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Closed form solutions to the coupled space-time fractional evolution equations in mathematical physics through analytical method

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Abstract

In this article, we consider the space-time fractional coupled modified Korteweg-de-Vries (mKdV) equations and the space-time fractional coupled Whitham-Broer-Kaup (WBK) equations which are important mathematical model to depict the propagation of wave in shallow water under gravity, combined formal solitary wave, internal solitary waves in a density and current stratified shear flow with a free surface, ion acoustic waves in plasma, turbulent motion, quantum mechanics and also in financial mathematics. We examine new, useful and further general exact wave solutions to the above mentioned space-time fractional equations by means of the generalized (G'/G)*-expansion method by using of fractional complex transformation and discuss the examined results with other method. This method is more general, powerful, convenient and direct and can be used to establish new solutions for other kind nonlinear fractional differential equations arising in mathematical physics.*

Keywords: Coupled mKdV equations; coupled WBK equation; nonlinear evolution equations; fractional differential equations.

I. Introduction

Nonlinear fractional differential equations (FNLDEs) are generalizations of classical differential equations with integer orders. Now-a-days, FNLDEs in mathematical physics perform an important role in different fields, as for instance, physics, signal processing, control theory, fractal dynamics, medicine, polymer rheology, aerodynamics, hydrology, pharmacy, material science, the modeling of earthquake, electricity, optical fibers, chemical kinematics, biology and so on. The fractional order models are more useful than integer order models in many cases of above fields. Therefore, obtaining exact solution to the fractional differential equations is an important task. Recently nonlinear fractional differential equations have been concerned much importance and it has gained reputation to the researchers.

Jumarie's modified Riemann-Liouville [XIX] derivative and Caputo [IX] derivative of definitions of fractional derivative are in common. Exact solutions of NFDEs are very much significant to identify the internal structure of intricate tangible events. Hence, for this demand some useful and effective methods have been created and enhanced for determining exact solution to the fractional evolution equations, for instance, the modified simple equation method (MSE) [XXI, XXXIV], the (G'/G)-expansion method [I, II, VI, XXXIII], the differential transformation method [XXVII], the finite element method [X], the exp-function method [XXXVI], the fractional sub-equation method [XXIX], the Adomin's decomposition method [XII], the Jacobi elliptic method [XXXVII], the variational iteration method [XVIII], the modified trial equation method [VIII], the fractional Riccati equation transformation method [XXII], the homotopy analysis method [V], the modified Kudryashov method [XI], the first integral method [XXIII, XXXII], the tanh-function method [XVII], etc.

In current literature, the space-time fractional coupled mKdV equations [XXIV, XXVIII] and the space-time fractional WBK equations are examined through the extended fractional sub-equation method [XXXI], the modified extended tanh-function method [IV, XXXV], the modified Kudryashov method [XXV], the variational iteration method [XXVI], the homotopy analysis method [XIV, XX], the exp-function method [VII, XIII], the Riccati equation transform method [XXX], the Adomin's decomposition method [III], and etc. So far of our knowledge the space-time fractional coupled mKdV equations and the space-time fractional coupled WBK equations have not been investigated by using the generalized (G'/G)-expansion method.

Therefore, our aim is to establish the further general and new exact wave solutions of the space-time fractional coupled mKdV equations and the space-time fractional coupled WBK equations by using the generalized (G'/G)-expansion method and provide the physical explanation of the obtained solutions for its definite values in graphically to analyze for both the evolution equations. The generalized (G'/G)-expansion method is more powerful, efficient, and rising method to determine new wave solutions to the FNLDEs. Its finding results are easy, more general, potential, useful, and no need to use the symbolic computation software to operate the algebraic equations.

The rest of the article is executed as follows: In section 2, we describe the Jumarie's modified Riemann-Liouville derivative. In section 3, we explain the outline of the generalized (G'/G)-expansion method. In section 4, we search the new and further general solutions to the fractional evolution equations which are mentioned above. In section 5, we argue the results and discussion and in section 6, we represent our conclusions.

II. Modified Riemann-Liouville Derivative

The Jumarie's modified Riemann-Liouville derivative of order α is defined as follows [XIX]:

$$D_x^\alpha u(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)}\frac{d}{dx}\int_0^x (x-\xi)^{-\alpha-1}\big(u(\xi)-u(0)\big)d\xi, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-\xi)^{-\alpha}\big(u(\xi)-u(0)\big)d\xi, & 0 < \alpha < 1, \\ \left[u^{(\alpha-n)}(x)\right]^{(n)}, & n \le \alpha < n+1, n \ge 1. \end{cases} \tag{2.1}$$

Some properties for the proposed modified Riemann-Liouville derivative are listed as follows:

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0, \tag{2.2}$$

$$D_x^\alpha\big(u(x)w(x)\big) = w(x)D_x^\alpha u(x) + u(x)D_x^\alpha w(x), \tag{2.3}$$

$$D_x^\alpha u[w(x)] = u'_w\ [w(x)]D_x^\alpha w(x), \tag{2.4}$$

$$D_x^\alpha u[w(x)] = D_w^\alpha u[w(x)]\big(w'(x)\big)^\alpha, \tag{2.5}$$

Remark 2.1: We will derive effective way for solving fractional partial differential equations using above formulae. In the above formulae (2.2)-(2.5), (2.2) is non-differentiable function and the function $u(x)$ is non-differentiable in (2.3) and (2.4) and differentiable in (2.5), also $w(x)$ is non-differentiable in (2.4), and $u(u)$ is differentiable in (2.4) but non-differentiable in (2.5). In this article, we apply formulae (2.3) and (2.5) to examine the solutions of mentioned fractional differential equations base on the generalized (G'/G)-expansion method. The above thoughts exposed that the use of the generalized (G'/G)-expansion method allow us to examine new exact close form solutions from the known seed solutions. To convert NFDEs into its differential partner easily, He et al. introduced the fractional complex transform in [XV, XVI].

Hence, above formulae take part an important role in fractional calculus and also fractional differential equations.

III. Delineation of the Method

We consider a general nonlinear fractional differential equation in the form:

$$H\Big(u, D_t^\alpha u, D_x^\beta u, D_y^\gamma u,\ D_z^\varepsilon u, D_t^{2\alpha} u, D_x^{2\beta} u, \dots\Big), \tag{3.1}$$

where $u = u(x,y,z,t)$ is an unidentified function, H is a polynomial in $u(x,y,z,t)$ and its fractional derivatives, which include the highest order derivative and nonlinear terms of the highest order wherein $\alpha, \beta, \gamma, \varepsilon$ are non-integer and the subscripts denote the partial derivatives. To obtain the solution of Eq. (3.1) by using the generalized (G'/G)-expansion method, we have to execute the subsequent steps:

Step 1: We consider the following traveling wave variable

$$u(x,y,z,t) = u(\xi),\ \xi = \frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{ny^{\gamma}}{\Gamma(1+\gamma)} + \frac{lz^{\beta}}{\Gamma(1+\beta)} + \frac{kt^{\alpha}}{\Gamma(1+\alpha)}, \qquad (3.2)$$

for real fractional differential equations and

$$u(x,y,z,t) = \varphi(\xi)e^{i\eta},$$
$$\xi = k\left(\frac{x^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{y^{\gamma}}{\Gamma(1+\gamma)} + \frac{z^{\beta}}{\Gamma(1+\beta)} - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right), \eta = \frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{ny^{\gamma}}{\Gamma(1+\gamma)} + \frac{lz^{\beta}}{\Gamma(1+\beta)} - \frac{wt^{\alpha}}{\Gamma(1+\alpha)}, \qquad (3.3)$$

for complex fractional differential equations, permits us to transform the Eq. (3.1) into the following ordinary differential equation (ODE):

$$Q(u, u', u'', \dots) = 0, \qquad (3.4)$$

where Q is a polynomial in $u(\xi)$ and its derivatives, wherein $u'(\xi) = \frac{du}{d\xi}$.

Step 2: According to the possibility of Eq. (3.4) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant may be zero for simplicity.

Step 3: We assume that the traveling wave solution of Eq. (3.4) can be written in the form:

$$u(\xi) = \sum_{i=0}^{N} a_i (h+M)^i + \sum_{i=1}^{N} b_i (h+M)^{-i} \qquad (3.5)$$

where either a_N or b_N may be zero, but both a_N and b_N cannot be zero at a time, a_i, $(i = 0, 1, 2 \dots N)$ and b_i, $(i = 1, 2, \dots N)$ and d are arbitary constants to be evaluated and $M(\xi)$ is given by

$$M(\xi) = (G'/G) \qquad (3.6)$$

where $G = G(\xi)$ satisfies the following auxiliary nonlinear ordinary differential equation: $PGG'' - QGG' - EG^2 - R(G')^2 = 0$ (3.7)

where the prime stands for derivative with respect to ξ; P, Q, R and E are real parameters.

Step 4: The positive integer N arises in Eq. (3.5) can be determined by homogeneous balancing the highest order nonlinear terms and the derivatives of highest order occur in Eq. (3.4).

Step 5: Inserting Eqs. (3.5), (3.6) and (3.7) into Eq. (3.4) with the value of N obtained in Step 4, we attain polynomials in $(h+M)^N$ $(N = 0,1,2,\dots)$ and

$(h+M)^{-N}$ $(N=1,2,3,\dots)$. Then, we collect each coefficient of the resulted polynomials to zero gives a set of algebraic equations for $a_i, (i=0,1,2,\dots)$, $b_i(i=1,2,3\dots)$, d and k.

Step 6: Suppose that the value of the constants $a_i, (i=0,1,2,\dots)$, $b_i(i=1,2,3\dots)$, d and k can be found by solving the algebraic equations obtained in Step 5. Since the general solution of equation (3.7) is well known to us, putting the values of $a_i, (i=0,1,2,\dots)$, $b_i(i=1,2,3\dots)$, d and k into Eq. (3.5), we attain more general type and new exact travelling wave solutions of the nonlinear fractional differential Eq. (3.1).

Using the general solution of Eq. (3.7), we attain the following solutions of Eq. (3.6):

Family 1: When $Q\neq 0, \psi = P-R$ and $\Omega = Q^2+4E(P-R)>0$,

$$M(\xi)=(G'/G)=\frac{Q}{2\,\psi}+\frac{\sqrt{\Omega}}{2\,\psi}\times\frac{r_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\,\xi\right)+r_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\,\xi\right)}{r_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\,\xi\right)+r_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\,\xi\right)} \tag{3.8}$$

Family 2: When $Q\neq 0, \psi = P-R$ and $\Omega = Q^2+4E(P-R)<0$,

$$M(\xi)=(G'/G)=\frac{Q}{2\,\psi}+\frac{\sqrt{-\Omega}}{2\,\psi}\times\frac{-r_1 sin\left(\frac{\sqrt{-\Omega}}{2P}\,\xi\right)+r_2 cos\left(\frac{\sqrt{-\Omega}}{2P}\,\xi\right)}{r_1 cos\left(\frac{\sqrt{-\Omega}}{2P}\,\xi\right)+r_2 sin\left(\frac{\sqrt{-\Omega}}{2P}\,\xi\right)} \tag{3.9}$$

Family 3: When $Q\neq 0, \psi = P-R$ and $\Omega = Q^2+4E(P-R)=0$,

$$M(\xi)=(G'/G)=\frac{Q}{2\,\psi}+\frac{r_2}{r_1+r_2\,\xi} \tag{3.10}$$

Family 4: When $Q=0, \psi = P-R$ and $\Delta = \psi\, E>0$,

$$M(\xi)=(G'/G)=\frac{\sqrt{\Delta}}{\psi}\times\frac{r_1 sinh\left(\frac{\sqrt{\Delta}}{P}\,\xi\right)+r_2 cosh\left(\frac{\sqrt{\Delta}}{P}\,\xi\right)}{r_1 cosh\left(\frac{\sqrt{\Delta}}{P}\,\xi\right)+r_2 sinh\left(\frac{\sqrt{\Delta}}{P}\,\xi\right)} \tag{3.11}$$

Family 5: When $Q=0, \psi = P-R$ and $\Delta = \psi\, E<0$,

$$M(\xi)=(G'/G)=\frac{\sqrt{-\Delta}}{\psi}\times\frac{-r_1 sin\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)+r_2 cos\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)}{r_1 cos\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)+r_2 sin\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)} \tag{3.12}$$

IV. Formulation of the solutions

In this section, we evaluate the new, useful and further general effective solutions to the space-time fractional coupled mKdV equations and the space-time fractional coupled WBK equations.

IV. I: The coupled mKdV equations

In this sub-section, we determine some useful close form traveling wave solutions to the space-time fractional coupled mKdV equations by making use of the generalized G'/G-expansion method. Let us suppose the space-time fractional coupled mKdV equations of the form:

$$D_t^{\alpha}w - \frac{1}{2}D_x^{3\alpha}w + 3\,w^2 D_x^{\alpha}w - \frac{3}{2}D_x^{2\alpha}w - 3\,D_x^{\alpha}(w\,\phi) + 3\,\lambda D_x^{\alpha}w = 0$$

$$D_t^{\alpha}\phi + D_x^{3\alpha}w + 3\phi\,D_x^{\alpha}\phi + 3\,D_x^{\alpha}w\,D_x^{\alpha}\phi - 3w^2 D_x^{\alpha}\phi - 3\lambda D_x^{\alpha}\phi = 0 \qquad (4.1.1)$$

where $\boldsymbol{\alpha}$ is fractional order derivative and $\boldsymbol{\lambda}$ are arbitrary constant. The coupled mKdV equations are mathematical equations that describe as mathematical model of motion in shallow water wave under gravity and wave propagation with different dispersion relations, ion acoustic waves in plasma, hydrodynamics, turbulent motion, quantum mechanics and also in financial mathematics. The travelling transformation (3.2) is used to convert the equation (4.1.1) into the following nonlinear ODE:

$$2kw' - m^3w''' + 6mw^2w' - 3m^2w'' - 6m\phi w' - 6m\phi' w + 6m\lambda w' = 0$$

$$k\phi' + m^3w''' + 3m\phi\phi' + 3m^2w'\phi' - 3mw^2\phi' - 3m\lambda\phi' = 0 \qquad (4.1.2)$$

Now, balancing the linear term of the highest order derivative term and the nonlinear term of the highest order occurring in (4.1.2), yields $N_1 = N_2 = 1$. Then the solution of Eq. (4.1.2) is of the form:

$$w = b_1(h+M)^{-1} + a_0 + a_1(h+M)$$

$$\phi = e_1(h+M)^{-1} + c_0 + c_1(h+M) \qquad (4.1.3)$$

where $a_0\,, a_1, b_1, c_0, c_1$ and e_1 are arbitrary constants to be determined, such that either a_1 or b_1 may be zero, but both a_1 and $b_1$cannot be zero at a time and also either c_1 or $e_1$may be zero, but both c_1 and e_1 cannot be zero at a time.

Inserting Eq. (4.1.3) together with Eqs. (3.6) and (3.7) into Eq. (4.1.2), the left hand side is converted into polynomials in $(h+M)^N$ $(N = 0,1,2, \dots)$ and $(h+M)^{-N}$ $(N = 1,2,3, \dots)$. We collect each coefficient of these resulted polynomials and setting them zero yields a set of simultaneous algebraic equations (for simplicity the equations are not present here) for $a_0\,, a_1, b_1, c_0, c_1, e_1\,, h, k$. Solving these algebraic equations with the help of symbolic computation software, such as, Maple, we obtain the following 02 (two) sets of solutions:

Set-1:

$$a_0 = \frac{P+m(2h\psi+Q)}{2P}, a_1 = 0, b_1 = -\frac{m(h^2\psi+hQ-E)}{P}, c_0 = \frac{2P\lambda+m(2h\psi+Q)}{2P}, e_1 = -\frac{m(h^2\psi+hQ-E)}{P}$$

and

$$c_1 = 0, k = \frac{m\left(3P^2-m^2\left(4E\psi+Q^2\right)\right)}{4P^2} \tag{4.1.4}$$

Set-2:

$$a_0 = \frac{P-m(2h\psi+Q)}{2P}, a_1 = \frac{m\psi}{P},$$
$$b_1 = 0, c_0 = \frac{2P\lambda-m(2h\psi+Q)}{2P}, c_1 = \frac{m\psi}{P}, e_1 = 0$$

and

$$k = \frac{m\left(3P^2-Qm-4m^2E\psi\right)}{4P^2} \tag{4.1.5}$$

where $\psi = P - R,\ P, Q, R, E$ and h are free parameters.

For simplicity we have discussed only on the solutions Set-1 of the mentioned equations is arranged in Eq. (4.1.4) as follows and other sets of solutions are omitted here.

When $Q \neq 0, \psi = P - R$ and $\Omega = Q^2 + 4E(P-R) > 0$, substituting the values of the constants arranged in Eq. (4.1.4) into Eq. (4.1.3) and simplifying, we attained the travelling wave solutions

$$w(\xi) = \frac{P+m(2h\psi+Q)}{2P} - \frac{m(h^2\psi+hQ-E)}{P}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{\Omega}}{2\psi}\frac{r_1\sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+r_2\cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{r_1\cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+r_2\sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)^{-1} \tag{4.1.6}$$

$$\phi(\xi) = \frac{2P\lambda+m(2h\psi+Q)}{2P} - \frac{m(h^2\psi+hQ-E)}{P}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{\Omega}}{2\psi}\frac{r_1\sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+r_2\cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{r_1\cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+r_2\sinh\left(\frac{\sqrt{\Omega}}{2P}k\xi\right)}\right)^{-1} \tag{4.1.7}$$

where

$$\xi = \frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{kt^{\alpha}}{\Gamma(1+\alpha)}.$$

Since r_1 and r_2 are integral constants, so we might choose arbitrarily their values. If we pick $r_1 = 0$ but $r_2 \neq 0$, then the solutions (4.1.6) and (4.1.7) is simplified as

$$w_1(x,t)=\frac{P+m(2h\psi+Q)}{2P}-\frac{m(h^2\psi+hQ-E)}{P}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{\Omega}}{2\psi}coth\left(\frac{\sqrt{\Omega}}{2P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)}+\frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

$$\phi_1(x,t)=\frac{2P\lambda+m(2h\psi+Q)}{2P}-\frac{m(h^2\psi+hQ-E)}{P}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{\Omega}}{2\psi}coth\left(\frac{\sqrt{\Omega}}{2P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)}+\frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

Again if we pick $r_2=0$ but $r_1\neq 0$, then the solutions (4.1.6) and (4.1.7) is simplified as

$$w_2(x,t)=\frac{P+m(2h\psi+Q)}{2P}-\frac{m(h^2\psi+hQ-E)}{P}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{\Omega}}{2\psi}tanh\left(\frac{\sqrt{\Omega}}{2P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)}+\frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

$$\phi_2(x,t)=\frac{2P\lambda+m(2h\psi+Q)}{2P}-\frac{m(h^2\psi+hQ-E)}{P}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{\Omega}}{2\psi}tanh\left(\frac{\sqrt{\Omega}}{2P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)}+\frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

When $Q\neq 0, \psi=P-R$ and $\Omega=Q^2+4E(P-R)<0$, inserting the values of the constants arranged in Eq. (4.1.4) into Eq. (4.1.3) and if we choose $r_1=0$ but $r_2\neq 0$ and simplifying, we obtained the travelling wave solutions as

$$w_3(x,t)=\frac{P+m(2h\psi+Q)}{2P}-\frac{m(h^2\psi+hQ-E)}{P}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{-\Omega}}{2\psi}cot\left(\frac{\sqrt{-\Omega}}{2P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)}+\frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

$$\phi_3(x,t)=\frac{2P\lambda+m(2h\psi+Q)}{2P}-\frac{m(h^2\psi+hQ-E)}{P}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{-\Omega}}{2\psi}cot\left(\frac{\sqrt{-\Omega}}{2P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)}+\frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

Again if we pick $r_2=0$ but $r_1\neq 0$, then we obtained the travelling wave solutions as

$$w_4(x,t)=\frac{P+m(2h\psi+Q)}{2P}-\frac{m(h^2\psi+hQ-E)}{P}\left(h+\frac{Q}{2\psi}-\frac{\sqrt{-\Omega}}{2\psi}tan\left(\frac{\sqrt{-\Omega}}{2P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)}+\frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

$$\phi_4(x,t)=\frac{2P\lambda+m(2h\psi+Q)}{2P}-\frac{m(h^2\psi+hQ-E)}{P}\left(h+\frac{Q}{2\psi}-\frac{\sqrt{-\Omega}}{2\psi}tan\left(\frac{\sqrt{-\Omega}}{2P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)}+\frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

When $Q \neq 0, \psi = P - R$ and $\Omega = Q^2 + 4E(P - R) = 0$, inserting the values of the constants arranged in Eq. (4.1.4) into Eq. (4.1.3) and if we pick $r_1 = 0$ but $r_2 \neq 0$ and simplifying, we obtained the travelling wave solutions as

$$w_5(x,t) = \frac{P + m(2h\psi + Q)}{2P} - \frac{m(h^2\psi + hQ - E)}{P}\left(h + \frac{Q}{2\psi} + \left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-1}\right)^{-1}$$

$$\phi_5(x,t) = \frac{2P\lambda + m(2h\psi + Q)}{2P} - \frac{m(h^2\psi + hQ - E)}{P}\left(h + \frac{Q}{2\psi} + \left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-1}\right)^{-1}$$

But if we pick $r_2 = 0$ but $r_1 \neq 0$, then we attained trivial solution, which is not recorded here.

When $Q = 0, \psi = P - R$ and $\Delta = \psi E > 0$, inserting the values of the constants arranged in Eq. (4.1.4) into Eq. (4.1.3) and if we pick $r_1 = 0$ but $r_2 \neq 0$ and simplifying, we obtained the travelling wave solutions as

$$w_6(x,t) = \frac{P + m(2h\psi + Q)}{2P} - \frac{m(h^2\psi + hQ - E)}{P}\left(h + \frac{\sqrt{\Delta}}{\psi} coth\left(\frac{\sqrt{\Delta}}{P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

$$\phi_6(x,t) = \frac{2P\lambda + m(2h\psi + Q)}{2P} - \frac{m(h^2\psi + hQ - E)}{P}\left(h + \frac{\sqrt{\Delta}}{\psi} coth\left(\frac{\sqrt{\Delta}}{P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

Again if we pick $r_2 = 0$ but $r_1 \neq 0$, then we obtained the travelling wave solutions as

$$w_7(x,t) = \frac{P + m(2h\psi + Q)}{2P} - \frac{m(h^2\psi + hQ - E)}{P}\left(h + \frac{\sqrt{\Delta}}{\psi} tanh\left(\frac{\sqrt{\Delta}}{P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

$$\phi_7(x,t) = \frac{2P\lambda + m(2h\psi + Q)}{2P} - \frac{m(h^2\psi + hQ - E)}{P}\left(h + \frac{\sqrt{\Delta}}{\psi} tanh\left(\frac{\sqrt{\Delta}}{P}\left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

When $Q = 0, \psi = P - R$ and $\Delta = \psi E < 0$, inserting the values of the constants arranged in Eq. (4.1.4) into Eq. (4.1.3) and if we pick $r_1 = 0$ but $r_2 \neq 0$ and simplifying, we obtained the travelling wave solutions as

$$w_8(x,t) = \frac{P + m(2h\psi + Q)}{2P} - \frac{m(h^2\psi + hQ - E)}{P}\left(h + \frac{\sqrt{-\Delta}}{\psi}cot\left(\frac{\sqrt{-\Delta}}{P}\left(\frac{mx^\varepsilon}{\Gamma(1+\varepsilon)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

$$\phi_8(x,t) = \frac{2P\lambda + m(2h\psi + Q)}{2P} - \frac{m(h^2\psi + hQ - E)}{P}\left(h + \frac{\sqrt{-\Delta}}{\psi}cot\left(\frac{\sqrt{-\Delta}}{P}\left(\frac{mx^\varepsilon}{\Gamma(1+\varepsilon)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

Again if we pick $r_2 = 0$ but $r_1 \neq 0$, then we obtained the travelling wave solutions as

$$w_9(x,t) = \frac{P + m(2h\psi + Q)}{2P} - \frac{m(h^2\psi + hQ - E)}{P}\left(h - \frac{\sqrt{-\Delta}}{\psi}tan\left(\frac{\sqrt{-\Delta}}{P}\left(\frac{mx^\varepsilon}{\Gamma(1+\varepsilon)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

$$\phi_9(x,t) = \frac{2P\lambda + m(2h\psi + Q)}{2P} - \frac{m(h^2\psi + hQ - E)}{P}\left(h - \frac{\sqrt{-\Delta}}{\psi}tan\left(\frac{\sqrt{-\Delta}}{P}\left(\frac{mx^\varepsilon}{\Gamma(1+\varepsilon)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}$$

Where

$$k = \frac{m\left(3P^2 - m^2\left(4E\psi + Q^2\right)\right)}{4P^2}.$$

It is remarkable to see that the traveling wave solutions $w_1 - w_9$, and $\phi_1 - \phi_9$ of the space-time fractional coupled mKdV equations are fresh and further more general and have not been familiar in the previous solutions. Obtained solutions occur to be convenient to search the demandable mathematical model of motion in shallow water wave under gravity and wave propagation, optics and ion acoustic waves in plasma, model of the fractional fluid mechanics system, model of wave particle duality is noteworthy, model of turbulent motion and also in financial mathematics.

IV.II: The Whitham-Broer-Kaup equations

In this sub-section, we evaluate some appropriate close form traveling wave solutions to the space-time fractional coupled Whitham-Broer-Kaup (WBK) equations by making use of the generalized G'/G-expansion method. Let us consider the space-time fractional coupled WBK equations of the form:

$$D_t^\alpha u + u\,D_x^\alpha u + D_x^\alpha v + \beta\,D_x^\alpha\,D_x^\alpha u = 0$$

$$D_t^\alpha v + D_x^\alpha(uv) - \beta\,D_x^\alpha\,D_x^\alpha v + \gamma D_x^\alpha D_x^\alpha\,D_x^\alpha u = 0;\; 0 < \alpha \leq 1, \tag{4.2.1}$$

where α are fractional order derivative and β and γ are arbitrary constants and also β and γ are stand for in different diffusion powers. The WBK equations are mathematical equations that describe the propagation of shallow water waves with different dispersion relations, internal solitary waves in a density and current stratified shear flow with a free surface, ion acoustic waves in plasma, turbulent motion and also in financial mathematics. The traveling transformation (3.2) is used to convert the equation (4.2.1) into the following nonlinear ODE:

$$k\,u' + m\,u\,u' + m\,v' + \beta\,m^2u'' = 0$$

$$k\,v' + m\,u\,v' + m\,v\,u' - \beta\,m^2v'' + \gamma\,m^3u''' = 0 \qquad (4.2.2)$$

Now, balancing the linear term of the highest order derivative term and the nonlinear term of the highest order occurring in (4.2.2), yields $N_1 = 1$ and $N_1 = 2$. Then the solution of Eq. (4.2.2) is the form:

$$u = b_1(h+M)^{-1} + a_0 + a_1(h+M)$$

$$v = e_2(h+M)^{-2} + e_1(h+M)^{-1} + c_0 + c_1(h+M) + c_2(h+M)^2 \qquad (4.2.3)$$

where $a_0, a_1, b_1, c_0, c_1, c_2, e_1$ and e_2 are arbitrary constants to be determined, such that either a_1 or b_1 may be zero, but both a_1 and b_1 cannot be zero at a time and also either c_2 or e_2 may be zero, but both c_2 and e_2 cannot be zero at a time.

Inserting Eq. (4.2.3) together with Eqs. (3.6) and (3.7) into Eq. (4.2.2), the left hand side is converted into polynomials in $(h+M)^N$ $(N = 0,1,2,\dots)$ and $(h+M)^{-N}$ $(N = 1,2,3,\dots)$. We collect each coefficient of these resulted polynomials and setting them zero yields a set of simultaneous algebraic equations (for simplicity the equations are not present here) for $a_0, a_1, b_1, c_0, c_1, c_2, e_1, e_2, h, k$. Solving these algebraic equations with the help of symbolic computation software, such as, Maple, we obtain the following 04 (four) sets of solutions.

Set-1.

$$a_0 = a_0,\; a_1 = \pm\frac{\psi L m}{P},\; c_0 = 0, b_1 = \frac{\pm\, m\, L\theta}{P}, c_1 = -\frac{\omega\, m^2 Y}{P^2},\; c_2 = \frac{m^2 Y(X - 2PR)}{P^2},$$

$$e_1 = \frac{m^2\theta Z(2h\psi + Q)}{P^2},\;\; e_2 = -\frac{m^2\theta^2 Z}{P^2}, h = h, k = -\left(m\,a_0 \pm \frac{m\,L(2h\psi + B)}{2P}\right) \quad (4.2.4)$$

Set-2.

$$a_0 = a_0, b_1 = 0, e_1 = 0, h = h, e_2 = 0,\ k = -\left(a_0 \pm \frac{m^2 L(2h\psi+Q)}{2P}\right),$$
$$a_1 = \pm\frac{mL\psi}{P}, c_1 =$$
$$-\frac{m^2}{P^2}\left((\pm 2\beta L - hL^2)\ (X - 2PR) +\ \psi QY\right), c_2 = \frac{m^2}{P^2} Y(X - 2PR),$$
$$c_0 = -\frac{m^2}{P^2}\left(\frac{L^2}{2}(h^2X - 4PR) + hQ(\psi - R) - \psi\left(\frac{L^2E}{2} \pm \beta L(hQ + E)\right) \pm \beta Lh^2(2PR - X)\right) \qquad (4.2.5)$$

Set-3.

$$a_0 = a_0,$$
$$a_1 = 0, h = h, c_1 = 0, c_2 = 0,\ k = -\left(a_0 \pm \frac{m^2 L(2h\psi+Q)}{2P}\right), b_1 = \pm\frac{mL\theta}{P},$$
$$e_2 = \frac{m^2\theta}{P^2}\left(\frac{L^2}{2} \pm \beta Lh^2\psi\right)$$
$$c_0 = \frac{m^2}{P^2}\left(\pm\beta Lh^2(X - 2PR) - \frac{L^2}{2}\theta\right), e_1 = \frac{m^2}{P^2}\left(h^3X(L^2 + 2\beta) + \frac{3}{2}L^2Qh^2\psi - 2\psi hEZ + \right.$$
$$\left. Y(Q(Qh - E) - 4Ph^3R)\right) \qquad (4.2.6)$$

Set-4.

$$h = -\frac{Q}{2\psi}, k = -ma_0, a_0 = a_0, c_1 = 0, e_1 = 0, a_1 == \pm\frac{mL\psi}{P},$$
$$b_1 = \pm\frac{mL\,(Q+4E\psi)}{2\psi P}, c_2 = \frac{m^2Y}{P^2}(X - 2PR),$$
$$c_0 = \frac{m^2}{4P^2}(4E\psi(\pm L + \beta)^2 + \beta(1 + \beta) - 2\gamma) \pm B^2L\left(\frac{1}{2} + (\pm B^2L - 3\beta)\right),$$
$$e_2 = \frac{m^2}{8P^2\psi}\left(L^2\left(4E^2(2P - X) - Q^2\left(2E\psi - \frac{Q^2}{4}\right)\right) \pm 8\beta E^2L(2X + \psi - 4PR) \pm L\beta B^4\right) \qquad (4.2.7)$$

where

$$L = 2\sqrt{\gamma + \beta^2}, X = P^2 + R^2,$$
$$\psi = P - R, \theta = h^2\psi + hQ - E, \omega = 2h(X - 2PR) + Q\psi,$$
$Y = \left(\pm\beta L - \frac{L^2}{2}\right), Z = \left(\pm\beta L + \frac{L^2}{2}\right)$ and a_0, P, Q, R, E, L, h are free parameters.

For simplicity we have discussed on the solution Set-1 of the mentioned equation, arranged by Eq. (4.2.4) as follows and other sets of solutions are omitted here.

When $Q \neq 0, \psi = P - R$ and $\Omega = Q^2 + 4E(P - R) > 0$, inserting the values of the constants arranged in Eq. (4.2.4) into Eq. (4.2.3) and simplifying, we obtained the travelling wave solutions

$$u(\xi) = a_0 \pm \frac{\psi L m}{P}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{r_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{r_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right) \pm \frac{m L\theta}{P}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{r_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{r_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)^{-1} \quad (4.2.8)$$

$$v(\xi) = -\frac{\omega m^2 Y}{P^2}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{r_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{r_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right) + \frac{m^2 Y(X - 2PR)}{P^2}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{r_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{r_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)^2 +$$
$$\frac{m^2\theta(2h\psi + Q)Z}{P^2}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{r_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{r_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)^{-1} - \frac{m^2\theta^2 Z}{P^2}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{r_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{r_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right) + r_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)^{-2} \quad (4.2.9)$$

where

$$\xi = \frac{m x^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{k t^{\alpha}}{\Gamma(1+\alpha)}.$$

Since r_1 and r_2 are integral constants, so we might choose arbitrarily their values. If we pick $r_1 = 0$ but $r_2 \neq 0$, then the solutions (4.2.8) and (4.2.9) is simplified as

$$u_1(\xi) = a_0 \pm \frac{\psi L m}{P}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} coth\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right) \pm \frac{m L\theta}{P}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} coth\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right)^{-1}$$

$$v_1(\xi) = -\frac{\omega m^2 Y}{P^2}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} coth\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right) + \frac{m^2 Y(X - 2PR)}{P^2}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} coth\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right)^2$$
$$+ \frac{m^2\theta(2h\psi + Q)Z}{P^2}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} coth\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right)^{-1} - \frac{m^2\theta^2 Z}{P^2}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} coth\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right)^{-2}$$

Again if we pick $r_2 = 0$ but $r_1 \neq 0$, then the solutions (4.2.8) and (4.2.9) is simplified as

$$u_2(\xi) = a_0 \pm \frac{\psi L m}{P}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} tanh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right) \pm \frac{m L\theta}{P}\left(h + \frac{Q}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} tanh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right)^{-1}$$

$$v_2(\xi) = -\frac{\omega m^2 Y}{P^2}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{\Omega}}{2\psi}tanh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right)+\frac{m^2Y(X-2PR)}{P^2}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{\Omega}}{2\psi}tanh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right)^2$$

$$+\frac{m^2\theta(2h\psi+Q)Z}{P^2}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{\Omega}}{2\psi}tanh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right)^{-1}-\frac{m^2\theta^2 Z}{P^2}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{\Omega}}{2\psi}tanh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)\right)^{-2}$$

When $Q \neq 0, \psi = P - R$ and $\Omega = Q^2 + 4E(P-R) < 0$, inserting the values of the constants arranged in Eq. (4.2.4) into Eq. (4.2.3) and if we pick $r_1 = 0$ but $r_2 \neq 0$ and simplifying, we obtained the travelling wave solutions as

$$u_3(\xi) = a_0 \pm \frac{\psi L m}{P}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{-\Omega}}{2\psi}cot\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)\pm\frac{m\,L\theta}{P}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{-\Omega}}{2\psi}cot\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)^{-1}$$

$$v_3(\xi) = -\frac{\omega m^2 Y}{P^2}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{-\Omega}}{2\psi}cot\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)+\frac{m^2Y(X-2PR)}{P^2}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{-\Omega}}{2\psi}cot\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)^2$$

$$+\frac{m^2\theta(2h\psi+Q)Z}{P^2}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{-\Omega}}{2\psi}cot\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)^{-1}-\frac{m^2\theta^2 Z}{P^2}\left(h+\frac{Q}{2\psi}+\frac{\sqrt{-\Omega}}{2\psi}cot\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)^{-2}$$

Again if we pick $r_2 = 0$ but $r_1 \neq 0$, then we obtained the travelling wave solutions as

$$u_4(\xi) = a_0 \pm \frac{\psi L m}{P}\left(h+\frac{Q}{2\psi}-\frac{\sqrt{-\Omega}}{2\psi}tan\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)\pm\frac{m\,L\theta}{P}\left(h+\frac{Q}{2\psi}-\frac{\sqrt{-\Omega}}{2\psi}tan\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)^{-1}$$

$$v_4(\xi) = -\frac{\omega m^2 Y}{P^2}\left(h+\frac{Q}{2\psi}-\frac{\sqrt{-\Omega}}{2\psi}tan\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)+\frac{m^2Y(X-2PR)}{P^2}\left(h+\frac{Q}{2\psi}-\frac{\sqrt{-\Omega}}{2\psi}tan\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)^2$$

$$+\frac{m^2\theta(2h\psi+Q)Z}{P^2}\left(h+\frac{Q}{2\psi}-\frac{\sqrt{-\Omega}}{2\psi}tan\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)^{-1}-\frac{m^2\theta^2 Z}{P^2}\left(h+\frac{Q}{2\psi}-\frac{\sqrt{-\Omega}}{2\psi}tan\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)\right)^{-2}$$

When $Q \neq 0, \psi = P - R$ and $\Omega = Q^2 + 4E(P-R) = 0$, inserting the values of the constants arranged in Eq. (4.2.4) into Eq. (4.2.3) and if we pick $r_1 = 0$ but $r_2 \neq 0$ and simplifying, we attained the travelling wave solutions as

$$u_5(\xi) = a_0 \pm \frac{\psi L m}{P}\left(h+\frac{Q}{2\psi}+\frac{1}{\xi}\right)\pm\frac{m\,L\theta}{P}\left(h+\frac{Q}{2\psi}+\frac{1}{\xi}\right)^{-1}$$

$$v_5(\xi) = -\frac{\omega m^2 Y}{P^2}\left(h+\frac{Q}{2\psi}+\frac{1}{\xi}\right)+\frac{m^2 Y(X-2PR)}{P^2}\left(h+\frac{Q}{2\psi}+\frac{1}{\xi}\right)^2$$
$$+\frac{m^2\theta(2h\psi+Q)Z}{P^2}\left(h+\frac{Q}{2\psi}+\frac{1}{\xi}\right)^{-1}-\frac{m^2\theta^2 Z}{P^2}\left(h+\frac{Q}{2\psi}+\frac{1}{\xi}\right)^{-2}$$

But if we pick $r_2 = 0$ but $r_1 \neq 0$, then we attained trivial solution, which is not recorded here.

When $Q = 0, \psi = P - R$ and $\Delta = \psi E > 0$, substituting the values of the constants arranged in Eq. (4.2.4) into Eq. (4.2.3) and if we pick $r_1 = 0$ but $r_2 \neq 0$ and simplifying, we obtained the travelling wave solutions as

$$u_6(\xi) = a_0 \pm \frac{\psi L m}{P}\left(h+\frac{\sqrt{\Delta}}{\psi}coth\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right) \pm \frac{m\,L\theta}{P}\left(h+\frac{\sqrt{\Delta}}{\psi}coth\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right)^{-1}$$

$$v_6(\xi) = -\frac{\omega m^2 Y}{P^2}\left(h+\frac{\sqrt{\Delta}}{\psi}coth\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right)+\frac{m^2 Y(X-2PR)}{P^2}\left(h+\frac{\sqrt{\Delta}}{\psi}coth\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right)^2$$

$$+\frac{m^2\theta(2h\psi+Q)Z}{P^2}\left(h+\frac{\sqrt{\Delta}}{\psi}coth\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right)^{-1}-\frac{m^2\theta^2 Z}{P^2}\left(h+\frac{\sqrt{\Delta}}{\psi}coth\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right)^{-2}$$

Again if we pick $r_2 = 0$ but $r_1 \neq 0$, then we obtained the travelling wave solutions as

$$u_7(\xi) = a_0 \pm \frac{\psi L m}{P}\left(h+\frac{\sqrt{\Delta}}{\psi}tanh\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right) \pm \frac{m\,L\theta}{P}\left(h+\frac{\sqrt{\Delta}}{\psi}tanh\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right)^{-1}$$

$$v_7(\xi) = -\frac{\omega m^2 Y}{P^2}\left(h+\frac{\sqrt{\Delta}}{\psi}r_1\,tanh\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right)+\frac{m^2 Y(X-2PR)}{P^2}\left(h+\frac{\sqrt{\Delta}}{\psi}r_1\,tanh\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right)^2$$

$$+\frac{m^2\theta(2h\psi+Q)Z}{P^2}\left(h+\frac{\sqrt{\Delta}}{\psi}tanh\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right)^{-1}-\frac{m^2\theta^2 Z}{P^2}\left(h+\frac{\sqrt{\Delta}}{\psi}tanh\left(\frac{\sqrt{\Delta}}{P}\xi\right)\right)^{-2}$$

When $Q = 0, \psi = P - R$ and $\Delta = \psi E < 0$, inserting the values of the constants arranged in Eq. (4.2.4) into Eq. (4.2.3) and if we pick $r_1 = 0$ but $r_2 \neq 0$ and simplifying, we obtained the travelling wave solutions as

$$u_8(\xi) = a_0 \pm \frac{\psi L m}{P}\left(h+\frac{\sqrt{-\Delta}}{\psi}cot\left(\frac{\sqrt{-\Delta}}{P}\xi\right)\right) \pm \frac{m\,L\theta}{P}\left(h+\frac{\sqrt{-\Delta}}{\psi}cot\left(\frac{\sqrt{-\Delta}}{P}\xi\right)\right)^{-1}$$

$$v_8(\xi) = -\frac{\omega m^2 Y}{P^2}\left(h+\frac{\sqrt{-\Delta}}{\psi}cot\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)\right)+\frac{m^2Y(X-2PR)}{P^2}\left(h+\frac{\sqrt{-\Delta}}{\psi}cot\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)\right)^2$$

$$+\frac{m^2\theta(2h\psi+Q)Z}{P^2}\left(h+\frac{\sqrt{-\Delta}}{\psi}cot\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)\right)^{-1}-\frac{m^2\theta^2 Z}{P^2}\left(h+\frac{\sqrt{-\Delta}}{\psi}cot\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)\right)^{-2}$$

Again if we pick $r_2 = 0$ but $r_1 \neq 0$, then we obtained the travelling wave solutions as

$$u_9(\xi) = a_0 \pm \frac{\psi Lm}{P}\left(h-\frac{\sqrt{-\Delta}}{\psi}tan\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)\right) \pm \frac{m\,L\theta}{P}\left(h-\frac{\sqrt{-\Delta}}{\psi}tan\left(\frac{\sqrt{-\Delta}}{P}\xi\right)\right)^{-1}$$

$$v_9(\xi) = -\frac{\omega m^2 Y}{P^2}\left(h-\frac{\sqrt{-\Delta}}{\psi}tan\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)\right)+\frac{m^2Y(X-2PR)}{P^2}\left(h-\frac{\sqrt{-\Delta}}{\psi}tan\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)\right)^2$$

$$+\frac{m^2\theta(2h\psi+Q)Z}{P^2}\left(h-\frac{\sqrt{-\Delta}}{\psi}tan\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)\right)^{-1}-\frac{m^2\theta^2 Z}{P^2}\left(h-\frac{\sqrt{-\Delta}}{\psi}tan\left(\frac{\sqrt{-\Delta}}{P}\,\xi\right)\right)^{-2}$$

In the above obtained solutions, $u_1 - u_9$ and $v_1 - v_9$ we want to use $\xi = \left(\frac{mx^{\varepsilon}}{\Gamma(1+\varepsilon)}+\frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right)$ but for simplicity we have omitted here, where $k = -\left(m\,a_0 \pm \frac{m\,L(2h\psi+B)}{2P}\right)$.

It is prominent to see that the traveling wave solutions $u_1 - u_9$ and $v_1 - v_9$ of the space-time fractional coupled WBK equation are fresh and further more general and have not been familiar in the previous solutions. Obtained solutions occur to be convenient to search the demandable model of the propagation of shallow water waves, internal solitary waves in density and current stratified shear flow with a free surface, periodic wave, combined formal solitary wave, ion acoustic waves in plasma, model of turbulent motion, model of the fractional fluid mechanics system, model of wave particle duality is noteworthy and also in financial mathematics.

V. Results and Discussion

It is remarkable to mention that some of the examined solutions attain good agreement with the already published solutions. A comparison between Zayed *et al.* solutions [XXXV] (investigate by the modified extended tanh-function method) and our obtained solutions of the space-time fractional coupled mKdV equations and the space-time fractional coupled WBK equations are shown in the following tables.

Table 1: Comparison between Zayed *et al.* solutions **(see appendix A)** and our obtained solutions of the space-time fractional coupled mKdV equations.

Zayed *et al.* solutions	Solutions obtained in this article
1. $u_1(\xi) = \frac{1}{2} + k\sqrt{-b}\, tanh\left(\sqrt{-b}\,\xi\right)$	1. If $Q = h = 0, \sqrt{\Omega} = 2P\sqrt{-b}, k = -\frac{mE\psi}{P^2 b}$ then the solution is $w_1(\xi) = \frac{1}{2} + k\sqrt{-b}\, tanh\left(\sqrt{-b}\,\xi\right)$
2. $v_1(\xi) = \lambda + k\sqrt{-b}\, tanh\left(\sqrt{-b}\,\xi\right)$	2. If $Q = h = 0, \sqrt{\Omega} = 2P\sqrt{-b}, k = -\frac{mE\psi}{P^2 b}$ then the solution is $\phi_1(\xi) = \lambda + k\sqrt{-b}\, tanh\left(\sqrt{-b}\,\xi\right)$
3. $u_2(\xi) = \frac{1}{2} + k\sqrt{-b}\, coth\left(\sqrt{-b}\,\xi\right)$	3. If $Q = h = 0, \sqrt{\Omega} = 2P\sqrt{-b}, k = -\frac{mE\psi}{P^2 b}$ then the solution is $w_2(\xi) = \frac{1}{2} + k\sqrt{-b}\, coth\left(\sqrt{-b}\,\xi\right)$
4. $v_2(\xi) = \lambda + k\sqrt{-b}\, coth\left(\sqrt{-b}\,\xi\right)$	4. If $Q = h = 0, \sqrt{\Omega} = 2P\sqrt{-b}, k = -\frac{mE\psi}{P^2 b}$ then the solution is $\phi_2(\xi) = \lambda + k\sqrt{-b}\, coth\left(\sqrt{-b}\,\xi\right)$
5. $u_3(\xi) = \frac{1}{4}\left[2 + \frac{1}{\sqrt{-b}}\, coth\left(\sqrt{-b}\,\xi\right)\right]$	5. If $Q = h = 0, \sqrt{\Delta} = P\sqrt{-b}, \frac{mE\psi}{P^2} = \frac{1}{4}$ then the solution is $w_7(\xi) = \frac{1}{4}\left[2 + \frac{1}{\sqrt{-b}}\, coth\left(\sqrt{-b}\,\xi\right)\right]$
6. $v_3(\xi) = \frac{\lambda}{4}\left[4 + \frac{1}{\sqrt{-b}}\, coth\left(\sqrt{-b}\,\xi\right)\right]$	6. If $Q = h = 0, \sqrt{\Delta} = P\sqrt{-b}, \frac{mE\psi}{P^2} = \frac{1}{4}$ then the solution is $\phi_7(\xi) = \frac{\lambda}{4}\left[4 + \frac{1}{\sqrt{-b}}\, coth\left(\sqrt{-b}\,\xi\right)\right]$
7. $u_4(\xi) = \frac{1}{4}\left[2 + \frac{1}{\sqrt{-b}}\, tanh\left(\sqrt{-b}\,\xi\right)\right]$	7. If $Q = h = 0, \sqrt{\Delta} = P\sqrt{-b}, \frac{mE\psi}{P^2} = \frac{1}{4}$ then the solution is $w_6(\xi) = \frac{1}{4}\left[2 + \frac{1}{\sqrt{-b}}\, tanh\left(\sqrt{-b}\,\xi\right)\right]$
8. $v_4(\xi) = \frac{\lambda}{4}\left[4 + \frac{1}{\sqrt{-b}}\, tanh\left(\sqrt{-b}\,\xi\right)\right]$	8. If $Q = h = 0, \sqrt{\Delta} = P\sqrt{-b}, \frac{mE\psi}{P^2} = \frac{1}{4}$ then the solution is $\phi_6(\xi) = \frac{\lambda}{4}\left[4 + \frac{1}{\sqrt{-b}}\, tanh\left(\sqrt{-b}\,\xi\right)\right]$

In addition in the table 1, we obtain further new exact solution solutions $w_3 - w_5, w_8, w_9, \phi_3 - \phi_5, \phi_8$ and ϕ_9 which are not reported in the

Zayed *et al.* solutions. When the arbitrary constants assume particular values the obtained solutions reduce to some special functions **(see table 1)**.

Table 2: Comparison between Zayed *et al.* solutions **(see appendix B)** and

Zayed *et al.* solutions	Solutions obtained in this article
1.If $b=-\frac{1}{2}$ then $v_2(\xi)=2k^2\beta^2 tan^2\left(\sqrt{-\frac{1}{2}}\,\xi\right)$	1. If $h=Q=\omega=0,\pm\beta=-\frac{L^2}{2},\sqrt{-\Omega}=2P\sqrt{-\frac{1}{2}},\psi^2=\frac{m^2(X-2PR)(L^2+L)}{8k^2\beta^2},$ then $v_4(\xi)=2k^2\beta^2 tan^2\left(\sqrt{-\frac{1}{2}}\,\xi\right)$
2.If $b=\frac{1}{4}$ then $v_2(\xi)=-k^2\beta^2 cot^2\left(\sqrt{\frac{1}{4}}\,\xi\right)$	2. If $h=Q=\omega=0,\pm\beta=-\frac{L^2}{2},\sqrt{-\Delta}=P\sqrt{\frac{1}{4}},\psi^2=\frac{m^2(X-2PR)(L^2+L)}{8k^2\beta^2},$ then $v_8(\xi)=-k^2\beta^2 cot^2\left(\sqrt{\frac{1}{4}}\,\xi\right)$
3. If $b=-\frac{1}{2}$ then $v_3(\xi)=2k^2\beta^2 cot^2\left(\sqrt{-\frac{1}{2}}\,\xi\right)$	3. If $h=Q=\omega=0,\pm\beta=-\frac{L^2}{2},\sqrt{-\Omega}=2P\sqrt{-\frac{1}{2}},\psi^2=\frac{m^2(X-2PR)(L^2+L)}{8k^2\beta^2},$ then $v_3(\xi)=2k^2\beta^2 cot^2\left(\sqrt{-\frac{1}{2}}\,\xi\right)$
4. If $b=\frac{1}{4}$ then $v_3(\xi)=-k^2\beta^2 tan^2\left(\sqrt{\frac{1}{4}}\,\xi\right)$	4. If $h=Q=\omega=0,\pm\beta=-\frac{L^2}{2},\sqrt{-\Delta}=P\sqrt{\frac{1}{4}},\psi^2=\frac{m^2(X-2PR)(L^2+L)}{8k^2\beta^2},$ then $v_9(\xi)=-k^2\beta^2 tan^2\left(\sqrt{\frac{1}{4}}\,\xi\right)$

our obtained solutions of the space-time fractional coupled WBK equations. In addition in the table 2, we see that only 04 (four) solutions u_1, v_1, u_2 and u_3 of the Zayed *et al.* which are not reported in our obtained solutions but it is remarkable to see that we are obtained 14 (fourteen) further new exact solutions u_1-u_9, v_1, v_2, v_5, v_6 and v_7 which are not reported in the Zayed *et al.* solutions. When the arbitrary constants assume particular values the obtained solutions reduce to some special functions **(see table 2)**.

Here it is noticed that, we have obtained more new wave solutions using by the generalized (G'/G)-expansion method which have not been reported in the previous literature. Hence, compare between our obtained solutions and their solutions, we state that our solutions are more general and huge amount of new exact travelling wave solutions.

VI. Conclusion

In this article, we have examined the new, useful and further general solutions to the space-time fractional coupled mKdV equations and the space-time fractional coupled WBK equations by means of the efficient and powerful technique known as the generalized (G'/G)-expansion method. These solutions are attained in general form and definite values of the included parameters yield diverse known soliton solutions. The obtained solutions might be useful to the analyzed the fluid flow, ion osculate waves in plasma, signal processing waves through optical fibers, fractional quantum mechanics, internal solitary waves in a density and current stratified shear flow with a free surface, and water wave mechanics specially shallow water waves under gravity and propagation for the both equations. We also have shown that the generalized (G'/G)-expansion method over the modified extended tanh-function method offers more general form and have established huge amount of new exact travelling wave solutions. The established results also show that the generalized (G'/G)-expansion method is more general, powerful, and efficient which can be helped for many other nonlinear fractional differential equations to obtain exact travelling wave solutions and the new solitary wave solutions in mathematical physics.

References

I. Alam, M.N. and Akbar, M.A. "The new approach of the generalized (G'/G)-expansion method for nonlinear evolution equations". Ain Shams Eng. J., Vol. 5, pp 595-603 (2014).

II. Alam, M.N. and Akbar, M.A. "Application of the new approach of generalized (G'/G)-expansion method to find exact solutions of nonlinear PDEs in mathematical physics". BIBECHANA, Vol. 10, pp 58-70 (2014).

III. Ahmad, J., Mushtaq, M. and Sajjad, N. "Exact solution of Whitham-Broer-Kaup shallow water equations". J. Sci. Arts, Vol. 1, No. 30, pp 5-12 (2015).

IV. Ali, A.H.A "The modified extended tanh-function method for solving coupled mKdV and coupled Hirota-Satsuma coupled KdV equations". Phys. Lett. A, Vol. 363, No. (5-6), pp 420-425 (2007).

V. Atchi, L.M. and Appan, M.K., "A Review of the homotopy analysis method and its applications to differential equations of fractional order". Int. J. Pure Appl. Math., Vol. 113, No. (10), pp 369-384 (2017).

VI. Bekir, A. and Guner, O. "Exact solutions of nonlinear fractional differential equation by (G'/G)-expansion method". Chin. Phys. B, Vol. 22, No. (11), pp 1-6 (2013).

VII. Bekir, A., Kaplan, M. "Exponential rational function method for solving nonlinear equations arising in various physical models". Chin. J. Phys. **54**(3), 365–370 (2016).

VIII. Bulut, H. Baskonus, H.M. and Pandir, Y. "The modified trial equation method for fractional wave equation and time fractional generalized Burgers equation". Abst. Appl. Anal., 2013, Article ID 636802, (2013).

IX. Caputo, M. and Fabrizio, M.A. "A new definition of fractional derivatives without singular kernel". Math. Comput. Model., Vol. 1, pp. 73-85 (2015).

X. Deng, W. "Finite element method for the space and time fractional Fokker-Planck equation". Siam J. Numer. Anal., Vol. 47, No. (1), pp 204-226 (2009).

XI. Ege, S.M. and Misirli, E. "Solutions of space-time fractional foam drainage equation and the fractional Klein-Gordon equation by use of modified Kudryashov method". Int. J. Res. Advent Tech., Vol. 2, No. (3), pp 384-388 (2014).

XII. El-Sayed, A.M.A., Behiry, S.H. and Raslan, W.E. "The Adomin's decomposition method for solving an intermediate fractional advection-dispersion equation". Comput. Math. Appl., Vol. 59, No. (5), pp 1759-1765 (2010).

XIII. El-Borai, M.M., El-Sayed, W.G. and Al-Masroub, R.M. "Exact solutions for time fractional coupled Whitham-Broer-Kaup equations via exp-function method". Int. Res. J. Eng. Tech., Vol. 2, No. (6), pp 307-315 (2015).

XIV. Gomez-Aguilar, J.F., Yepez-Martnrez, H., Escober-Jimenez, R.F., Olivarer-Peregrino, V.H., Reyes, J.M. and Sosa, I.O. "Series solution for the time-fractional coupled mKdV equation using the homotopy analysis method". Math. Prob. Eng., Vol. 2016, Article ID 7047126, 8 pages 2016.

XV. He, J.H., Elagan, S.K. and Li, Z.B. "Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus". Phys. Lett. A, Vol. 376, No. (4), pp 257-259 (2012).

XVI. He, J.H. "Asymptotic methods for solitary solutions and compacts". Abst. Appl. Anal., Volume 2012, Article ID 916793, 130 pages (2012).

XVII. Helal, M.A. and Mehanna, M.S. "The tanh-function method and Adomin decomposition method for solving the foam drainage equation". App. Math. Comput., vol. 190, No. (1), 599-609 (2007).

XVIII. Inc, M. "The approximate and exact solutions of the space and time-fractional Burgers equations with initial conditions by the variational iteration method". J. Math. Anal. Appl., Vol. 345, No. (1), pp 476-484 (2008).

XIX. Jumarie, G. "Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results". Comput. Math. Appl., Vol. 51, No. (9-10), pp. 1367-1376 (2006).

XX. Kadem, A. and Baleanu, D. "On fractional coupled Whitham-Broer-Kaup equations". Rom. J. Phys., Vol. 56, No. (5-6), pp 629-635 (2011).

XXI. Kaplan, M. Bekir, A. Akbulut, A. and Aksoy, E. "The modified simple equation method for nonlinear fractional differential equations". Rom. J. Phys., Vol. 60, No. (9-10), pp 1374-1383 (2015).

XXII. Lu, B. "Backlund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations". Phys. Lett. A, Vol. 376, pp 2045-2048 (2012).

XXIII. Lu, B. "The first integral method for some time fractional differential equations". J. Math. Appl., Vol. 395, pp. 684-693 (2012).

XXIV. Lu, D., Yue, C. and Arshad, M. "Traveling wave solutions of space-time fractional generalized fifth-order KdV equation". Advances Math. Phys., Volume 2017, Article ID 6743276, 6 pages, (2017).

XXV. Moatimid, G.M., El-Shiekh, R.M., Ghani, A. and Al-Nowehy, A.A.H. "Modified Kudryashov method for finding exact solutions of the (2+1) dimensional modified Korteweg-de Varies equations and nonlinear Drinfeld-Sokolov system". American J. Comput. Appl. Math., Vol. 1, No. 1 (2011).

XXVI. Odibat, Z. and Monani, S. "The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics". Coumpt. Math. Appl., Vol. 58, No. (11-12), pp 2199-2208 (2009).

XXVII. Rabtah, A.A., Erturk, R.S. and Momani, S. "Solution of fractional oscillator by using differential transformation method". Comput. Math. Appl., Vol. 59, pp 1356-1362 (2010).

XXVIII. Saad, M., Ehgan, S.K., Hamed, Y.S. and Sayed, M. "Using a complex transformation to get an exact solution for fractional generalized coupled mKdV and KDV equations". Int. J. Basic Appl. Sci., Vol. 13, No. (01), pp 23-25 (2014).

XXIX. Wang, G.W. and Xu, T.Z. "The modified fractional sub-equation method and its applications to nonlinear fractional partial differential equations". Rom. J. Phys., Vol. 59, No. (7-8), pp 636-645 (2014).

XXX. Yan, Z. and Zhang, H. "New explicit solitary wave solutions and periodic wave solutions for Whitham-Broer-Kaup equations in shallow water". Phys. Lett. A, Vol. 285, No. (5-6), pp 355-362 (2001).

XXXI. Yepez-Martinez, H., J.M. Reyes and I.O. Sosa, "Fractional sub-equation method and analytical solutions to the Hirota-Satsuma coupled KdV equation and mKdv equation". British J. Math. Coumpt. Sci., Vol. 4, No. (4), pp 572-589 (2014).

XXXII. Younis, M. "The first integral method for time-space fractional differential equations". J. Adv. Phys., Vol. 2, pp 220-223 (2013).

XXXIII. Younis, M. and Zafar, A. "Exact solutions to nonlinear differential equations of fractional order via (G'/G)-expansion method". Appl. Math., Vol. 2014, No. (5), pp 1-6 (2014).

XXXIV. Zayed, E.M.E, Amer, Y.A. and Al-Nowehy, A.G. "The modified simple equation method and the multiple ex-function method for solving nonlinear fractional Sharma-Tasso-Olver equation". Acta Mathematicae Applicatae Sinica, English Series, Vol. 32, No. (4), pp 793-812 (2016).

XXXV. Zayed, E.M.E., Amer, Y.A and Shohib, R.M.A. "The fractional complex transformation for nonlinear fractional partial differential equations in the mathematical physics". J. Association Arab Uni. Basic Appl. Sci., Vol. 19, pp 59-69 (2016).

XXXVI. Zheng, B. "Exp-function method for solving fractional partial differential equations". Sci. World J., DOI: 10.1155/2013/465723 (2013).

XXXVII. Zheng, B. and Feng, Q. "The Jacobi elliptic equation method for solving fractional partial differential equations". Abst. Appl. Anal., 2014, 9 pages, Article ID 249071 (2014).

Appendix-A

Zayed *et al.* solutions (investigated by the modified extended tanh-function method [XXXV]) for the space-time fractional coupled mKdV equations are as follows:

$$u_1(\xi) = \frac{1}{2} + k\sqrt{-b}\, tanh\left(\sqrt{-b}\,\xi\right) \tag{A.1}$$

$$v_1(\xi) = \lambda + k\sqrt{-b}\, tanh\left(\sqrt{-b}\,\xi\right) \tag{A.2}$$

$$u_2(\xi) = \frac{1}{2} + k\sqrt{-b}\, tanh\left(\sqrt{-b}\,\xi\right) \tag{A.3}$$

$$v_2(\xi) = \lambda + k\sqrt{-b}\, tanh\left(\sqrt{-b}\,\xi\right) \tag{A.4}$$

$$u_3(\xi) = \frac{1}{4}\left[2 + \frac{1}{\sqrt{-b}}\, coth\left(\sqrt{-b}\,\xi\right)\right] \tag{A.5}$$

$$v_3(\xi) = \frac{\lambda}{4}\left[4 + \frac{1}{\sqrt{-b}}\, coth\left(\sqrt{-b}\,\xi\right)\right] \tag{A.6}$$

$$u_4(\xi) = \frac{1}{4}\left[2 + \frac{1}{\sqrt{-b}}\, tanh\left(\sqrt{-b}\,\xi\right)\right] \tag{A.7}$$

$$v_4(\xi) = \frac{\lambda}{4}\left[4 + \frac{1}{\sqrt{-b}}\, tanh\left(\sqrt{-b}\,\xi\right)\right] \tag{A.8}$$

Appendix-B

Zayed *et al.* solutions (investigated by the modified extended tanh-function method [XXXV]) for the space-time fractional coupled WBK equations are as follows:

$$u_1(\xi) = \frac{-2k\beta\,(b-1)\sqrt{-b}}{3b}\left\{1 - \left[tanh\left(\sqrt{-b}\,\xi\right) + coth\left(\sqrt{-b}\,\xi\right)\right\}\right. \tag{B.1}$$

$$v_1(\xi) = \frac{2k^2\beta^2\left(2b^2-b-1\right)}{3b}\left\{2 - \left[tanh^2\left(\sqrt{-b}\,\xi\right) + coth^2\left(\sqrt{-b}\,\xi\right)\right\}\right. \tag{B.2}$$

$$u_2(\xi) = \frac{2k\beta\,(b-1)}{3\sqrt{b}}\left\{-\sqrt{2} + \left[tan\left(\sqrt{b}\,\xi\right) + cot\left(\sqrt{b}\,\xi\right)\right\}\right. \tag{B.3}$$

$$v_2(\xi) = \frac{-2k^2\beta^2\,(b-1)}{9b}\left\{(4b-1)tan^2\left(\sqrt{b}\,\xi\right) - (2b+1)cot^2\left(\sqrt{b}\,\xi\right)\right\} \tag{B.4}$$

$$u_3(\xi) = \frac{-2k\beta\,(b-1)}{3\sqrt{b}}\left\{\sqrt{2} - \left[tan\left(\sqrt{b}\,\xi\right) + cot\left(\sqrt{b}\,\xi\right)\right\}\right. \tag{B.5}$$

$$v_3(\xi) = \frac{-2k^2\beta^2\,(b-1)}{9b}\left\{(4b-1)cot^2\left(\sqrt{b}\,\xi\right) - (2b+1)tan^2\left(\sqrt{b}\,\xi\right)\right\} \tag{B.6}$$