overline{W'(z)} + \phi(z) + z\overline{\phi'(z)} + \psi(z)]_P^Q, \quad (8)$$

$$M = [KW(z)\overline{W'(z)} + 2Re\{k(z) - z\psi(z) - z\bar{z}\phi(z) - Kz\overline{W(z)W'(z)}\}]_P^Q \quad (9)$$

Where  $K'(z) = \psi(z)$ .

#### THE PROBLEM AND ITS SOLUTION

Let us suppose that an infinite dielectric medium of permeability  $\epsilon$  contains a hole in form of Pascals Limacon and the hole is filled up by the air. The medium is subjected to an electric fie  $E = E_x + iE_y$ , uniform at infinity. Moreover at infinity there act tensions  $T_1$  and  $T_2$  in the directions of  $x$  and  $y$  axes respectively.

We consider the transformation

$$z = g(\zeta) = R(\zeta + m\zeta^2), R > 0, 0 \leq m \leq \frac{1}{2}, \zeta = \sigma e^{i\theta} \quad (10)$$

Which maps the region outside the hole in the  $z$ -plane on to the region exterior to the unit circle in the  $\zeta$ -plane.

The tangential component of the electrical intensity vector and the normal component of the electrical displacement vector are continuous at the interface. These boundary conditions lead to the complex potential to assume in the form

$$[W(\zeta)] = \begin{cases} \bar{E}\zeta + AE\zeta^{-1}, & |\zeta| > 1 \\ -D\bar{E}\zeta, & |\zeta| < 1 \end{cases} \quad (11)$$

With

$$A = \frac{1-\epsilon}{1+\epsilon}, B = \frac{2\epsilon}{1+\epsilon}. \quad (12)$$

Again since the stresses and displacements are single valued and the hole is free from external stresses,  $\phi(\zeta)$  and  $\psi(\zeta)$  can be taken as

$$\begin{aligned} \phi(\zeta) &= R\Gamma\zeta + \phi_0(\zeta), \\ \psi(\zeta) &= R\Gamma\zeta + \psi_0(\zeta) \end{aligned} \quad (13)$$

Where  $\phi_0(\zeta)$  and  $\psi_0(\zeta)$  are holomorphic for  $|\zeta| > 1$  and where one can assume  $\phi_0(\infty) = 0$ . Moreover  $\Gamma$  and  $\Gamma'$  are constants related to the conditions at infinity.

The boundary condition satisfied by  $\phi(\zeta)$  and  $\psi(\zeta)$  at the interface

$\zeta = \frac{1}{\bar{\zeta}} = \sigma e^{i\theta}$  is given by

$$KW(a) \frac{w'(a)}{g'(a)} + \overline{\phi(a)} + \overline{g(a)} \frac{\phi'(a)}{g'(a)} + \psi(a) = \bar{f} \quad (14)$$

$$KW(a) \frac{w'(a)}{g'(a)} + \phi(a) + g(a) \frac{\phi'(a)}{g'(a)} + \overline{\psi(a)} = f \quad (15)$$

Putting (13) in the equations (14) and (15) we note that  $\phi_0(\zeta)$  and  $\psi_0(\zeta)$  satisfying the boundary conditions (14) and (15), the only difference being that  $f$  must be replaced by  $f_0$  where

$$f_0 = f - R\Gamma \left( a + \frac{g(a)}{g'(a)} \right) - \frac{\Gamma'R}{a} \quad (16)$$

Since the hole is free from stresses,  $f = 0$ .

Thus, we have finally

$$KW(a) \frac{w'(a)}{g'(a)} + \overline{\phi_0(a)} + \overline{g(a)} \frac{\phi_0'(a)}{g'(a)} + \psi_0(a) = -R\Gamma \left( a + \frac{g(a)}{g'(a)} \right) - \frac{\Gamma'R}{a} \quad (17)$$

And its conjugate is

$$KW(a) \frac{w'(a)}{g'(a)} + \phi_0(a) + g(a) \frac{\phi_0'(a)}{g'(a)} + \overline{\psi_0(a)} = -R\bar{\Gamma} \left( \frac{1}{a} + \frac{\overline{g(a)}}{g'(a)} \right) - \Gamma'R a \quad (18)$$

Multiplying both sides of (17) and (18) by  $\frac{1}{2\pi i} \frac{da}{a\zeta}$  and integrating around the unit circle we get

$$\phi_0(\zeta) = \frac{\kappa}{R} \frac{AE^2}{\zeta + 2m} - \bar{\Gamma}' R \frac{1}{\zeta} \quad (19)$$

$$\begin{aligned}\psi_0(\zeta) = & -\bar{\Gamma}R\frac{1}{\zeta} - \bar{\Gamma}R\frac{\zeta+m}{\zeta^2(1+2m\zeta)} - \frac{K}{R}\frac{AE^2}{\zeta^3(1+2m\zeta)} - \frac{K(1-A^2)E\bar{E}}{R\zeta(1+2m\zeta)} \\ & + \frac{K}{R}\bar{E}^2\left(\frac{\zeta}{1+2m\zeta} - \frac{1}{\zeta+2m}\right) - \frac{\zeta+m}{\zeta^2(1+2m\zeta)}\phi'_0(\zeta)\end{aligned}\quad (20)$$

Substituting these values of  $\phi_0(\zeta)$  and  $\psi_0(\zeta)$  in (13) we obtain

$$\phi(\zeta) = R\Gamma\zeta + \frac{K}{R}\frac{AE^2}{(\zeta+2m)} - \Gamma'R\frac{1}{\zeta}, \quad (21)$$

$$\begin{aligned}\psi(\zeta) = & R\Gamma'\zeta - \bar{\Gamma}R\frac{1}{\zeta} - \bar{\Gamma}R\frac{\zeta+m}{\zeta^2(1+2m\zeta)} - \frac{K}{R}\frac{AE^2}{\zeta^3(1+2m\zeta)} - \frac{K(1-A^2)E\bar{E}}{R\zeta(1+2m\zeta)} \\ & + \frac{K}{R}\bar{E}^2\left(\frac{\zeta}{1+2m\zeta} - \frac{1}{\zeta+2m}\right) - \frac{\zeta+m}{\zeta^2(1+2m\zeta)}\phi'_0(\zeta)\end{aligned}\quad (22)$$

The conditions at infinity lead to

$$\Gamma = \bar{\Gamma} = \frac{1}{8}(T_1 + T_2) - \frac{KE\bar{E}}{2R^2}, \quad \Gamma' = \bar{\Gamma}' = \frac{1}{2}(T_2 - T_1). \quad (23)$$

Substituting these values of  $\Gamma'$  and  $\bar{\Gamma}'$  in (21) and (22), we have

$$\phi(\zeta) = \frac{RT_1}{8}\left(\zeta + \frac{2}{\zeta}\right) + \frac{RT_2}{8}\left(\zeta - \frac{2}{\zeta}\right) - \frac{KE\bar{E}\zeta}{2R} + \frac{K}{R}\frac{AE^2}{(\zeta+2m)}, \quad (24)$$

$$\begin{aligned}\psi(\zeta) = & \frac{RT_1}{8}\left\{-2\zeta - \frac{1}{\zeta} - \frac{\zeta+m}{\zeta^2(1+2m\zeta)}\right\} + \frac{RT_2}{8}\left\{2\zeta - \frac{1}{\zeta} - \frac{\zeta+m}{1+2m\zeta}\right\} \\ & + \frac{KE\bar{E}}{2R}\left\{\zeta + \frac{\zeta+m}{\zeta^2(1+2m\zeta)}\right\} - \frac{K}{R}\frac{AE^2}{\zeta^3(1+2m\zeta)} - \frac{K(1-A^2)E\bar{E}}{R\zeta(1+2m\zeta)} \\ & + \frac{K}{R}\bar{E}^2\left(\frac{\zeta}{1+2m\zeta} - \frac{1}{\zeta+2m}\right) - \frac{\zeta+m}{\zeta^2(1+2m\zeta)}\phi'_0(\zeta)\end{aligned}\quad (25)$$

The stresses can be obtained from (2) and (3).

Now the circumferential stress on the boundary of the hole is of great interest and is given by

$$\tau_{\theta\theta} = 4K\frac{w'(\zeta)}{g'(\zeta)}\frac{\overline{w'(\zeta)}}{\overline{g'(\zeta)}} + 8R\varrho\left\{\frac{\phi'(\zeta)}{g'(\zeta)}\right\} \quad (26)$$

Accordingly we have for various angles

$$\begin{aligned}\tau_{\theta\theta} = & T_1\left\{\frac{(1-2\cos 2\theta)(1+2m\cos\theta)+4m\sin\theta\sin 2\theta}{1+4m^2+4m\cos\theta}\right\} + T_2\left\{\frac{(1+2\cos 2\theta)(1+2m\cos\theta)+4m\sin\theta\sin 2\theta}{1+4m^2+4m\cos\theta}\right\} - \\ & \frac{4KE\bar{E}(2+2m\cos\theta+A^2)}{R^2(1+4m^2-4m\cos\theta)} - \frac{4KA\bar{E}^2}{R^2}\frac{\cos\theta}{1-4m^2+4m\cos\theta} + \frac{4KA\bar{E}^2}{R^2(1-4m^2+4m\cos\theta)^3}\{\cos 2\theta(1+4m^2+ \\ & 4m\cos\theta)^2 - 2[(\cos\theta+2m)^2 - \sin^2\theta](1+2m\cos\theta) + 8m\sin^2\theta(\cos\theta+m)\} \quad (27)\end{aligned}$$

To find the nature of the circumferential stress numerically we consider the following two cases:

#### CASE 1:

Let the pole be subjected to longitudinal tension  $T$  along the direction of Y-axis only, so that  $T_1 = 0$  and  $T_2 = T$ . Then we from (27)

$$\begin{aligned} \widehat{\theta\theta} = \frac{\tau_{\theta\theta}}{T} = & \left\{ \frac{((1+2\cos 2\theta)(1+2m\cos\theta)+4m\sin\theta\sin 2\theta)}{1+4m^2+4m\cos\theta} \right\} - \frac{4KE\bar{E}(2+2m\cos\theta+A^2)}{R^2T(1+4m^2+4m\cos\theta)} \\ & - \frac{4KA\bar{E}^2}{R^2T(1+4m^2+4m\cos\theta)} + \frac{4KA\bar{E}^2}{R^2T(1+4m^2+4m\cos\theta)^3} [\cos 2\theta(1+4m^2+4m\cos\theta)^2 - 2\{(\cos\theta + \\ & 2m)^2 - \sin^2\theta\}(1+2m\cos\theta) + 8m\sin^2\theta(\cos\theta + m)] \end{aligned} \quad (28)$$

For numerical results, we choose  $m = 0.25$ ,  $\epsilon = 0.5$ ,  $R = 1$ ,  $\frac{KE^2}{T} = \frac{K\bar{E}^2}{T} = 2$ .

The values of  $\widehat{\theta\theta}$  for different values of  $\theta$  are plotted in Fig-1 by continuous curve. It is seen that the circumferential stress is harmonic in nature attaining its maximum at  $\theta = \pi$  and minimum at  $\theta = \frac{\pi}{5}, \frac{9\pi}{5}$ .

#### CASE 2:

Suppose that the plate is subjected to transverse tension  $T$ . In this case  $T_1 = T$  and  $T_2 = 0$  and we have from (27)

$$\begin{aligned} \widehat{\theta\theta} = \frac{\tau_{\theta\theta}}{T} = & \left\{ \frac{((1-2\cos 2\theta)(1+2m\cos\theta)+4m\sin\theta\sin 2\theta)}{1+4m^2+4m\cos\theta} \right\} + T_2 \left\{ \frac{((1+2\cos 2\theta)(1+2m\cos\theta)+4m\sin\theta\sin 2\theta)}{1+4m^2+4m\cos\theta} \right\} - \\ & \frac{4KE\bar{E}(2+2m\cos\theta+A^2)}{R^2(1+4m^2+4m\cos\theta)} - \frac{4KA\bar{E}^2}{R^2(1+4m^2+4m\cos\theta)} + \frac{4KA\bar{E}^2}{R^2(1+4m^2+4m\cos\theta)^3} \{ \cos 2\theta(1+4m^2 + \\ & 4m\cos\theta)^2 - 2\{(\cos\theta + 2m)^2 - \sin^2\theta\}(1+2m\cos\theta) + 8m\sin^2\theta(\cos\theta + m) \} \end{aligned}$$

The circumferential stress  $\widehat{\theta\theta}$  is shown in Fig-1 by dotted curve and it is also seen that the stress is harmonic in nature attaining its maximum at  $\theta = \pi$  and minimum at

$$\theta = \frac{\pi}{5}, \frac{9\pi}{5}.$$

Comparing the above two particular cases, it follows for longitudinal tension,  $\widehat{\theta\theta}$  is greater for longitudinal tension than that for transverse tension in the region  $0 < \theta < \frac{\pi}{5}$

And  $\frac{9\pi}{5} < \theta < 2\pi$  and lower for  $\frac{\pi}{5} < \theta < \frac{9\pi}{5}$ .

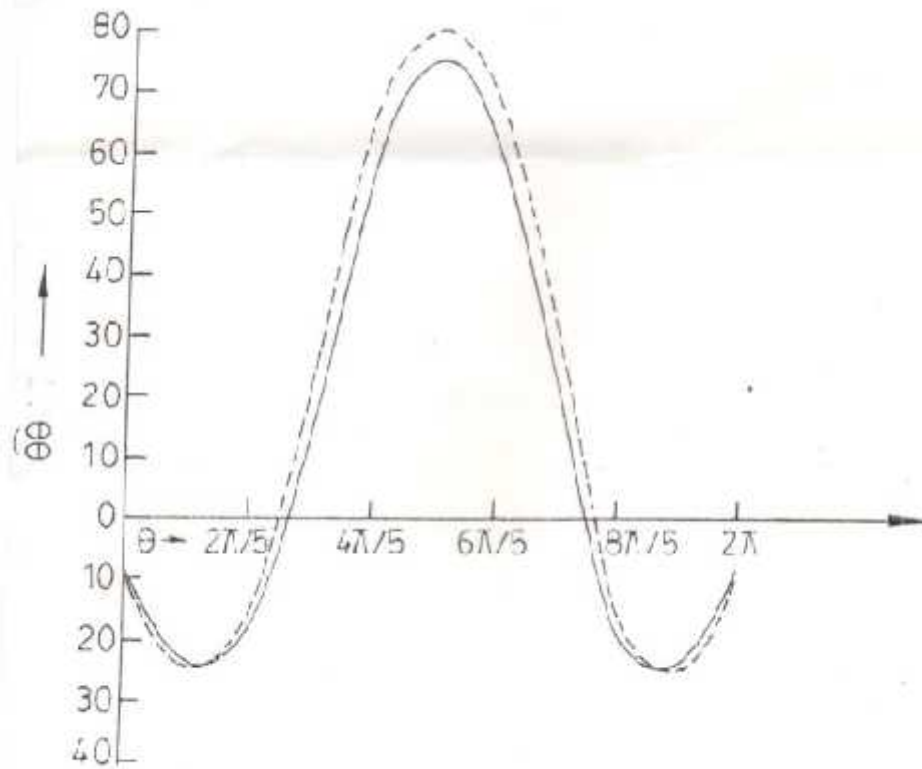


Fig 1

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