



## ON TRANSVERSAL VIBRATIONS OF AN AXIALLY MOVING BEAM UNDER INFLUENCE OF VISCOUS DAMPING

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### Abstract

*In this paper, a transversal vibration of an axially moving beam under the influence of viscous damping has been studied. The axial velocity of the beam is assumed to be positive, constant and small compared to wave-velocity. The beam is moving in a positive horizontal direction between the pair of pulleys and the length between the two pulleys is fixed. From a physical viewpoint, this model describes externally damped transversal motion for a conveyor belt system. The beam is assumed to be externally damped, where there is no restriction on the damping parameter which can be sufficiently large in contrast to much research material. The straightforward expansion method is applied to obtain approximated analytic solutions. It has been shown that the obtained solutions have not been broken out for any parametric values of the small parameter  $\epsilon$ . The constructed solutions are uniform and have been damped out. Even though there are several secular terms in the solutions, but they are small compared to damping.*

**Keywords:** Moving beam, Viscous Damping, Secular terms, Eigen functions, Straight-forward expansion method

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### I. Introduction

The class of the oscillatory systems is generally accepting almost all the systems, such as physical, structural, and mechanical systems, of the physical world around us. Axially translating elastic systems are also one of them. These axially moving systems have received great importance and research attention for the last 50 years. Axially translating systems have been observed in many practical and engineering applications. For example, magnetic tapes, conveyor belt systems, data

*Khalid H. Malik et al*

saving devices, pipes conveying fluids, and elevator cable systems, and all such kind of systems are bound to vibrations. Many factors such as the eccentricity of a pulley, non-uniform material properties, the irregular speed of the driving motor, and environmental disturbances influence the dynamics of these devices. The vibrations contributed by all these factors can lead to severe damage to the system. It is a common experience that many mechanical or physical structure's severe failures are caused by vibration. The Tacoma Narrows suspension bridge in Washington DC, United State of America, is a classic example of a structural collapse due to 42 mile-per-hour wind. From this viewpoint, it becomes necessary to design systems where unnecessary noise and vibrations can be reduced by means of solid procedures. Damping devices can be used to control the vibrations in the mechanical systems, see for instance [I, II, III, IV, X]. In [V] authors studied the transversal vibrations of the string-like equation with non-classical boundary conditions. One end of the string was kept fixed and spring-dashpot system attached at other boundary and some specific initial conditions has been taken. The vertical displacement of the string under the effect of boundary damping has been constructed by a two timescales perturbation method with the conjunction of the method of characteristic coordinates. It has been found that the amplitude of oscillations suppressed due to boundary damping. In [VI] authors studied the string-like equation with the influence of viscous damping. Both ends of the string kept fixed and general initial conditions have been taken. A formal asymptotic approximation of the exact solutions has been constructed by the method of two timescales perturbation method with the conjunction of the Laplace transform method. It has been found that the truncation of modes is applicable for certain parametric values of damping. In [VII] authors studied the beam-like equation under the effect of boundary damping. One end of the beam is assumed to be simply supported and a spring-dashpot system attached at other boundary. Damping generated by the dashpot is considered to be small. A formal asymptotic approximation of the solutions has been obtained by the method of two timescales perturbation method. It has been shown how different oscillations modes are damped. A beam like equation under the effect of material damping has been studied in [VIII]. An asymptotic approximation of the Initial-boundary value problem has been constructed by the method of multiple scales perturbation method. It has been shown that all oscillation modes are damped out. In [X] authors studied the transverse vibrations of the string and a beam on a semi-infinite domain. To control the undesired vibration in the string (or beam) dampers are used at the boundary. The explicit solutions have been computed by the D-Alembert method and Laplace transforms method. Reflection of waves for different types of boundary conditions has been visualized.

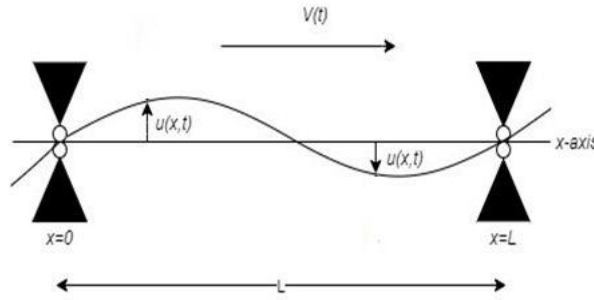
In this paper, a transversal vibration of an axially moving beam-like equation under the influence of viscous damping has been studied. From a physical point of view, this model describes the externally damped transversal motion of the conveyor belt system or band-saw blade. This mathematical model is a linear-homogeneous fourth-order partial differential equation. The beam is assumed to be externally damped, where there is no restriction on the damping parameter. This external damping refers to the viscous medium such as chain moving in an oily engine or any other such viscous medium where viscosity plays its dominant role. The damping parameter can

*Khalid H. Malik et al*

have larger values instead of smaller values as has been done in a lot of research in literature. The axial velocity of the beam is assumed small as compared to wave-velocity and to be positive. A formal asymptotic approximation of solutions has been constructed by the straightforward expansion method.

## II. Governing equations of motion

The equation of motion related to axially moving Euler-Bernoulli beam with the appropriate initial and boundary conditions will be studied. The beam is moving at the uniform constant axial speed  $\bar{V}$  between a pair of pulleys those are at distance  $L$  from each other, as shown in Figure 1. A stretched beam is simply supported at  $X = 0$  and  $X = L$ . Consider  $U(X, T)$  is the transverse displacement field variable,  $T$  is the time,  $L$  is the distance between two boundaries/supports, a cross-sectional area  $A$  of the beam, uniform axially moving beam of mass density  $\rho$ , flexural rigidity  $EI$ , a moment of inertia  $I$ , damping coefficient  $\delta$ , and uniform tension  $P$ . The functions  $F(X)$  and  $G(X)$  state the displacement and velocity at  $X = 0$ , respectively. Assumed that  $\rho, A, EI, P, V$  and  $\delta$  are positive. Gravity and other external forces are neglected. The equation which describes the displacement of the beam under viscous damping is given by:



**Fig.1:** The schematic model of damped axially moving beam

$$\rho A (U_{TT} + 2\bar{V}U_{XT} + \dot{V}U_X + V^2U_{XX}) - PU_{XX} + EIU_{XXXX} + \delta(U_T + VU_X) = 0 \quad (1)$$

$$\text{BC'S: } U(0, T) = U_X(0, T) = U(L, T) = U_X(L, T) = 0, \quad T \geq 0 \quad (2)$$

$$\text{IC'S: } U(X, 0) = F(X), \quad U_T(X, 0) = G(X) \quad (3)$$

It is significant to mention that the term  $2\bar{V}U_{XT}$  in Eq. (1) is the effect of the Coriolis acceleration affected by a beam element traveling with a velocity  $\bar{V}$  in a frame at the actual position of the element and revolving at an angular speed  $U_{XT}$ . This term is also known as the gyroscopic term. The term  $\bar{V}^2U_{XX}$  is due to the centripetal acceleration affected by the element in the same revolving frame due to the tangential velocity  $\bar{V}$  on a way of estimate curvature  $U_{XX}$ .

To write the Equations of motion (1) to (3) in dimensionless form, we use the following transformations as under,

$$u(x, t) = \frac{U(X, T)}{L}, x = \frac{X}{L}, t = \frac{cT}{L}, V^* = \frac{\bar{v}}{c}, \delta^* = \frac{\delta L}{\rho c}, f(x) = \frac{F(X)}{L}, g(x) = \frac{G(X)}{c}$$

where  $c = \sqrt{\frac{P}{\rho}}$  is a wave speed.

Based upon the above transformations the dimensionless form of initial-boundary value problem (1) to (3) is given as under:

$$u_{tt} + 2Vu_{xt} + (V^2 - 1)u_{xx} + \mu u_{xxxx} + \delta(u_t + Vu_x) = 0 \quad (4)$$

$$u(0, t) = u_x(0, t) = u(1, t) = u_x(1, t) = 0, t \geq 0 \quad (5)$$

$$u(x, 0) = f(x), u_t(x, 0) = g(x), 0 < x < 1 \quad (6)$$

In this study, we have assumed that the axial velocity of the beam is small in comparison to wave speed that is,  $V \ll c$ , thus it is reasonable to write  $V = O(\varepsilon) = \varepsilon V_0$ . Substituting this assumption in Eq. (4) – (6), it follows that:

$$u_{tt} - u_{xx} + \mu u_{xxxx} + \delta u_t = -2\varepsilon V_0 u_{xt} - \varepsilon \delta V_0 u_x - \varepsilon^2 V_0^2 u_{xx} \quad (7)$$

$$\text{BC's: } u(0, t; \varepsilon) = u_x(0, t; \varepsilon) = u(1, t; \varepsilon) = u_x(1, t; \varepsilon) = 0 \quad (8)$$

$$\text{IC's: } u(x, 0; \varepsilon) = f(x), u_t(x, 0; \varepsilon) = g(x) \quad (9)$$

Where  $\varepsilon$  is small dimensionless parameter  $0 < \varepsilon \ll 1$ .

### III. Straightforward expansion method

It can be observed that the unknown function  $u(x, t)$  does not only depend on independent variable  $x$  and  $t$ , but also depends on the small parameter  $\varepsilon$ , so that  $u(x, t) \Rightarrow u(x, t; \varepsilon)$ . Therefore, it is reasonable to use a straightforward expansion method. To further investigate the initial-boundary value problem given in Eq's. (7) – (9), the straightforward expansion method is utilized, where  $u(x, t; \varepsilon)$  can be expanded into the asymptotic series as

$$u(x, t; \varepsilon) = u_0(x, t) + \varepsilon u_1(x, t) + \varepsilon^2 \dots \quad (10)$$

Now, by utilizing Eq. (10) into initial-boundary value problem given in Eq. (7) to (9) and by comparing coefficients of  $\varepsilon^0$  and neglecting  $\varepsilon^2$  and higher-order terms, we get the following problem of  $O(1)$ -problem with associated conditions:

**The  $O(1)$ -problem:**

$$u_{0tt} - u_{0xx} + \delta u_{0t} + \mu u_{0xxxx} = 0 \quad (11)$$

$$u_0(0, t) = u_{0x}(0, t) = u_0(1, t) = u_{0x}(1, t) = 0 \quad (12)$$

$$u_0(x, 0) = f(x), u_{0_t}(x, 0) = g(x) \quad (13)$$

and similarly by collecting the coefficients of terms of  $\varepsilon^1$ , we have following  $O(\varepsilon)$ -problem as under

**The  $O(\varepsilon)$ -problem:**

$$u_{1_{tt}} - u_{1_{xx}} + \delta u_{1_t} + \mu u_{1_{xxxx}} = -2v_0 u_{0_{xt}} - \delta v_0 u_{0_x} \quad (14)$$

$$u_1(0, t) = u_{1_x}(0, t) = u_1(1, t) = u_{1_x}(1, t) = 0 \quad (15)$$

$$u_1(x, 0) = f(x), u_{1_t}(x, 0) = g(x) \quad (16)$$

**The solution of the  $O(1)$ - a problem given in Eq. (11)-(13)**

The separation of variables method is useful to find the exact analytic solution of the initial-boundary value problem given in Eq. (11) to (13), where partial differential equation (PDEs) is linear and homogeneous with linear and homogeneous boundary conditions (BCs). The PDE is given in Eq. (11) can simply be remodeled into ODE, by using the separation of variables method, as follows as the product of the functions of the one variable:

$$u_0(x, t) = \psi(x)\phi(t) \quad (17)$$

Substituting Eq. (17) and its required derivatives into  $O(1)$ -problem and its associated conditions, it follows:

$$\ddot{\phi}(t) + \delta \dot{\phi}(t) + \lambda \phi(t) = 0 \quad (18)$$

Here negative sign  $(-\lambda)$  is employed for convenience, so the time depending part oscillates for  $\lambda > 0$ .

and space-dependent part is a boundary value problem, given as:

$$\psi^{(iv)}(x) - \frac{1}{\mu} \psi''(x) - \frac{\lambda}{\mu} \psi(x) = 0 \quad (19)$$

$$\psi(0) = \psi'(0) = \psi(1) = \psi'(1) = 0 \quad (20)$$

*Analysis of the time-dependent equation*

Let  $\phi(t) = e^{\alpha t}$  (where  $\alpha \in \mathbb{R}$ ), be the solution of Eq. (18), therefore the characteristic equation is

$$\alpha^2 + \delta \alpha + \lambda = 0 \quad (21)$$

Solving Eq. (21), the roots are as follows

$$\alpha_{1,2} = -\frac{\delta}{2} \pm \sqrt{\frac{\delta^2}{4} - \lambda} \quad (22)$$

The nature of the roots in Eq. (22) depends on the discriminant  $b^2 - 4ac$ . In this regard, the following are three cases.

**Case. 1:** For  $\frac{\delta^2}{4} - \lambda = 0$ , then  $\alpha_{1,2} = -\frac{\delta}{2}$ . Hence the roots are real and repeated, so the solution is,

$$\phi(t) = e^{-\frac{\delta}{2}t}(c_1 + c_2 t) \quad (23)$$

For  $t$  to be sufficiently large, it can easily be observed that  $\phi(t) \equiv 0$ . It means that the time-dependent part is bounded for large times  $t$ .

**Case 2:** For  $\frac{\delta^2}{4} - \lambda > 0$ ,  $\delta > 2\sqrt{\lambda}$ , then the roots are real and distinct, that is.

$$\alpha_1 = -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} - \lambda}, \quad \alpha_2 = -\frac{\delta}{2} - \sqrt{\frac{\delta^2}{4} - \lambda}$$

Therefore, the solution of the time-depending part is given by

$$\phi(t) = e^{-\frac{\delta}{2}t} \left( c_1 e^{\left(\sqrt{\frac{\delta^2}{4} - \lambda}\right)t} + c_2 e^{\left(-\sqrt{\frac{\delta^2}{4} - \lambda}\right)t} \right) \quad (24)$$

If time is sufficiently large, then the time-dependent solution  $\phi(t) \equiv 0$ .

Note that  $\frac{\delta^2}{4} - \lambda < \frac{\delta^2}{4}$  for  $\lambda > 0$ .

**Case 3:** For  $\frac{\delta^2}{4} - \lambda < 0 \Rightarrow 0 < \delta < 2\sqrt{\lambda}$ , for  $\lambda > 0$ , then

$$\alpha_1 = -\frac{\delta}{2} + i\sqrt{\lambda - \frac{\delta^2}{4}} \text{ and } \alpha_2 = -\frac{\delta}{2} - i\sqrt{\lambda - \frac{\delta^2}{4}}$$

Hence, the roots are complex conjugate with real part negative, so the general solution is;

$$\phi(t) = e^{-\frac{\delta}{2}t} \left( c_1 \cos\left(\sqrt{\lambda - \frac{\delta^2}{4}}t\right) + c_2 \sin\left(\sqrt{\lambda - \frac{\delta^2}{4}}t\right) \right) \quad (25)$$

For  $t$  to be sufficiently large, it can easily be observed that  $T(t) \equiv 0$ . It means that the time-dependent part is bounded for large time  $t$ .

**Analysis of the space-dependent equation**

Let  $\psi(x) = e^{\alpha x}$  ( $\alpha$  to be determined), be the solution of Eq. (19). Therefore, the characteristic equation is:

$$\mu\alpha^4 - \alpha^2 - \lambda = 0 \quad (26)$$

If  $\lambda < 0$  and  $\lambda = 0$  then  $\psi(x) \equiv 0 \Rightarrow \lambda < 0$  and  $\lambda = 0$  are not eigenvalues. The nontrivial solutions of Eq. (19) are obtained when  $\lambda > 0$ . If  $\lambda > 0$ , then the roots are as  $\pm\alpha$  and  $\pm i\beta$ , where the nontrivial solution of Eq. (19) is given by

$$\psi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x) + c_3 \cos(\beta x) + c_4 \sin(\beta x) \quad (27)$$

$$\alpha = \sqrt{\frac{1 + \sqrt{1 + 4\lambda\mu}}{2\mu}} \quad (28)$$

and

$$\beta = \sqrt{\frac{\sqrt{1 + 4\lambda\mu} - 1}{2\mu}} \quad (29)$$

Applying the boundary conditions, the following system is obtained into matrix form:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \beta \\ \cosh(\alpha) & \sinh(\alpha) & \cos(\beta) & \sin(\beta) \\ \alpha \sinh(\alpha) & \alpha \cosh(\alpha) & -\beta \sin(\beta) & \beta \cos(\beta) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (30)$$

Letting  $\det A = 0$ , non-trivial solutions are obtained when  $c_1, c_2, c_3, c_4 \neq 0$ :

$$\psi_n(x) = \alpha \sin(\beta x) - \beta \sinh(\alpha x) - \frac{\alpha \sin(\beta) - \beta \sinh(\alpha)}{\cos(\beta) - \cosh(\alpha)} (\cos(\beta x) - \cosh(\alpha x)) \quad (31)$$

where  $\psi_n(x)$  are the eigenfunction of the BVP

$$\psi^{(iv)}(x) - \frac{1}{\mu} \psi''(x) - \frac{\lambda}{\mu} \psi(x) = 0 \quad (32)$$

BCs:

$$\psi(0) = \psi(1) = \psi'(0) = \psi'(1) = 0, 0 < x < 1 \quad (33)$$

$$\psi_m^{(iv)}(x) - \frac{1}{\mu} \psi_m''(x) - \frac{\lambda}{\mu} \psi_m(x) = 0 \quad (34)$$

$$\psi_n^{(iv)}(x) - \frac{1}{\mu} \psi_n''(x) - \frac{\lambda}{\mu} \psi_n(x) = 0 \quad (35)$$

The frequency equation is given as:

$$f_{\mu}(\lambda) = 2\alpha\beta(1 - \cosh(\alpha)\cos(\beta)) + (\alpha^2 - \beta^2)\sinh(\alpha)\sin(\beta) = 0 \quad (36)$$

The frequency Eq. (36) is a nonlinear transcendental equation and cannot be solved exactly. From the frequency Eq. (36), the eigenvalues are determined numerically and are listed in the following table.

**Table 1:** First five eigenvalues of the frequency equation (36) ( $f_{\mu}(\lambda) = 0$ )

$n$	$\lambda_n$ at $\mu = 0.1$	$\lambda_n$ at $\mu = 0.01$	$\lambda_n$ at $\mu = 0.001$
1	61.8205	16.8038	11.3595
2	426.3927	83.2918	46.7804
3	1560.6132	244.5954	110.2646
4	4165.9405	570.6941	208.408
5	9177.5211	1155.1228	326.2054

From these Table.1 it can be observed that, if  $\mu \rightarrow 0$  then  $\lambda_n \rightarrow (n\pi)^2$  (Beam tends to string).

#### ***The general solution of the $O(1)$ -problem***

**Case 3:**  $\frac{\delta^2}{4} - \lambda_n < 0$

By using the superposition principle, and combining the product solutions as given in Eq. (17), it yields

$$u_0(x, t) = \sum_{n=1}^{\infty} e^{-\frac{\delta^2}{4}t} \left( A_{n0} \cos\left(\left(\sqrt{\lambda_n - \frac{\delta^2}{4}}\right)t\right) + B_{n0} \sin\left(\left(\sqrt{\lambda_n - \frac{\delta^2}{4}}\right)t\right) \right) \psi_n(x) \quad (42)$$

where  $A_{n0}$  is same as mentioned in Eq. (38) and

$$B_{n0} = \frac{\int_0^1 \left( g(x) + \frac{\delta}{2} f(x) \right) \psi_n(x) dx}{\sqrt{\lambda_n - \frac{\delta^2}{4}} \int_0^1 \psi_n^2(x) dx} \quad (43)$$

#### ***The solution to the $O(\varepsilon)$ -problem***

To solve the  $O(\varepsilon)$  – problem (14) - (16); an Eigen-function expansion method is used:

$$u_1(x, t) = \sum_{n=1}^{\infty} y_n(t) \psi_n(x) \quad (44)$$

where  $y_n(t)$  are unknown generalized Fourier coefficients.

*Khalid H. Malik et al*



**Case 3:**  $\frac{\delta^2}{4} - \lambda_n < 0, \lambda_n > 0$

The solution of  $O(\varepsilon)$ -problem consists of a homogeneous solution and the particular solution is,

$$u_1(x, t) = \sum_{n=1}^{\infty} e^{-\frac{\delta}{2}t} \left[ A_{n1} \cos(\omega_n t) + B_{n1} \sin(\omega_n t) + \frac{2v_0\omega_n}{\sigma_m^2 \left( \lambda_m - \frac{\delta^2}{4} - \omega_n^2 \right)} (A_{n0} \sin(\omega_n t) - B_{n0} \cos(\omega_n t)) \theta_{nm} \right] \psi_n(x) \quad (45)$$

where  $A_{n0}$ ,  $B_{n0}$ ,  $A_{n1}$  and  $B_{n1}$  are the same as mentioned in Eqs. (38), (46) respectively.

#### IV. Results and discussion

In this section the main focus on interpreting, commenting and explaining the obtained results in this paper. Complete analytic solution of the  $O(1)$  –problem has been used and discusses the influence of the small parameter  $0 < \varepsilon \ll 1$ , and the parameter of the damping on the translating Euler-Bernoulli-beam moving system has discussed in detail.

In this research work, we conclude that:

- i. The straightforward expansion method is applicable for the damped equation but will fail for the un-damped cases.
- ii. The solutions are bounded for all times  $t$
- iii. The damping term dominates the linear and trigonometric functions and suppresses the oscillations
- iv. ,  $u(x, t; \varepsilon) \rightarrow 0$  as  $t \rightarrow \infty$

**Case 3.**  $\frac{\delta^2}{4} - \lambda_n < 0, \lambda = \lambda_n$ ; is given in Table. 1

$$u(x, t; \varepsilon) = \sum_{n=1}^{\infty} e^{-\frac{\delta}{2}t} (A_{n0} \cos(\omega_n t) + B_{n0} \sin(\omega_n t)) \psi_n(x) + \varepsilon \left( \sum_{n=1, n \neq m}^{\infty} e^{-\frac{\delta}{2}t} \left( A_{n1} \cos(\omega_n t) + B_{n1} \sin(\omega_n t) + \frac{2v_0\omega_n}{\sigma_m^2 \left( \lambda_m - \frac{\delta^2}{4} - \omega_n^2 \right)} (A_{n0} \sin(\omega_n t) - B_{n0} \cos(\omega_n t)) \theta_{nm} \right) \psi_n(x) \right)$$

In this case, the solution is bounded due to the dominance of the damping term  $e^{-\frac{\delta}{2}t}$ . Solution  $u(x, t) \rightarrow 0$  as  $t \rightarrow \infty$ .

## V. Conclusion

In this paper, the Beam-like equation under the influence of viscous damping has been studied. The general initial conditions are considered for displacement and velocity. To construct the solution of the initial-boundary value problem analytically, the straightforward expansion method together with the classical separation of variables method has been used. By the use of the separation of variables method the two ordinary differential equations have been obtained: the time-dependent ordinary differential equation and the space-dependent ordinary differential equation. The nature of the solution of the time-dependent part depends on the nature of the discriminant: discriminant  $< 0$  means under damping, discriminant  $> 0$  means over damping, and discriminant  $= 0$  means critical damping. It is interesting to note that the solution of the space-depending part is a trivial solution for both cases when the separation constant  $\lambda < 0$  or  $\lambda = 0$ , and is non-trivial or non-zero when  $\lambda > 0$ . The solution of the space-depending part (boundary-value problem) is constructed, whereas eigenvalues and eigenfunctions are obtained. The damped amplitude-response of the system is obtained under certain cases of damping. It is concluded that the response of the system remains bounded due to the dominance of the damping on all terms.

## Conflict of Interest:

No conflict of interest regarding this article

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*Khalid H. Malik et al*

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