ISSN (Online) : 2454 -7190 ISSN (Print) 0973-8975
J.Mech.Cont.& Math. Sci., Vol.-13, No.-4, September-October (2018) Pages 1-11

Approximate Solution of Strongly Forced Nonlinear Vibrating Systems Which Vary With Time

[1]*Pinakee dey, [2]Nasir Uddin, [3]Md Asaduzzaman, [4]Sanjay kumar saha, [5]M. A. Sattar

[1,2,4,5] Department of Mathematics, Mawlana Bhashani Science and Technology University, Santosh, Tangail-1902, Bangladesh.

[3] Department of Computer science & Engineering, Ranada Prasad Shaha University,Narayangang, Bangladesh.

*Corresponding author : Dr. Pinakee Dey

E mail: pinakee_68@yahoo.com,

Abstract

Based on the combined work of extended Krylov-Bogoliubov-Mitropolskii method and harmonic balance (HB) method an analytical technique is presented to determine approximate solutions of nonlinear differential systems whose coefficients change slowly and periodically with time. Furthermore, a non-autonomous case also investigated in which an external force acts in this system. Formulation as well as determination of the solution is systematic and easier than the existing procedures. The method is illustrated by suitable examples.

Keywords : Asymptotic solution**,** Forced nonlinear oscillation, Varying coefficient, Unperturbed equation, KBM method, HB method.

I Introduction

Krylov-Bogoliubov-Mitropolskii (KBM) [VIII],[VII],[X] method is particularly convenient and is the widely used technique to obtain the approximate solutions. Originally the method developed for systems with periodic solutions, was later extended by Popov [III] for damped nonlinear oscillations. Arya and Bojadziev [IV] studied a second order time dependent differential equation with damping, slowly varying coefficients and small time delay in which a non-periodic external force acted. Murty [II] has developed a unified KBM method for solving second order nonlinear systems which cover the undamped, damped and over-damped cases. Shamsul [VI] has presented a unified method for solving an *n*-th order differential equation (autonomous) characterized by oscillatory, damped oscillatory and non-oscillatory processes. In other recent papers, a simple technique [I], [IX] has been presented to determine periodic solutions of nonlinear differential equations by using a truncated Fourier series in which there exists a strongly nonlinear term based on the Harmonic balance (HB) method. Recently, Roy and Shamsul [V] found an asymptotic solution of a differential system in which the coefficient changes in an exponential order of slowly varying time. In this paper, we have used combining the methods of extended KBM [VIII],[VII],[X] and Harmonic balance (HB) method [I],[IX] to find the solution of strongly nonlinear non-autonomous differential system in which an external force acts.

II Method

Let us consider the nonlinear non autonomous differential system with slowly varying coefficients

$$\ddot{x} + (c_1 + c_2 \cos\tau + c_3^2 \sin 2\tau)x = -\varepsilon\, f(x,\tau,\nu t), \qquad \tau = \varepsilon t \tag{1}$$

where the over-dots denote differentiation with respect to *t*, ε is a small parameter, c_1, c_2 and c_3 are constants, $c_2 = c_3 = \mathrm{O}(\varepsilon)$, *f* is a given nonlinear function. Setting $\omega^2(\tau) = (c_1 + c_2 \cos\tau + c_3^2 \sin 2\tau)$, $\omega(\tau)$ is known as frequency and ν is the frequency of the external acting force.

Putting $\varepsilon = 0$ and $\tau = \tau_0 =$ constant, in Eq.(1), we obtain the unperturbed solution of (1) in the form

$$x(t,0) = x_{1,0} e^{\lambda_1(\tau_0)t} + x_{-1,0} e^{\lambda_2(\tau_0)t}, \tag{2}$$

When $\varepsilon \neq 0$, we seek a solution of Eq. (1) in the form

$$x(t,\varepsilon) = x_1(t,\tau) + x_{-1}(t,\tau) + \varepsilon u_1(x_1,x_{-1},t,\tau) + \varepsilon^2 u_2(x_1,x_{-1},t,\tau) +, \tag{3}$$

where x_1 and x_{-1} satisfy the equations

$$\begin{aligned} \dot{x}_1 &= \lambda_1(\tau)x_1 + \varepsilon X_1(x_1,x_{-1},\tau) + \varepsilon^2 X_1(x_1,x_{-1},\tau), \\ \dot{x}_{-1} &= \lambda_2(\tau)x_{-1} + \varepsilon X_{-1}(x_1,x_{-1},\tau) + \varepsilon^2 X_{-1}(x_1,x_{-1},\tau). \end{aligned} \tag{4}$$

Differentiating $x(t,\varepsilon)$ two times with respect to *t*, substituting for the derivatives $\ddot{x}$ and *x* in the original equation (1) and equating the coefficient of ε, we obtain

$$\begin{aligned} &\left(\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1}\Omega x_{-1}\right)X_1 + \left(\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1}\Omega x_{-1}\right)X_{-1} + \lambda_1' x_1 + \lambda_2' x_{-1} - \lambda_2 X_1 - \lambda_1 X_{-1} + \\ &+ \left(\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1}\Omega x_{-1} - \lambda_1\right)\left(\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1}\Omega x_{-1} - \lambda_2\right)u_1 \\ &= -f^{(0)}(x_1,x_{-1},\tau,\nu t), \end{aligned} \tag{5}$$

Where

$$\lambda_1' = \frac{d\lambda_1}{d\tau},\ \lambda_2' = \frac{d\lambda_2}{d\tau},\ \Omega x_1 = \frac{\partial}{\partial x_1}, \Omega x_{-1} = \frac{\partial}{\partial x_{-1}},\ f^{(0)} = f(x_0, \dot{x}_0, \tau, \nu t)$$

Herein it is assumed that $f^{(0)}$ can be expanded in Taylor's series

$$f^{(0)} = \sum_{r_1, r_2 = 0}^{\infty} F_{r_1, r_2}(\tau) x_1^{r_1} x_{-1}^{r_2} \tag{6}$$

To obtain a solution of (1), we impose a restriction that $u_1 \cdots$ exclude the terms $x_1^{i_1} x_{-1}^{i_2}, i_1 - i_2 = \pm 1 \quad i_1, i_2 = 0, 1, 2 \cdots$. The assumption assures that $u_1 \cdots$ are free from secular type terms $te^{-\lambda_1 t}$. This restriction guarantees that the solution always excludes *secular*-type terms or the first harmonic terms, otherwise a sizeable error would occur [VI]. We shall be able to transform (3) to the exact form of the KBM [VIII],[VII],[X] solution by substituting $x_1 = \rho e^{i\phi}/2$ and $x_{-1} = \rho e^{-i\phi}/2$. Herein, ρ and ϕ are respectively amplitude and phase variables (see Shamsul [VI]). Under this assumption, we shall able to find the unknown functions u_1 and X_1, X_{-1}.

III Example

As example of the above procedure, let us consider a nonlinear non-autonomous system with slowly varying coefficients

$$\ddot{x} + (c_1^2 + c_2 \cos\tau + c_3^2 \sin 2\tau)x = -\varepsilon x^3 + +\varepsilon E \cos\nu t, \tag{7}$$

Here over dots denote differentiation with respect to t. $x_0 = x_1 + x_{-1}$ and the function $f^{(0)}$ becomes,

$$f^{(0)} = -(x_1^3 + 3x_1^2 x_{-1} + 3x_1 x_{-1}^2 + x_{-1}^3) + \frac{\varepsilon E}{2}\left(e^{i\nu t} + e^{-i\nu t}\right). \tag{8}$$

Following the assumption (discussed in section 2) excludes we substitute in (5) and separate it into two parts as

$$\begin{aligned}&\left(\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1} \Omega x_{-1}\right) X_1 + \left(\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1} \Omega x_{-1}\right) X_{-1} + \lambda_1' x_1 + \lambda_2' x_{-1} - \lambda_2 X_1 - \lambda_1 X_{-1} \\ &= -\left(3x_1^2 x_{-1} + 3x_1 x_{-1}^2\right) + \frac{\varepsilon E}{2}\left(e^{i\nu t} + e^{-i\nu t}\right)\end{aligned} \tag{9}$$

and

$$\left(\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1} \Omega x_{-1} - \lambda_1\right)\left(\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1} \Omega x_{-1} - \lambda_2\right) u_1 = -(x_1^3 + x_{-1}^3) \tag{10}$$

The particular solution of (10) is

$$u_1 = -x_1^3 / 2\lambda_1(3\lambda_1 - \lambda_2) - x_{-1}^3 / 2\lambda_2(3\lambda_2 - \lambda_1) \tag{11}$$

The particular solutions of (9) is

and

$$\begin{aligned} X_1 &= -\lambda_1' x_1 /(\lambda_1 - \lambda_2) - 3x_1^2 x_{-1} / 2\lambda_1 + Ee^{i\nu t} / 2(i\nu - \lambda_2) \\ X_{-1} &= \lambda_2' x_{-1} /(\lambda_1 - \lambda_2) - 3x_1 x_{-1}^2 / 2\lambda_2 - Ee^{-i\nu t} / 2(i\nu + \lambda_1) \end{aligned} \qquad (12)$$

Substituting the functional values of X_1 and X_{-1} (12) into (4) and rearranging, we obtain

and

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + \varepsilon\left(-\lambda_1' x_1 /(\lambda_1 - \lambda_2) - 3x_1^2 x_{-1} / 2\lambda_1\right) + Ee^{i\nu t} / 2(i\nu - \lambda_2) \\ \dot{x}_{-1} &= \lambda_2 x_{-1} + \varepsilon\left(\lambda_2' x_{-1} /(\lambda_1 - \lambda_2) - 3x_1 x_{-1}^2 / 2\lambda_2 - Ee^{-i\nu t} / 2(i\nu + \lambda_1)\right) \end{aligned} \qquad (13)$$

The variational equations of ρ and ϕ in the real form (ρ and ϕ are known as amplitude and phase) which transform (13) to

and

$$\begin{aligned} \dot{\rho} &= -\varepsilon\rho\omega' / 2\omega - \varepsilon E \sin(\phi - \nu t) / \nu + \omega \\ \dot{\phi} &= \omega - \nu + 3\varepsilon\rho^2 / 8\omega - \varepsilon E \cos(\phi - \nu t) / a(\nu + \omega) \end{aligned} . \qquad (14)$$

where

$$\omega^2 = (c_1 + c_2 \cos\tau + c_3^2 \sin 2\tau)$$

The variational equation (14) is in the form of the KBM solution**.** The variational equations for amplitude and phase are usually appeared in a set of first order differential equations and solved by the numerical technique (see Shamsul [VI]).

Therefore, the improved solution of the equation (7) is

$$x(t,\, \varepsilon) = \rho \cos\varphi + \varepsilon\, u_1 +, \qquad (15)$$

where $\varphi = \omega t + \phi$ (see **Appendix A**) and ρ, ϕ are the solutions of the equation (14) and u_1 is given by Eq. (11).

IV Results and Discussions

In this article an analytic technique has been presented to obtain the first order analytical approximate solutions of nonlinear differential systems with constant and varying coefficients based on the extended KBM method (by Popov) and HB method. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However owing to the rapidly growing algebraic complexity for the derivation of the function, the solution is in general confined to a low order, usually the first. In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one can easily compare the approximate solution to the numerical solution (considered to be exact). Due to such a comparison concerning of this paper, we refer to the works of Murty [II], and Shamsul [VI] have been compared to the corresponding numerical solution. In this paper we have also compared the perturbation solutions (15) and of *Duffing's* equation (7) to those obtained by Range-kutta (Fourth-order) procedure.

First of all, x is calculated by (15) with initial conditions $\rho = 1.00000,\ \varphi = -.001038$ [$x(0) = 1.00000,\ \dot{x}(0) = 0.00000$] for $e = .1,\ h = .05,\ E = 0.5.$ $\omega^2 = \omega_0(c_1 + c_2 \cos\tau + c_3^2 \sin 2\tau)$. Then corresponding numerical solutions is also computed by Runge-Kutta fourth-order method. The result is shown in Fig.1(a). Also we plot existing perturbation solution in Fig.1(b) with initial conditions $\rho = 1.0000,\ \varphi = -.070993$ [$x(0) = 1.00000,\ \dot{x}(0) = 0.00000$] for $e = .1\ h = .05,\ E = 0.5.$

$\omega^2 = \omega_0(c_1 + c_2 \cos\tau + c_3^2 \sin 2\tau)$. We see that in Fig. 1(a) the perturbation solution nicely agree with the numerical solution, but in this situation existing perturbation solution in Fig.1(b) does not give satisfactory result. The corresponding numerical solutions have also been computed by Runge-Kutta fourth-order method. From Fig. 2(a), Fig. 3(a), Fig. 4(a) and Fig. 5(a) we observe that the approximate solutions agree with numerical results nicely but in Fig. 2(b), Fig. 3(b), Fig. 4(b) and Fig. 5(b) do not agree and the solution fails to give desired result.

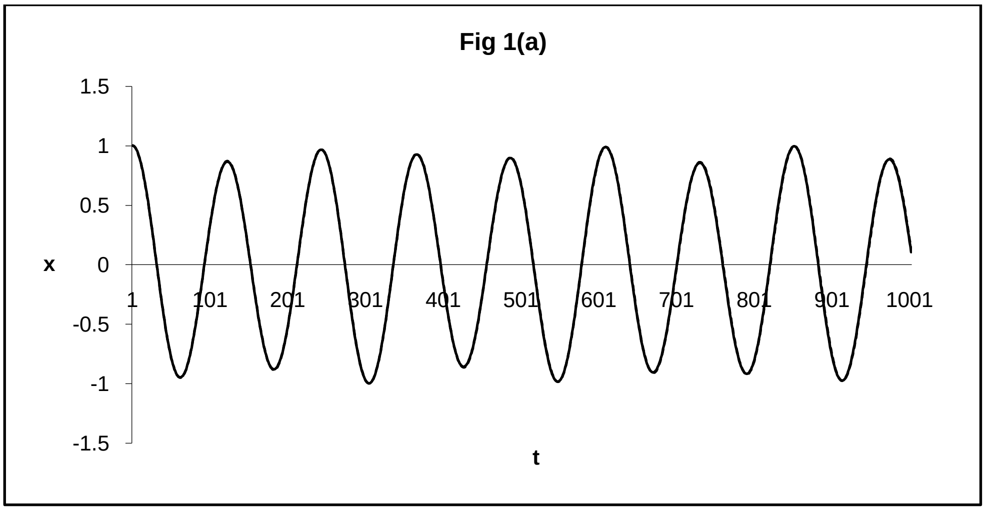

Fig 1(a): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $\rho = 1.00000,\ \varphi = -.001038$ [$x(0) = 1.00000,\ \dot{x}(0) = 0.00000$] for $e = .1,\ h = .05,\ E = 0.5,\ \nu = .6.$

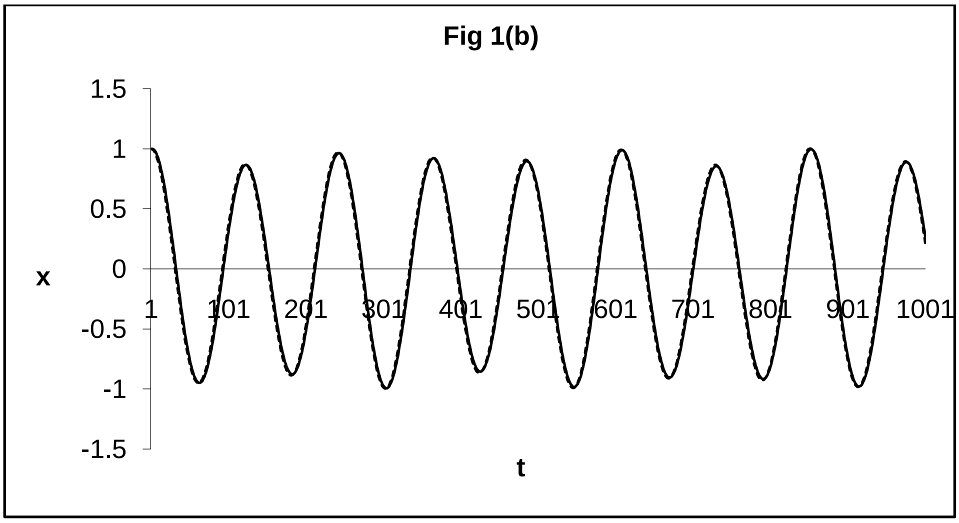

Fig.1(b): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $\rho = 1.0000,\ \varphi = -.070993$ $[x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ for $e = .1\ \ h = .05,\ E = 0.5,\ \nu = .6.$

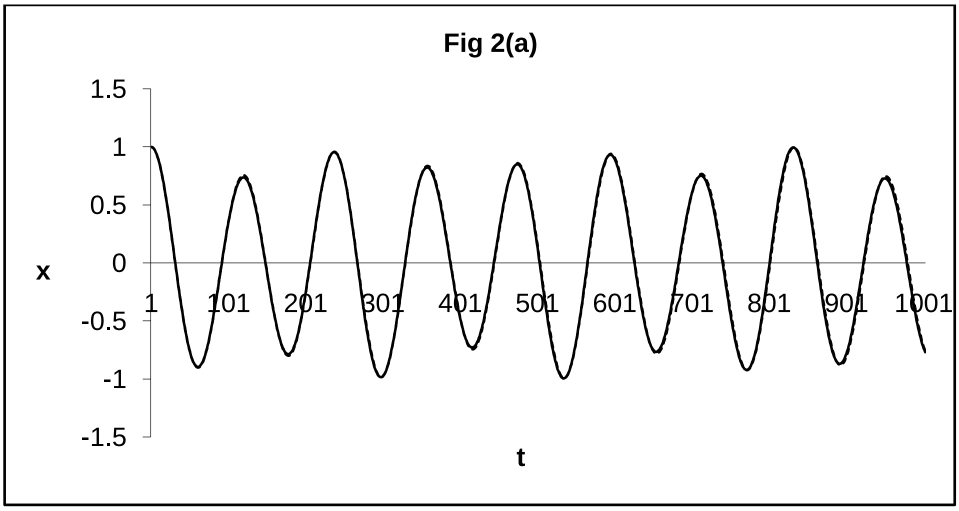

Fig 2(a): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $\rho = 1.00000,\ \varphi = 0.001076$ $[x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ for $e = .2\ \ h = .05,\ E = 0.5,\ \nu = .6.$

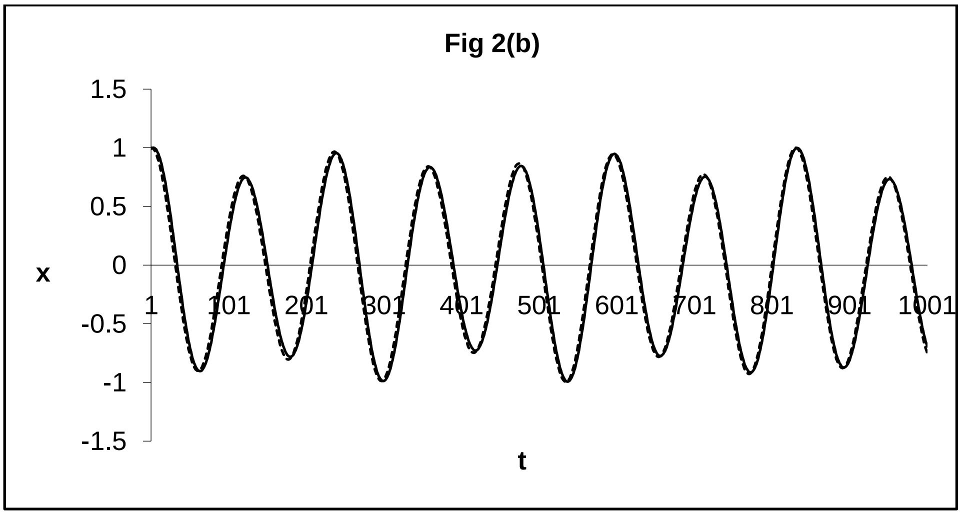

Fig 2(b): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $\rho = 1.00000, \ \varphi = -.130940$ $[x(0) = 1.00000, \ \dot{x}(0) = 0.00000]$ for $e = .2, \ h = .05, \ E = 0.5, \nu = .6.$

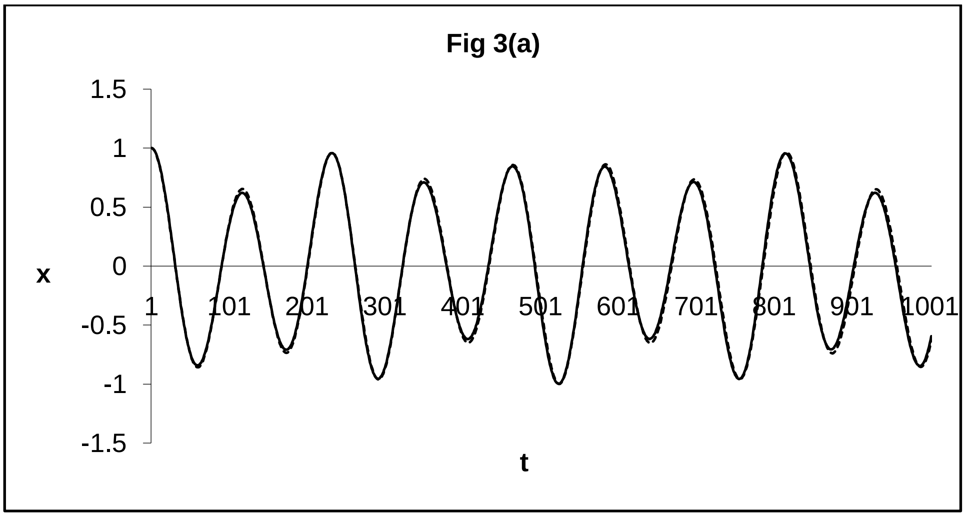

Fig.3(a): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $\rho = 1.0000, \ \varphi = -.001112$ $[x(0) = 1.00000, \ \dot{x}(0) = 0.00000]$ for $e = .3 \ \ h = .05, \ E = 0.5, \ \nu = .6.$

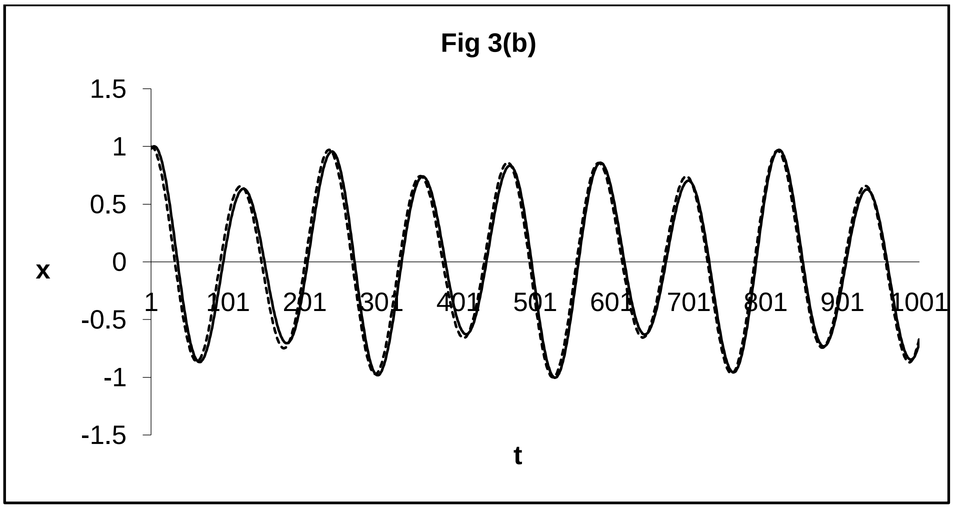

Fig 3(b): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $\rho = 1.00000, \varphi = -.182232$ $[x(0) = 1.00000, \dot{x}(0) = 0.00000]$ for $e = .3, \; h = .05, \; E = 0.5, \; \nu = .6.$

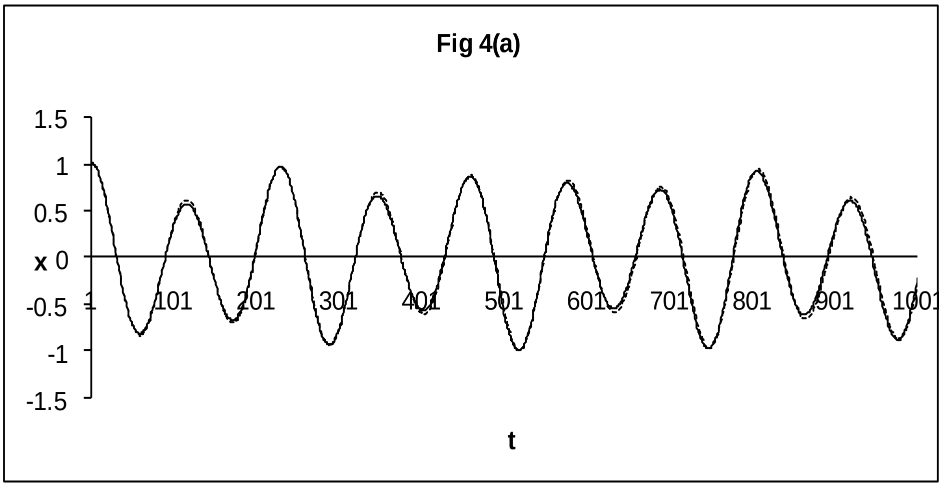

Fig.4(a): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $\rho = 1.0000, \; \varphi = -.001131$ $[x(0) = 1.00000, \; \dot{x}(0) = 0.00000]$ for $e = .35 \;\; h = .05, \; E = 0.5, \; \nu = .6.$

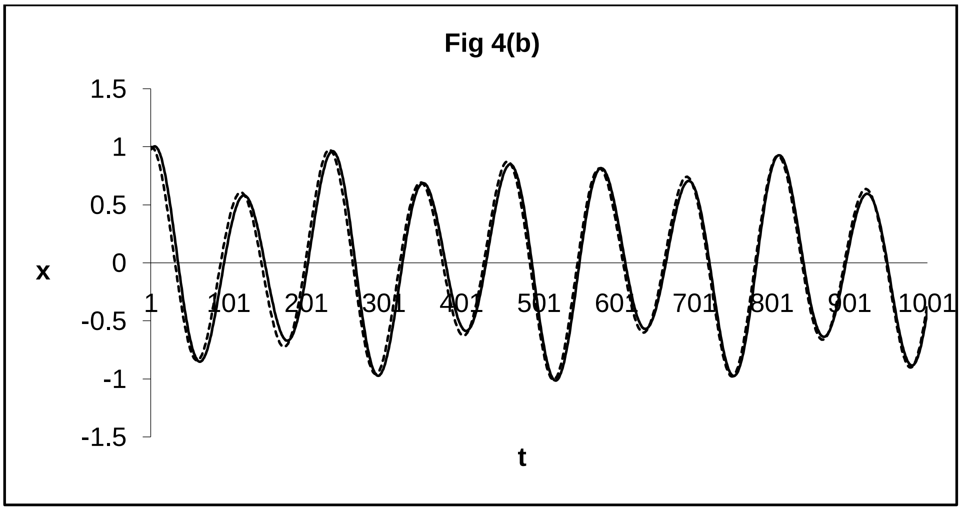

Fig 4(b): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $\rho = 1.00000,\ \varphi = -.205198$ $[x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ for $e = .35,\ h = .05,\ E = 0.5,\ \nu = .6.$

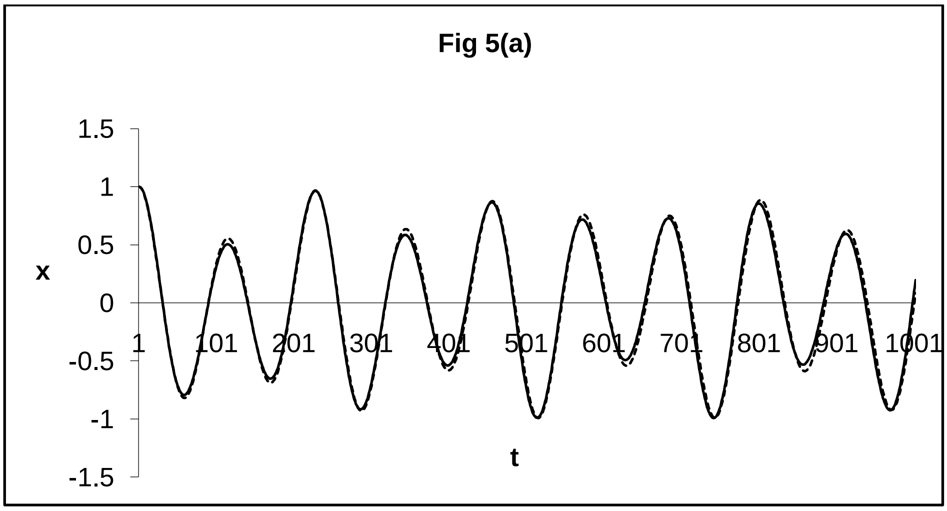

Fig. 5(a): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $\rho = 1.0000,\ \varphi = -.001150$ $[x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ for $e = .4\ \ h = .05,\ E = 0.5,\ \nu = .6.$.

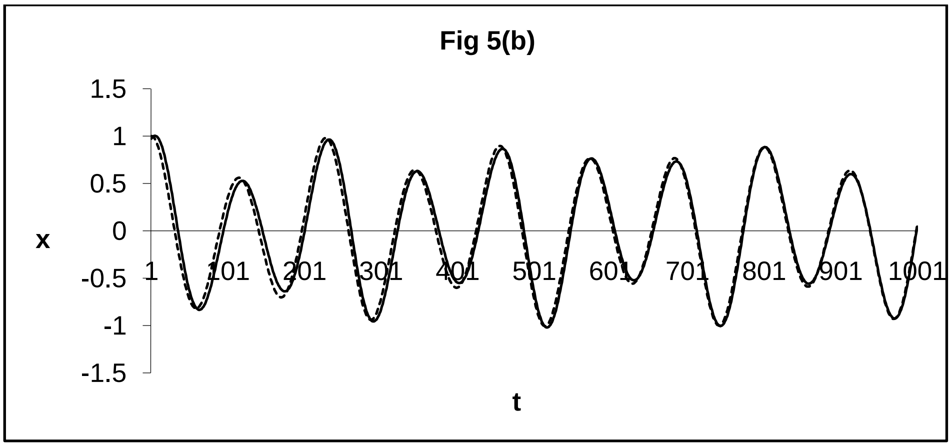

Fig 5(b): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $\rho = 1.00000,\ \varphi = -.226618$ $[x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ for $e = .4\ \ h = .05,\ E = 0.5,\ \nu = .6.$

V Conclusion

An approximate solution of nonlinear non autonomous deferential system with slowly varying coefficients has been found. This improved method gives better results than classical and extended KBM method. The solution for different initial condition shows good coincidence with corresponding numerical solution.

Appendix A

Let us consider nonlinear differential equation

$$\ddot{x} + (c_1^2 + c_2 \cos\tau + c_3^2 \sin 2\tau)x = -\varepsilon x^3 + +\varepsilon E \cos \nu t, \tag{A.1}$$

According to [I],[IX], there exists a periodic solution of **harmonic balance** method of Eq.(S.1) in the form,

$$x(t,0) = a\cos\varphi + a^3 c_3 \cos 3\varphi + \tag{A.2}$$

where a and c_3 are constants.

By substituting Eq.(A.2) in to Eq. (A.1) and equating the coefficient of $\cos\varphi$, we obtain

$$-\dot{\varphi}^2 + \omega^2 = \frac{3\varepsilon a^2}{4}(1 + a^2 c_3) \tag{A.3}$$

Here $c_3 = 0$ then the Eq. (A.3), become

$$-\dot{\varphi}^2 + \omega^2 = \frac{3\varepsilon a^2}{4} \tag{A.4}$$

Simplifying this equation (A.4), we get

$$\dot{\varphi} = 4\omega\left(\sqrt{(1 - 3\varepsilon a^2 / 4\omega^2}\right) \tag{A.5}$$

References

I. C. W. Lim and B. S. Wu. "A new analytical approach to the Duffing-harmonic oscillator", Physics Letters A, 311, 365-373, 2003.

II. I. S. N. Murty, "A unified Krylov-Bogoliubov method for second order nonlinear systems", Int. J. nonlinear Mech. 6. 45-53. 1971.

III. I. P. Popov. "A generalization of the Bogoliubov asymptotic method in the theory of nonlinear oscillations", Dokl.Akad. Nauk SSSR 111, 308-310 (in Russian) 1956.

IV. J. C. Arya and G. N. Bojadziev, "Damped oscillating systems modeled by hyperbolic differential equations with slowly varying coefficients", Acta Mechanica, 35, 215-221, 1980.

V. K.C. Roy and M. Shamsul Alam. "Effect of higher approximation of Krylov- Bogoliubov-Mitropolskii solution and matched asymptotic solution of a differential system with slowly varying coefficients and damping near to a turning point", Vietnam Journal of Mechanics, VAST, 26,182-192, 2004.

VI. M. Shamsul Alam,. "Unified Krylov-Bogoliubov-Mitropolskii method for solving n-th order nonlinear system with slowly varying coefficients", Journal of Sound and Vibration, 256. 987-1002, 2003.

VII. N. N. Bogoliubov and Yu. Mitropolskii."Asymptotic methods in the theory of nonlinear oscillations", Gordan and Breach, New York, 1961.

VIII. N. N. Krylov and N. N. Bogoliubov, "Introduction to nonlinear mechanics". Princeton University Press, New Jersey, 1947.

IX. R. E. Mickens. "Oscillation in Planar Dynamic Systems", World Scientific, Singapore, (1996).

X. Yu. Mitropolskii. "Problems on asymptotic method of non-stationary oscillations" (in Russian), Izdat, Nauka, Moscow, 1964.