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# The unique symmetric positive solutions for nonlinear fourth order arbitrary two-point boundary value problems: A fixed point theory approach

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## Abstract

*In this paper, we explore the existence and uniqueness of positive solutions for the following nonlinear fourth order ordinary differential equation*

$$u^{(4)}(t) = f(t,u(t)), \quad t \in [a, b],$$

*withthe following arbitrary two-point boundary conditions:*

$$u(a) = u(b) = u'(a) = u'(b) = 0,$$

*where,* $a, b$ *are two arbitrary constants satisfying* $b > 0, a = 1 - b$ *and* $f \in C([a,b] \times [0,\infty),[0,\infty))$*.Here we also demonstrate that under certain assumptions the above boundary value problem exist a unique symmetric positive solution. The analysis of this paper is based on a fixed point theorem in partially ordered metric spaces due to Amini-Harandi and Emami. The results of this paper generalize the results of several authors in literature. Finally, we provide some illustrative examples to support our analytic proof.*

**Keywords :** Arbitrary two-point boundary conditions; Nonlinear fourth order ordinary differential equation; Unique symmetric positive solutions; Fixed point theorem;

## I. Introduction

It is well known that the fixed point technique is the most important technique for checking the existence and uniqueness of solutions for nonlinear boundary value problems. In the last few decades, two-point, three-point and fourpoint boundary value problems for fourth order nonlinear ordinary differential equationshas extensively been studied by using various techniques, see for instance [I, III, IV, VI-XI, XIII-XX] and references therein. But there have only a small number of works about the existence and uniqueness of solutions for thenonlinear boundary value problem (for short BVP) with arbitrary point boundary conditions,see for instance [XXI] and references therein.From this context, in this paper we establish the criteria for the existence and uniqueness ofsymmetric positive solution to the following nonlinear fourth order arbitrary two-point boundary value problem by applying a fixed point theorem in partially ordered metric space due to Amini-Harandi and Emami[II]:

$$u^{(4)}(t) = f\left(t,u(t)\right), \quad t \in [a, b], \tag{1.1}$$

$$u(a) = u(b) = u'(a) = u'(b) = 0, \tag{1.2}$$

where, $a, b$ are two arbitrary constants satisfying $b > 0,\ a = 1 - b$ and

$$f \in C\left([a,b] \times [0,\infty),[0,\infty)\right).$$

By the above considered BVP it is possible to describe the bending of an elastic beam clamped at both arbitrarily chosen endpoints. Further physical interpretation of that elastic beam equation can be found in the work of Zill and Cullen [V, pp. 237-243].

Recently, Caballero *et al.*[VIII] and Zhai*et al.*[III] studied the following fourth order particular two-point boundary value problems by applying a fixed point theorem in partially ordered metric space due to Amini-Harandi and Emami[II] and a fixed point theorem of general $\alpha$ - concave operators [III] respectively:

$$\begin{cases} u^{(4)}(t) = f\left(t,u(t)\right), & t \in [0, 1] \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \tag{1.3}$$

where, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

In this paper, we generalize the works of Caballero *et al.*[VIII] and Zhai*et al.*[III] in case of arbitrariness of boundary points.The rest of this paper is furnished as follows:

In Section 2, we provide some basic concepts, a lemma and a fixed point theorem due to Amini-Harandi and Emami [II]. In Section 3, we state and prove our main results, which provide us the techniques to check the existence and uniqueness of symmetric positive solutions of fourth order arbitrary two-point BVPs under some certain assumptions.In Section 4, we give some examples which help us to illustrate our main results. Finally, we give a conclusion.

**II. Preliminary Notes**

In this section we provide some basic concepts and a fixed point theorem due to Amini-Harandi and Emami[II], which areessential to establishour main results.

**Definition 2.1.**Let $(B,\|.\|)$be a real Banach spaceand $K$ be a nonempty closed convex subset of $B$. Then we say that $K$ is a *cone* on $B$ if it is satisfies the following properties:

(i) $\eta r \in K$ for $r \in K, \eta \geq 0;$ (ii) $r, -r \in K$ implies $r = \theta$, where $\theta$ denotes the null element of $B$.

**Definition 2.2.**Let $C[a,b]$denote the Banach space of continuous functions on $[a,b]$with uniform norm $\|p\| = \sup_{a \leq t \leq b} |p(t)|,\ p \in C[a,b], b > 0, a = 1-b$. Then the function $p \in C[a,b]$is said to be *symmetric*if $p(t) = p(1-t),\ t \in [a,b]$.

**Definition 2.3.**A solution $u(t)$of the BVP defined by (1.1) and (1.2) is said to be *symmetric solution* if $u(t)$is a symmetric function, i.e., $u(t) = u(1-t),\ \textit{for all}\, t \in [a,b]$.

**Definition 2.4.**A solution $u(t)$of the BVP defined by (1.1) and (1.2) is said to be *positive solution* if $u(t) > 0$ for all $t \in (a,b)$.

Now we state a fixed point theorem due to Amini-Harandi and Emami[II].

Let $\mathscr{M}$ denote the class of all functions of type $\beta : [a, \infty) \to [a,b)$ satisfying the condition

$$\beta(t_n) \to b \text{ implies } t_n \to a,\ n \in \mathbb{N} \text{ (set of natural numbers)}. \tag{2.1}$$

**Theorem 2.1**[II].*Let $(P, \leq)$ be a partially ordered set and suppose that there exists a metric $\delta$ in $P$ such that $(P, \delta)$ is a complete metric space. Let $A : P \to P$ be a non-decreasing mapping such that there exists an element $p_0 \in P$ with $p_0 \leq Ap_0$. Suppose that there exists $\beta \in \mathscr{M}$ such that*

$$\delta(Ap, Aq) \leq \beta(\delta(p,q)) \cdot \delta(p,q),\ \textit{for any}\ p,q \in P\ \textit{with}\ p \geq q. \tag{2.2}$$

*Assume that either $A$ is continuous or $P$ is such that*

*if $\{p_n\}$ is a non-decreasing sequence in*

$$P\ \textit{such that}\ p_n \to p, \textit{then}\ p_n \leq p\ \ \textit{for all}\ n \in \mathbb{N}. \tag{2.3}$$

*Besides, suppose that*

$$\text{for each } p, q \in P, \text{there exists } r \in P \text{ which is comparable to } p \text{ and } q. \tag{2.4}$$

*Then* $A$ *has a unique fixed point.*

**Remark 2.1.** In this paper, we will work with a subset of the Banach space $C[a,b]$, where $b > 0$ and $a = 1 - b$. This space will be considered with the standard metric

$$\delta(p,q) = \sup_{a \le t \le b} |p(t) - q(t)|. \tag{2.5}$$

It can also be equipped with a partial order given by

$$p, q \in C[a,b],\ p \le q \Leftrightarrow p(t) \le q(t),\ \text{for } t \in [a,b]. \tag{2.6}$$

According to the work of Nieto and Rodriguez-Lopez [XII], it is easy to prove that $(C[a,b], \le)$ with the above mentioned metric satisfies the condition defined by (2.3) of Theorem 2.1. Furthermore, for $p, q \in C[a,b]$, the function $\max(p,q) \in C[a,b]$, and $(C[a,b], \le)$ satisfies the condition defined by (2.4) of Theorem 2.1.

**Lemma 2.1.** *Assume that* $b > 0$ and $a = 1 - b$. *If* $h(t) \in C[a,b]$, for all $t \in [a,b]$, *then the unique solution of following nonlinear fourth order arbitrary two-point BVP*

$$\begin{cases} u^{(4)}(t) = h(t), & t \in [a, b],\ b > 0, a = 1 - b \\ u(a) = u(b) = u'(a) = u'(b) = 0. \end{cases} \tag{2.7}$$

*is* $u(t) = \int_a^b G(t,s) h(s) ds, \quad t \in [a,b],$

*where,* $G(t,s)$ *is the Green's function of homogeneous fourth order arbitrary two-point BVP*

$$\begin{cases} u^{(4)}(t) = 0, & t \in [a, b],\ b > 0, a = 1 - b, \\ u(a) = u(b) = u'(a) = u'(b) = 0, \end{cases} \tag{2.8}$$

*i.e.,*

$G(t,s)$

$$=\frac{1}{6(b-a)^3}\begin{cases}\begin{bmatrix}\left[s^3\left(a^3-3a^2b\right)+6s^2a^2b^2-3s\left(a^3b^2+a^2b^3\right)+2a^3b^3\right]\\ +t\begin{bmatrix}6s^3ab-3s^2\left(a^3+a^2b+4ab^2\right)+6s\left(a^3b+a^2b^2+ab^3\right)\\ -3\left(a^3b^2+a^2b^3\right)\end{bmatrix}\\ +t^2\begin{bmatrix}-3s^3\left(a+b\right)+6s^2\left(a^2+ab+b^2\right)\\ -3s\left(4a^2b+ab^2+b^3\right)+6a^2b^2\end{bmatrix}\\ +t^3\left[2s^3-3s^2\left(a+b\right)+6sab+\left(b^3-3ab^2\right)\right]\end{bmatrix}; & a\le t\le s\le b,\\ \begin{bmatrix}\left[t^3\left(a^3-3a^2b\right)+6t^2a^2b^2-3t\left(a^3b^2+a^2b^3\right)+2a^3b^3\right]\\ +s\begin{bmatrix}6t^3ab-3t^2\left(a^3+a^2b+4ab^2\right)+6t\left(a^3b+a^2b^2+ab^3\right)\\ -3\left(a^3b^2+a^2b^3\right)\end{bmatrix}\\ +s^2\begin{bmatrix}-3t^3\left(a+b\right)+6t^2\left(a^2+ab+b^2\right)\\ -3t\left(4a^2b+ab^2+b^3\right)+6a^2b^2\end{bmatrix}\\ +s^3\left[2t^3-3t^2\left(a+b\right)+6tab+\left(b^3-3ab^2\right)\right]\end{bmatrix}; & a\le s\le t\le b.\end{cases} \quad (2.9)$$

**Proof.**To prove this lemma it is sufficient to show that the corresponding homogeneous differential equation of the BVP (2.8)has a Green's function $G(t,s)$ defined by (2.9).

The general solution of the differential equation of BVP (2.8) is

$$u(t)=A+Bt+Ct^2+Dt^3, \text{ where } A,B,C,D \text{ are arbitrary constants.} \quad (2.10)$$

Using the boundary conditions of BVP (2.8) in (2.10), we easily obtained that $A=B=C=D=0,$ and which ensure that the trivial solution of the BVP (2.8) is $u(t)=0$. Therefore, the BVP (2.8) has aunique Green's function and which is as follows:

$$G(t,s)=\begin{cases}A_1+A_2t+A_3t^2+A_4t^3; & a\le t\le s\le b,\\ B_1+B_2t+B_3t^2+B_4t^3; & a\le s\le t\le b.\end{cases} \quad (2.11)$$

Now, by the properties of Green's function for the BVP (2.8) and with its boundary conditions we obtained the following equations:

$$B_1+B_2s+B_3s^2+B_4s^3=A_1+A_2s+A_3s^2+A_4s^3, \quad (2.12)$$

$$B_2 + 2B_3 s + 3B_4 s^2 = A_2 + 2A_3 s + 3A_4 s^2, \quad (2.13)$$

$$2B_3 + 6B_4 s = 2A_3 + 6A_4 s, \quad (2.14)$$

$$6B_4 = 6A_4, \quad (2.15)$$

$$A_1 + A_2 a + A_3 a^2 + A_4 a^3 = 0, \quad (2.16)$$

$$B_1 + B_2 b + B_3 b^2 + B_4 b^3 = 0, \quad (2.17)$$

$$A_2 + 2A_3 a + 3A_4 a^2 = 0, \quad (2.18)$$

$$B_2 + 2B_3 b + 3B_4 b^2 = 0. \quad (2.19)$$

After solving the equations (2.12) to (2.19), we yield the following values,

$$A_1 = \frac{1}{6(b-a)^3}\left[ s^3\left(a^3 - 3a^2 b\right) + 6s^2 a^2 b^2 - 3s\left(a^3 b^2 + a^2 b^3\right) + 2a^3 b^3 \right],$$

$$A_2 = \frac{1}{6(b-a)^3}\left[ 6s^3 ab - 3s^2\left(a^3 + a^2 b + 4ab^2\right) + 6s\left(a^3 b + a^2 b^2 + ab^3\right) - \left(3a^3 b^2 + 3a^2 b^3\right) \right],$$

$$A_3 = \frac{1}{6(b-a)^3}\left[ -3s^3(a+b) + 6s^2\left(a^2 + ab + b^2\right) - 3s\left(4a^2 b + ab^2 + b^3\right) + 6a^2 b^2 \right],$$

$$A_4 = \frac{1}{6(b-a)^3}\left[ 2s^3 - 3s^2(a+b) + 6sab + \left(b^3 - 3ab^2\right) \right],$$

$$B_1 = \frac{1}{6(b-a)^3}\left[ s^3\left(b^3 - 3ab^2\right) + 6s^2 a^2 b^2 - 3s\left(a^3 b^2 + a^2 b^3\right) + 2a^3 b^3 \right],$$

$$B_2 = \frac{1}{6(b-a)^3}\left[ 6s^3 ab - 3s^2\left(4a^2 b + ab^2 + b^3\right) + 6s\left(a^3 b + a^2 b^2 + ab^3\right) - \left(3a^3 b^2 + 3a^2 b^3\right) \right]$$

$$B_3 = \frac{1}{6(b-a)^3}\left[ -3s^3(a+b) + 6s^2\left(a^2 + ab + b^2\right) - 3s\left(a^3 + a^2 b + 4ab^2\right) + 6a^2 b^2 \right],$$

and

$$B_4 = \frac{1}{6(b-a)^3}\left[ 2s^3 - 3s^2(a+b) + 6sab + \left(a^3 - 3a^2 b\right) \right].$$

Now, putting the values of $A_1, A_2, A_3, A_4, B_1, B_2, B_3, \text{and}\, B_4$ in (2.11), we get our desired unique Green's function $G(t,s)$, which confirm that

$$u(t) = \int_a^b G(t,s)h(s)ds,$$

is the unique solution of the BVP (2.7).
This completes the proof.

■

**Remark 2.2.**By Lemma 2.1, we can convert the BVP defined by (1.1) and (1.2) as the following integral equation

$$u(t)=\int_a^b G(t,s)\, f\left(s,u(s)\right)ds, \text{ for all } t\in[a,b], \tag{2.20}$$

where $G(t,s)$ is the Green's function given by (2.9). It is also noted that, the Green's function $G(t,s)$ have the following properties:

(I) $G(t,s)$ is continuous on $[a,b]\times[a,b]$,

(II) $G(a,s)=G(b,s)=G'(a,s)=G'(b,s)=0,$ for all $s\in[a,b]$, and

(III) $G(t,s)\ge 0,$ for all $t,s\in[a,b]$.

Obviously, $u\in[a,b]\times[a,b]$ is a solution of the BVP (1.1) and (1.2), if and only if it is a solution of the integral equation (2.20). Furthermore, if we consider a cone $K$ on $C[a,b]$ and define an integral operator $A:K\to K$ by

$$Au(t)=\int_a^b G(t,s)\, f\left(s,u(s)\right)ds, \text{ for all } u\in K, \tag{2.21}$$

then it is easy to see that the BVP (1.1) and (1.2) has a solution $u=u(t)$ if and only if $u$ is a fixed point of the operator $A$ defined by (2.21).

## III. Main Results

In this section, we state and prove ourmain results, which analytically prove the existence and uniqueness of symmetric positive solutions of our BVP defined by (1.1) and (1.2).

**Theorem 3.1.***Let* $\mathcal{E}$ *be the class of all non-decreasing functions* $\phi:[a,\infty)\to[a,\infty)$ *satisfying (i) for any* $w>0, \phi(w)<w,$ *and (ii)* $\beta(w)=\dfrac{\phi(w)}{w}\in\mathcal{M}$ *. If the boundary value problem defined by (1.1) and (1.2) satisfy the following assumptions:*

$(A_1)$ $f:[a,b]\times[0,\infty)\to[0,\infty)$ *is continuous, where* $a$ *and* $b$ *are two arbitrary constants*
*satisfying* $b>0, a=1-b,$

$(A_2)$ $f(t,u(t))$ *is non-decreasing with respect to the second variable for each* $t \in [a,b]$,

$(A_3)$ *suppose that there exists* $a < \lambda \le \dfrac{1}{\left(\sup_{a\le t\le b}\int_a^b G(t,s)ds\right)}$, *such that for* $p,q \in [a,\infty)$ *with* $p \le q$,

$$\left(f(t,q) - f(t,p)\right) \le \lambda\phi(q-p),\ with\ \phi \in \mathcal{E}\ , \tag{3.1}$$

*where,* $G(t,s)$ *is the Green's function defined as Lemma2.1, then the boundary value problem defined by (1.1) and (1.2) has a unique non-negative solution.*

**Proof.**If we define a cone on $C[a,b]$ by the following set:

$$K = \left\{p(t) \in C[a,b] : p(t) \ge 0 \text{ for } t \in [a,b]\right\}, \tag{3.2}$$

then, it is clear that $(K,\delta)$ with $\delta(p,q) = \sup_{a\le t\le b}\left\{\left|p(t) - q(t)\right| : t \in [a,b]\right\}$ is a complete metric space satisfying the conditions (2.3) and (2.4) of Theorem 2.1.

From the Remark 2.2, it is obvious that the integral operator $A$ defined by (2.21) applies the cone $K$ into itself, since according to the assumptions both of the functions $G(t,s)$ and $f(t,u)$ are continuous.

Now, we prove that the assumptions of the Theorem 2.1 are satisfied by the integral operator $A$ defined by (2.21).

First, we prove that the operator $A$ is non-decreasing.

Since $f(t,u)$ is non-decreasing with respect to the second variable, then for $p,q \in K, p \ge q$ and $t \in [a,b]$, we have

$$\begin{aligned} Ap(t) &= \int_a^b G(t,s) f(s,p(s))ds \\ &\ge \int_a^b G(t,s) f(s,q(s))ds = Aq(t). \end{aligned} \tag{3.3}$$

Thus, the operator $A$ is non-decreasing.

Again, since $G(t,s)$ and $f(t,u)$ are continuous, thenthe operator $A$ is so.

Now, for, $p,q \in K, p > q$ and $t \in [a,b]$, from our assumptions we yield the following estimate:

$$
\begin{aligned}
\delta(Ap,Aq) &= \sup_{a\le t\le b}\left|Ap(t)-Aq(t)\right| \\
&= \sup_{a\le t\le b}\left(Ap(t)-Aq(t)\right) \\
&= \sup_{a\le t\le b}\int_a^b G(t,s)\Big(f\big(s,p(s)\big)-f\big(s,q(s)\big)\Big)ds \\
&\le \sup_{a\le t\le b}\int_a^b G(t,s)\lambda\phi\big(p(s)-q(s)\big)ds \\
&< \sup_{a\le t\le b}\int_a^b G(t,s)\lambda\phi\big(\delta(p,q)\big)ds \\
&= \lambda\phi\big(\delta(p,q)\big)\left(\sup_{a\le t\le b}\int_a^b G(t,s)ds\right) \\
&\le \phi\big(\delta(p,q)\big)\frac{1}{\left(\sup_{a\le t\le b}\int_a^b G(t,s)ds\right)}\left(\sup_{a\le t\le b}\int_a^b G(t,s)ds\right) \\
&= \phi\big(\delta(p,q)\big)=\frac{\phi\big(\delta(p,q)\big)}{\delta(p,q)}\cdot\delta(p,q).
\end{aligned}
\tag{3.4}
$$

Hence, for $p,q\in K, p>q$ and $t\in[a,b]$, (3.4) yields

$$\delta(Ap,Aq)\le\beta\big(\delta(p,q)\big)\cdot\delta(p,q). \tag{3.5}$$

It is also clear that the inequality (3.5) holds for $p=q$.

Therefore, contractive condition of Theorem 2.1 is satisfied for $p\ge q$.

Again, since both of the functions $G(t,s)$ and $f(t,u)$ are non-negative, then we get

$$A(0)=\int_a^b G(t,s)\,f(s,0)\,ds\ge 0. \tag{3.6}$$

We have already confirm that the cone $K$ satisfy the assumptions (2.3) and (2.4) of Theorem 2.1.

Hence, all the assumptions of the Theorem 2.1 are satisfied by the integral operator $A$ .

Therefore, according to the Theorem 2.1, we can say that the integral operator $A$ defined by (2.21) has a unique fixed point and which confirm that the BVP defined by (1.1) and (1.2) has a unique non-negative solution.

This completes the proof.

■

In the next theorem, we will prove that the BVP defined by (1.1) and (1.2) exist a unique positive solution.

**Theorem 3.2.** *If the boundary value problem defined by (1.1) and (1.2) satisfy the assumptions* $(A_1),(A_2)$ *and* $(A_3)$ *of Theorem 3.1 and the following additional assumption:*

$(A_4)$ *there exists a certain* $t_0 \in [a,b]$ *such that* $f(t_0,0) \neq 0$,

*then the boundary value problem defined by (1.1) and (1.2) has a unique positive solution.*

**Proof.** Since, all the assumptions of Theorem 3.1 are satisfied, thus we get a unique non-negative solution $u(t)$ for the BVP (1.1) and (1.2) and this solution $u(t)$ must satisfy the following integral equations:

$$u(t) = \int_a^b G(t,s) f(s,u(s)) ds. \tag{3.7}$$

To complete the proof it is enough to show that $u(t)$ is a positive solution of the BVP (1.1) and (1.2). We will prove this by a contradiction.

Suppose that there exists a $t^* \in (a,b)$ such that $u(t^*) = 0$, i.e.,

$$u(t^*) = \int_a^b G(t^*,s) f(s,u(s)) ds = 0. \tag{3.8}$$

Now, since $u$ is nonnegative, the function $f$ is non-decreasing with respect to the second variable and $G(t,s) \geq 0$, thus we get

$$0 = u(t^*) = \int_a^b G(t^*,s) f(s,u(s)) ds \geq \int_a^b G(t^*,s) f(s,0) ds \geq 0, \tag{3.9}$$

and this yields

$$\int_a^b G(t^*,s) f(s,0) ds = 0. \tag{3.10}$$

Hence from the non-negativity of $G(t,s)$ and $f(t,u(t))$, we have

$$G(t^*,s) \cdot f(s,0) = 0 \text{ a.e}(s) (\text{almost everywhere}(s)). \tag{3.11}$$

But, $G(t^*,s)$ expressed as polynomial, hence $G(t^*,s) \neq 0$ a.e$(s)$, and this imply that

$$f(s,0) = 0 \text{ a.e}(s). \tag{3.12}$$

Now, from the assumption $(A_4)$, we get $f(t_0,0) > 0$ for a certain $t_0 \in [a,b]$ and hence the continuity of $f$ provide us a set $D \subset [a,b]$ with $t_0 \in D$ and $\mu(D) > 0$, where $\mu$ is the Lebesgue measure, such that $f(t,0) > 0$, for any $t \in D$. This leads a contradiction in (3.12). Thus, $u(t)$ is positive for all $t \in (a,b)$. This completes the proof. ■

Finally, we will provethe criteria for the existence and uniqueness of symmetric positive solution to the BVP (1.1) and (1.2) and the following theorem will establish this criteria.

**Theorem 3.3.** *If the boundary value problem defined by (1.1) and (1.2) satisfy the assumptions* $(A_1),(A_2)$ *and* $(A_3)$ *of Theorem 3.1,* $(A_4)$ *of Theorem 3.2 and the following additional assumption:*

$(A_5)$ $f(t,u(t)) = f(1-t,u(t))$ *for each* $(t,u(t)) \in [a,b]\times[a,\infty)$,

*then the boundary value problem defined by (1.1) and (1.2) has a unique symmetric positive solution.*

**Proof.** From the proof of Theorem 3.2, it is clear that the BVP defined by (1.1) and (1.2) exists a unique positive solution under our considered assumptions. So, to complete the proof of this theorem it is enough to show that the existed solution is symmetric.

According to the proof of Theorem 3.1, if we replace the cone $K$ by $J$ and define the cone $J$ on $C[a,b]$ as follows

$$J = \{p(t)\in C[a,b] : p(t) > 0 \text{ and } p(t) \text{ is symmetric for } t\in[a,b]\}, \quad (3.13)$$

then, it is obvious that $J$ is closed subset of $C[a,b]$ and $(J,\delta)$ is a complete metric space with the induced metric $\delta$ define by

$$\delta(p,q) = \sup_{a\le t\le b}|p(t)-q(t)|, \text{for } p(t),q(t)\in J \text{ and } t\in[a,b]. \quad (3.14)$$

Furthermore, $J$ with the induced partially ordered set $(C[a,b],\le)$ satisfy the condition(2.3) of Theorem 2.1, and it is also clear that the function $\max\{p(t),q(t)\}\in J$, for $p(t),q(t)\in J$, $t\in[a,b]$ and hence, $(J,\delta)$ satisfies the condition (2.4) of Theorem 2.1.

Now, as in the Remark 2.2 we define an integral operator by

$$Ap(t) = \int_a^b G(t,s)\, f(s,p(s))\,ds, \text{ for all } p\in K. \quad (3.15)$$

Finally, under our assumptions we prove that the operator $A$ maps $J$ into itself, which ensure the symmetric property of the solution of VBP (1.1) and (1.2).

From (3.15), we get

$$Ap(1-t) = \int_a^b G(1-t,s)\, f(s,p(s))\,ds. \quad (3.16)$$

Putting $s = 1-r$ in (3.16) and for $t\in[a,b]$, $b>0$, $a=1-b$, we obtain

$$\begin{aligned} Ap(1-t) &= -\int_b^a G(1-t,1-r)\, f(1-r,p(1-r))\,dr \\ &= \int_a^b G(1-t,1-r)\, f(1-r,p(1-r))\,dr. \end{aligned} \quad (3.17)$$

Now, if we replace $t$ and $s$ by $1-t$ and $1-s$ in the Green's function $G(t,s)$ defined by (2.9), then we can easily prove that, $G(t,s)=G(1-t,1-s)$, hence from (3.17), we have

$$Ap(1-t)=\int_a^b G(t,r)\,f\left(1-r,p(1-r)\right)dr. \tag{3.18}$$

Applying our assumption $(A_5)$ and the symmetric property of $p$, from (3.18) we obtain

$$\begin{aligned} Ap(1-t) &= \int_a^b G(t,r)\,f\left(r,p(r)\right)dr \\ &= \int_a^b G(t,s)\,f\left(s,p(s)\right)ds \\ &= Ap(t) \end{aligned} \tag{3.19}$$

Hence, the integral operator $A$ maps $J$ into itself. Therefore, the solution leads by the integral operator $A$ is symmetric.
This completes the proof.

■

**Remark 3.1.**Our Theorem 3.1, Theorem 3.2 and Theorem 3.3 generalized directly Theorem 3.1, Theorem 3.2 and Theorem 4.2 of Caballero *et al.*[VIII] respectively in case of arbitrariness of boundary points, as because we established our theorems under arbitrary two-point boundary conditions, whereas Caballero *et al.*[VIII] used particulartwo-point boundary conditions. Our results also generalized the results of Zhai*et al.*[III] in case of arbitrariness of boundary points, but they used different fixed point theorems.

## IV. Examples

In this section, we provide some examples to illustrate our main results.

**Example 4.1.**Consider the following nonlinear fourth order arbitrary two-point boundary value problem:

$$\begin{cases} u^{(4)}(t)=d+\tau\tan^{-1}\left(u(t)\right),\ t\in[a,b],\ b>0,\ a=1-b, \text{ and } d,\ \tau>0 \\ u(a)=u(b)=u'(a)=u'(b)=0. \end{cases} \tag{4.1}$$

Now, by using our Theorem 3.1 and Theorem 3.2, we justify that the BVP (4.1) exist a unique non-negative solution and a unique positive solution respectively.
For the above BVP if we consider

$$f\left(t,u(t)\right)=d+\tau\tan^{-1}\left(u(t)\right), \tag{4.2}$$

then, it is easy to prove that the assumptions $(A_1)$ and $(A_2)$ of Theorem 3.1are satisfied by the function $f\left(t,u(t)\right)$.

Before justifying the function $f(t,u(t))$ satisfies the assumption $(A_3)$ of Theorem 3.1, we will prove that the function $\phi:[a,\infty)\to[a,\infty)$ defined by $\phi(w)=\tan^{-1}w$, satisfies

$$\phi(w_1)-\phi(w_2)\le\phi(w_1-w_2),\ \text{for}\ w_1\ge w_2. \tag{4.3}$$

As $\phi$ is non-decreasing and $w_1 \ge w_2$, so if we put

$$\phi(w_1)=\tan^{-1}w_1=\alpha_1 \text{ and } \phi(w_2)=\tan^{-1}w_2=\alpha_2,$$

then we get $\alpha_1\ge\alpha_2$ and $\alpha_1,\alpha_2\in\left[0,\frac{\pi}{2}\right)$. Consequently, $\tan\alpha_1,\tan\alpha_2\in[0,\infty)\subseteq[a,\infty)$.

Hence, from $\tan(\alpha_1-\alpha_2)=\dfrac{\tan\alpha_1-\tan\alpha_2}{1+\tan\alpha_1\cdot\tan\alpha_2}$, we yield

$$\tan(\alpha_1-\alpha_2)\le\tan\alpha_1-\tan\alpha_2. \tag{4.4}$$

Now, applying non-decreasing property of $\phi$ in the inequality (4.4), we obtain

$$\begin{aligned}&\alpha_1-\alpha_2\le\tan^{-1}(\tan\alpha_1-\tan\alpha_2)\\ \Rightarrow\ &\phi(w_1)-\phi(w_2)\le\tan^{-1}(w_1-w_2)=\phi(w_1-w_2).\end{aligned} \tag{4.5}$$

This proves our assertion.

For $p\le q$ and $t\in[a,b]$, we have

$$\begin{aligned}f(t,q)-f(t,p)&=\tau\left(\tan^{-1}q-\tan^{-1}p\right)\\ &\le\tau\tan^{-1}(q-p)\\ &=\tau\phi(q-p).\end{aligned} \tag{4.6}$$

Now using the definition of $\mathscr{E}$ we prove that $\phi\in\mathscr{E}$. It is clear that $\phi$ maps $[a,\infty)$ into itself. Since $\phi'(p)=\dfrac{1}{1+p^2}$ and $\phi$ is non-decreasing, then by putting $\varphi(p)=p-\phi(p)$, we have

$$\varphi'(p)=1-\frac{1}{1+p^2}>0,\ \text{for}\ p>0,$$

hence $\varphi$ is strictly increasing, and thus $\phi(p)<p$, for $p>0$.

If we consider $\beta(p)=\dfrac{\phi(p)}{p}=\dfrac{\tan^{-1}p}{p}$ and $\beta(t_n)\to b$, then the sequence $\{t_n\}$ must be bounded, otherwise $t_n\to\infty$ and this leads a contradiction. Now, we have to prove that $t_n \to a.$ Suppose that $t_n \not\to a$. Then we get an $\varepsilon > 0$ such that, for each $n \in \mathbb{N}$ there exist $q_n \geq n$ with $t_{q_n} \geq \varepsilon.$ By the bounded properties of $\{t_n\}$ we yield a convergent subsequence $\{t_{r_n}\}$ of $\{t_{q_n}\}$. If we consider $t_{r_n} \to c,$ then from $\beta(t_n) \to b,$ we obtain

$$\frac{\tan^{-1}(t_{r_n})}{t_{r_n}} \to \frac{\tan^{-1} c}{c} = b.$$

This gives us a unique solution, $c = 0$, i.e., $t_{r_n} \to 0,$ and this contradicts the fact that $t_{q_n} \geq \varepsilon$. Hence our supposition is not correct. Therefore, $t_n \to a$. This proves that $\phi \in \mathscr{E}$. Thus the assumption $(A_3)$ of Theorem 3.1 is satisfied by considering $a < \lambda \leq \frac{1}{\left(\sup_{a \leq t \leq b} \int_a^b G(t,s) ds\right)}$, where $G(t,s)$ is the Green's function of the corresponding homogeneous BVP of the BVP (4.1). Therefore, according to the Theorem 3.1, we can say that the BVP defined by (4.1) exista unique non-negative solution.

Now, we verify that the according to the Theorem 3.2, the BVP by (4.1) exists a unique positive solution and for this it is enough to prove that the assumption $(A_4)$ of Theorem 3.2 is satisfied by the function $f(t,u(t))$.

From (4.2), we get

$$f(t,0) = d > 0.$$

This confirms the assumption $(A_4)$ of Theorem 3.2 hold.

Therefore, according to the Theorem 3.2, we can say that the BVP defined by (4.1) exist a unique positive solution.

**Example 4.2.** Consider the following nonlinear fourth order arbitrary two-point boundary value problem:

$$\begin{cases} u^{(4)}(t) = d + \tau \sin(\pi t) \tan^{-1}(u(t)), \ t \in [a,b], \ b > 0, \ a = 1-b, \text{ and } d, \ \tau > 0 \\ u(a) = u(b) = u'(a) = u'(b) = 0. \end{cases} \tag{4.7}$$

Now, by using our Theorem 3.3, we justify that the BVP (4.7) exist a unique symmetric positive solution.

For the BVP (4.7) if we consider

$$f(t,u(t)) = d + \tau \sin(\pi t) \tan^{-1}(u(t)), \tag{4.8}$$

then, it is easy to prove that the assumptions $(A_1)$ and $(A_2)$ of Theorem 3.1 as well as of Theorem 3.3 are satisfied by the function $f(t,u(t))$.

Now, as in the Example 4.1, taking a function $\phi:[a,\infty)\to[a,\infty)$ defined by $\phi(w)=\tan^{-1}w$, we can prove that $\phi\in\mathscr{E}$ and for $p\le q,\ t\in[a,b]$, we obtain

$$\begin{aligned} f(t,q)-f(t,p) &= \tau\sin(\pi t)\cdot\left(\tan^{-1}q-\tan^{-1}p\right) \\ &\le \tau\sin(\pi t)\cdot\tan^{-1}(q-p) \\ &\le \tau\cdot\tan^{-1}(q-p). \end{aligned} \tag{4.9}$$

Thus, if we consider

$a<\lambda\le\dfrac{1}{\left(\sup_{a\le t\le b}\int_a^b G(t,s)ds\right)}$, where $G(t,s)$ is the Green's function of the corresponding homogeneous BVP of the BVP (4.2), then the assumption $(A_3)$ of Theorem 3.1 as well as of Theorem 3.3 is satisfied by $f(t,u(t))$.

Now, we have

$$f(t,0)=d+\tau\sin(\pi t)\tan^{-1}(0)=d>0. \tag{4.10}$$

This confirms the validity of the assumption $(A_4)$ of Theorem 3.2 as well as of Theorem 3.3.

Finally, we will check the validity of the assumption $(A_5)$ of Theorem 3.3 for the function $f(t,u(t))$.

If we replace $t$ by $1-t$ in (4.8) with respect the first variable only, then for each $(t,u(t))\in[a,b]\times[a,\infty)$, we have

$$\begin{aligned} f(1-t,u(t)) &= d+\tau\sin(\pi(1-t))\tan^{-1}(u) \\ &= d+\tau\sin(\pi t)\tan^{-1}(u) \\ &= f(t,u(t)), \end{aligned}$$

i.e., $f(t,u(t))$ is symmetric with respect to the first variable.

Hence, the assumption $(A_5)$ of Theorem 3.3 holds for the function $f(t,u(t))$.

Therefore, according to the Theorem 3.3, we can say that the BVP defined by (4.2) exist a unique symmetric positive solution.

## V. Conclusion

In this paper, we established general approaches for checking the existence and uniqueness of symmetric positive solutions of nonlinear fourth order arbitrary two-point boundary value problem defined by (1.1) and (1.2) applying a fixed point theorem in partially ordered metric space due to Amini-Harandi and Emami[II]. By our Theorem 3.1 one can check the existence of unique non-negative solution of the BVP (1.1) and (1.2). Theorem 3.2 has been used to examine the existence of unique positive solution of the BVP (1.1) and (1.2) whereas Theorem 3.3 has been used to check the existence of unique symmetric positive solutionof that BVP. The results of this paper generalized the corresponding results of Caballero *et al.*[VIII] and Zhai*et al.*[III]. Finally, by Example 4.1 and Example 4.2, we verified our results.

## Authors' Contributions

All authors read and approved the final version of the manuscript.

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