



## ERROR ANALYSIS OF CLOSED NEWTON-COTES CUBATURE SCHEMES FOR DOUBLE INTEGRALS

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### Abstract

*Numerical integration is one of the fundamental tools of numerical analysis to cope with the complex integrals which cannot be evaluated analytically, and for the cases where the integrand is not mathematically known in closed form. The quadrature rules are used for approximating single integrals, whereas cubature rules are used to evaluate integrals in higher dimensions. In this work, we consider the closed Newton-Cotes cubature schemes for double integrals and discuss consequent error analysis of these schemes in terms of the degree of precision, local error terms for the basic form approximations, composite forms and the global error terms. Besides, the computational cost of the implementation of these schemes is also presented. The theorems proved in this work are a pioneering investigation on error analysis of such schemes in the literature.*

**Keywords:** Cubature, Double integrals, closed Newton-Cotes, Precision, Order of accuracy, Local error, Computational cost.

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### I. Introduction

Mathematical models arise from the translation of the fundamental laws of science and engineering for various physical systems under consideration [III]. Such models often arrive in complex forms. The mainstream analytical methods are not always applicable to solve such systems, and in some situations, the functions are not known in the closed-form [III], [VIII]. Iterative and approximation tools of numerical analysis are used to cope with such problems [XIII], [XVI]. Numerical integration in the same zeal provides means for the efficient evaluation of complex integrals [I]. The very basic methods of numerical integration are due to Isaac Newton and Roger Cotes, well-known as Newton-Cotes rules [III]. The Newton-Cotes rules are quite

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frequently used for the evaluation of single integrals, i.e. integrating a function of one variable over a subset of reals [I],[III]. There have also been enormous modifications of closed and open Newton-Cotes rules for single integrals. The contributions by [VI], [VII] improved the conventional rules with two units of precision. There have also been derivative-based improvements [II], [IV], [V], [XVIII], [XVII], [XV]. The closed Newton-Cotes quadrature rules and the improvements have also been used to solve integral equations [XIV], carry out numerical simulation of switched reluctance machines [XII], efficient evaluation of Riemann-Stieltjes integral [X], [XI], evaluation of numerical cubature with derivative-based schemes [IX], etc. However, for the integration in higher dimensions the basic closed Newton-Cotes schemes are although available in the literature [Pal book], but the error analysis is not presented for the basic schemes.

In this research work, we discuss error analysis of the basic closed Newton-Cotes cubature schemes for double integrals. Theorems regarding the degree of precision, order of accuracy and local error terms are proved. The extension of the basic cubature schemes into composite forms is also carried out and the computational costs are discussed as a pioneering analysis in this work.

## II. Closed Newton-Cotes Schemes for Single Integral

The first three closed Newton-Cotes quadrature rules in basic form with local error terms for numerical integration are defined as equations (1)-(3) [III] :

$$\int_a^b f(x) dx = \frac{b-a}{2} [f(a) + f(b)] - \frac{(b-a)^3}{12} f''(\xi) \quad (1)$$

$$\int_a^b f(x) dx = \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi) \quad (2)$$

$$\int_a^b f(x) dx = \frac{b-a}{8} \left[ f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{(b-a)^5}{6480} f^{(4)}(\xi) \quad (3)$$

where  $\xi \in (a, b)$ , and are known as trapezoidal, Simpson's 1/3 and Simpson's 3/8 rules. The composite forms of these three rules with global error terms can be defined as (4)-(6) [III]:

$$\int_a^b f(x) dx = \frac{h}{2} [f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b)] - \frac{h^2}{12} (b-a) f''(\eta(x)) \quad (4)$$

$$\int_a^b f(x) dx = \frac{h}{3} \left[ f(a) + 2 \sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + 4 \sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + f(b) \right] - \frac{(b-a)}{180} h^4 f^{(4)}(\eta(x)) \quad (5)$$

$$\int_a^b f(x) dx = \frac{3h}{8} \left[ f(a) + 3 \sum_{i \neq 3k}^{\frac{n}{3}-1} f(x_i) + 2 \sum_{j=1}^{\frac{n}{3}-1} f(x_{3j}) + f(b) \right] - \frac{h^4}{80} (b-a)^4 f^{(4)}(\eta(x)) \quad (6)$$

where  $\eta(x) \in (a, b)$ ,  $h = \frac{b-a}{n}$ ,  $x_i = a + ih$ , and  $i = 0, 1, 2, \dots, n$

### III. Closed Newton-Cotes Schemes for Double Integral

The general form of the double integrals defined over rectangles in two dimensions is defined as

$$\int_c^d \left( \int_a^b f(x, y) dx \right) dy \quad (7)$$

Here we consider the evaluation of the double integral over a rectangle  $x = a, x = b, y = c, y = d$ .

Pal M. [XIII] discussed the extension of quadrature schemes (1)-(3) for double integrals, and the basic forms of closed Newton-Cotes double integral cubature schemes of Trapezoid, Simpson's 1/3 and Simpson's 3/8-type, respectively, CNCT, CNCS13 and CNCS38, can be described as (8)-(10) [XIII]:

$$\int_c^d \int_a^b f(x, y) dx dy \approx \frac{(b-a)(d-c)}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \quad (8)$$

$$\int_c^d \int_a^b f(x, y) dx dy = \frac{(b-a)(d-c)}{36} \left[ \begin{aligned} &f(a, c) + f(a, d) + f(b, c) + f(b, d) \\ &+ 4 \left\{ f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) \right\} \\ &+ f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) \\ &+ 16f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \end{aligned} \right] \quad (9)$$

$$\int_c^d \int_a^b f(x, y) dx dy = \frac{(b-a)(d-c)}{64} \left[ \begin{aligned} &f(a, c) + f(a, d) + f(b, c) + f(b, d) \\ &+ 3 \left\{ f\left(a, \frac{2c+d}{3}\right) + f\left(b, \frac{2c+d}{3}\right) + f\left(a, \frac{c+2d}{3}\right) + f\left(b, \frac{c+2d}{3}\right) \right\} \\ &+ f\left(\frac{2a+b}{3}, c\right) + f\left(\frac{2a+b}{3}, d\right) + f\left(\frac{a+2b}{3}, c\right) + f\left(\frac{a+2b}{3}, d\right) \\ &+ 9 \left\{ f\left(\frac{2a+b}{3}, \frac{2c+d}{3}\right) + f\left(\frac{2a+b}{3}, \frac{c+2d}{3}\right) \right\} \\ &+ f\left(\frac{a+2b}{3}, \frac{2c+d}{3}\right) + f\left(\frac{a+2b}{3}, \frac{c+2d}{3}\right) \end{aligned} \right] \quad (10)$$

### IV. Proposed Error Analysis and Main Results

We derive the degree of precision and local error terms of the cubature schemes (8)-(10), and discuss their extension in composite forms with global error terms. The degrees of precisions of CNCT, CNCS13 and CNCS38 cubature schemes for double integrals, i.e. (8)-(10) are discussed and proved in Theorem 1.

**Theorem 1.** The degree of precision of CNCT, CNCS13 and CNCS38 cubature schemes for double integrals, as defined in (8)-(10), respectively are one, three and three.

#### Proof of Theorem 1.

Following the definition of degree of precision as mentioned in [IX], the exact results from (7) for  $f(x, y) = (x y)^n$  with  $n = 0, 1, 2, 3, 4$  are:

$$\int_c^d \int_a^b x^0 y^0 dx dy = (b-a)(d-c) \quad (11)$$

$$\int_c^d \int_a^b (xy) dx dy = \frac{(b^2-a^2)(d^2-c^2)}{4} \quad (12)$$

$$\int_c^d \int_a^b (x^2y^2) dx dy = \frac{(b^3-a^3)(d^3-c^3)}{9} \quad (13)$$

$$\int_c^d \int_a^b (x^3y^3) dx dy = \frac{(b^4-a^4)(d^4-c^4)}{16} \quad (14)$$

$$\int_c^d \int_a^b (x^4y^4) dx dy = \frac{(b^5-a^5)(d^5-c^5)}{25} \quad (15)$$

Approximate results using CNCT scheme from (8) for  $f(x, y) = (xy)^n$  with  $n = 0, 1, 2$  are:

$$CNCT(x^0y^0) = (b-a)(d-c) \quad (16)$$

$$CNCT(xy) = \frac{(b^2-a^2)(d^2-c^2)}{4} \quad (17)$$

$$CNCT(x^2y^2) = \frac{(b-a)(b^2+a^2)(d-c)(d^2+c^2)}{4} \quad (18)$$

Comparison of (11)-(13) with (16)-(18) gives,

For CNCT scheme, and  $f(x, y) = (xy)^n$  with  $n \leq 1$ ,

$$\int_c^d \left( \int_a^b (xy)^n dx \right) dy - CNCT(xy)^n = 0 \quad (19)$$

But, for  $f(x, y) = x^2y^2$ , we have

$$\int_c^d \left( \int_a^b x^2y^2 dx \right) dy - CNCT(x^2y^2) \neq 0 \quad (20)$$

So, from (19)-(20), it appears that the degree of precision of CNCT double integral schemes is 1.

Approximate results using CNCS13 scheme from (9) for  $f(x, y) = (xy)^n$  with  $n = 0, 1, 2, 3, 4$  are:

$$CNCS13(x^0y^0) = (b-a)(d-c) \quad (21)$$

$$CNCS13(xy) = \frac{(b^2-a^2)(d^2-c^2)}{4} \quad (22)$$

$$CNCS13(x^2y^2) = \frac{(b^3-a^3)(d^3-c^3)}{9} \quad (23)$$

$$CNCS13(x^3y^3) = \frac{(b^4-a^4)(d^4-c^4)}{16} \quad (24)$$

$$CNCS13(x^4y^4) = \frac{(a-b)(c-d) \left\{ 4 \left( \frac{a+b}{2} \right)^4 + a^4 + b^4 \right\} \left\{ 4 \left( \frac{c+d}{2} \right)^4 + c^4 + d^4 \right\}}{36} \quad (25)$$

Comparison of (11)-(15) with (21)-(25) gives,

For CNCS13, and  $f(x, y) = (xy)^n$ ,  $n \leq 3$ ,

$$\int_c^d \left( \int_a^b (xy)^n dx \right) dy - CNCS13(xy)^n = 0 \quad (26)$$

But if  $f(x, y) = x^4 y^4$ , then

$$\int_c^d \left( \int_a^b x^4 y^4 dx \right) dy - CNCS13(x^4 y^4) \neq 0 \quad (27)$$

So, from (26)-(27), the degree of precision of CNCS13 double integral scheme is 3.

Similarly, for CNCS38 cubature scheme (10) for double integrals, we have for  $f(x, y) = (xy)^n$  and  $n \leq 3$ ,

$$\int_c^d \left( \int_a^b (xy)^n dx \right) dy - CNCS38(xy)^n = 0 \quad (28)$$

But, for  $f(x, y) = x^4 y^4$ , the approximation using (10) is:

$$CNCS38(x^4 y^4) = \frac{(a-b)(c-d) \left\{ \begin{aligned} &3\left(\frac{a}{3} + \frac{2b}{3}\right)^4 + 3\left(\frac{2a}{3} + \frac{b}{3}\right)^4 + a^4 + b^4 \\ &- \left\{ 3\left(\frac{c}{3} + \frac{2d}{3}\right)^4 + 3\left(\frac{2c}{3} + \frac{d}{3}\right)^4 + c^4 + d^4 \right\} \end{aligned} \right\}}{64} \quad (29)$$

From (15) and (29), we have for  $n = 4$ :

$$\int_c^d \left( \int_a^b x^4 y^4 dx \right) dy - CNCS38(x^4 y^4) \neq 0 \quad (30)$$

Therefore, from (28) and (30), the degree of precision of CNCS38 double integral scheme is 3.  $\square$

The local error terms of (8)-(10) are presented with proof in theorems 2-4, respectively for CNCT, CNCS13 and CNCS38 double integral schemes.

**Theorem 2.** Let  $a, b, c, d$  be finite real numbers, and  $f(x, y)$  is along with its second-order partial derivatives exist and are continuous in  $[a, b] \times [c, d]$ , then the CNCT double integral scheme in basic form with the local error term is defined as:

$$\begin{aligned} \int_c^d \int_a^b f(x, y) dx dy &= CNCT + R_{CNCT}[f] \\ &= \frac{(b-a)(d-c)}{4} \left[ \begin{aligned} &f(a, c) + f(a, d) \\ &+ f(b, c) + f(b, d) \end{aligned} \right] - \frac{(b-a)^3(d-c)}{12} f_{xx}(\xi, \eta) - \frac{(b-a)(d-c)^3}{12} f_{yy}(\xi, \eta) \end{aligned}$$

where  $\xi \in (a, b)$  and  $\eta \in (c, d)$

Proof of Theorem 2.

The second-order partial derivative term in Taylor's series of  $f(x, y)$  [III,] [XIII] about  $(x_0, y_0)$  is:

$$\frac{1}{2!} \left[ \begin{aligned} &(x - x_0)^2 \frac{\partial^2 f}{\partial x^2}(x_0, y_0) + 2(x - x_0)(y - y_0) \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ &+ (y - y_0)^2 \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{aligned} \right] \quad (31)$$

Using (31), the local error term of CNCT double integral scheme (8) can be represented as:

$$R_{CNCT} [f] = \frac{1}{2!} \left[ \int_c^d \int_a^b x^2 dx dy - CNCT(x^2) \right] f_{xx}(\xi, \eta) + \left[ \int_c^d \int_a^b xy dx dy - CNCT(xy) \right] f_{yx}(\xi, \eta) + \frac{1}{2!} \left[ \int_c^d \int_a^b y^2 dx dy - CNCT(y^2) \right] f_{yy}(\xi, \eta) \quad (32)$$

Using the exact results, which are:

$$\int_c^d \int_a^b x^2 dx dy = \frac{(b^3 - a^3)(d - c)}{3}, \int_c^d \int_a^b xy dx dy = \frac{(b^2 - a^2)(d^2 - c^2)}{4},$$

$$\int_c^d \int_a^b y^2 dx dy = \frac{(b - a)(d^3 - c^3)}{3}, \text{ and the approximate CNCT results:}$$

$$CNCT(x^2) = \frac{(b - a)(d - c)(b^2 + a^2)}{2}, CNCT(xy) = \frac{(b^2 - a^2)(d^2 - c^2)}{4},$$

$CNCT(y^2) = \frac{(b - a)(d - c)(d^2 + c^2)}{2}$  in (32), the local error term for CNCT double integral scheme takes the form

$$R_{CNCT} [f] = -\frac{(b - a)^3(d - c)}{12} f_{xx}(\xi, \eta) - \frac{(b - a)(d - c)^3}{12} f_{yy}(\xi, \eta) \quad (33)$$

where  $\xi \in (a, b)$  and  $\eta \in (c, d)$   $\square$

**Theorem 3.** Let  $a, b, c, d$  be finite real numbers, and  $f(x, y)$  is along with its fourth-order partial derivatives exist and are continuous in  $[a, b] \times [c, d]$ , then the CNCS13 double integral scheme in basic form with the local error term is defined as:

$$\int_c^d \int_a^b f(x, y) dx dy =$$

$$\frac{(b - a)(d - c)}{36} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right.$$

$$\left. + 4 \left\{ f\left(a, \frac{c + d}{2}\right) + f\left(b, \frac{c + d}{2}\right) \right\} + 16f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) \right]$$

$$- \frac{(b - a)^5(d - c)}{2880} f_{xxxx}(\xi, \eta) - \frac{(b - a)(d - c)^5}{2880} f_{yyyy}(\xi, \eta)$$

where  $\xi \in (a, b)$  and  $\eta \in (c, d)$

Proof of Theorem 3.

The fourth-order partial derivative term in Taylor's series of  $f(x, y)$  [III],[XIII] about  $(x_0, y_0)$  is:

$$\frac{1}{4!} \left[ (x - x_0)^4 \frac{\partial^4 f}{\partial x^4}(x_0, y_0) + 4(x - x_0)^3(y - y_0) \frac{\partial^4 f}{\partial x^3 \partial y}(x_0, y_0) \right.$$

$$+ 6(x - x_0)^2(y - y_0)^2 \frac{\partial^4 f}{\partial x^2 \partial y^2}(x_0, y_0)$$

$$\left. + 4(x - x_0)(y - y_0)^3 \frac{\partial^4 f}{\partial x \partial y^3}(x_0, y_0) + (y - y_0)^4 \frac{\partial^4 f}{\partial y^4}(x_0, y_0) \right] \quad (34)$$

Using (34), the local error term of CNCS13 double integral scheme (9) can be represented as:

$$\begin{aligned}
 R_{CNCS13}[f] = & \frac{1}{4!} \left[ \int_c^d \int_a^b x^4 dx dy - CNCS13(x^4) \right] f_{xxxx}(\xi, \eta) \\
 & + \frac{1}{6} \left[ \int_c^d \int_a^b x^3 y dx dy - CNCS13(x^3 y) \right] f_{yxxx}(\xi, \eta) + \frac{1}{4} \left[ \int_c^d \int_a^b x^2 y^2 dx dy - \right. \\
 & CNCS13(x^2 y^2) \left. \right] f_{yyxx}(\xi, \eta) + \frac{1}{6} \left[ \int_c^d \int_a^b x y^3 dx dy - CNCS13(x y^3) \right] f_{yyyx}(\xi, \eta) \\
 & + \frac{1}{4!} \left[ \int_c^d \int_a^b y^4 dx dy - CNCS13(y^4) \right] f_{yyyy}(\xi, \eta)
 \end{aligned} \quad (35)$$

The exact integrals in (35) are:

$$\begin{aligned}
 \int_c^d \int_a^b x^4 dx dy &= \frac{(b^5 - a^5)(d - c)}{5}, \int_c^d \int_a^b (x^3 y) dx dy = \frac{(b^4 - a^4)(d^2 - c^2)}{8}, \\
 \int_c^d \int_a^b (x^2 y^2) dx dy &= \frac{(b^3 - a^3)(d^3 - c^3)}{9}, \int_c^d \int_a^b (x y^3) dx dy = \frac{(b^2 - a^2)(d^4 - c^4)}{8}, \text{ and} \\
 \int_c^d \int_a^b y^4 dx dy &= \frac{(b - a)(d^5 - c^5)}{5}. \text{ The approximate CNCS13 scheme results for the} \\
 &\text{integrals in (35) are:}
 \end{aligned}$$

$$\begin{aligned}
 CNCS13(x^4) &= \frac{(a-b)(c-d) \left\{ 24 \left( \frac{a+b}{2} \right)^4 + 6a^4 + 6b^4 \right\}}{36}, CNCS13(x^3 y) = \frac{(b^4 - a^4)(d^2 - c^2)}{8}, \\
 CNCS13(x^2 y^2) &= \frac{(b^3 - a^3)(d^3 - c^3)}{9}, CNCS13(x y^3) = \frac{(b^2 - a^2)(d^4 - c^4)}{8}, \text{ and} \\
 CNCS13(y^4) &= \frac{(a-b)(c-d) \left\{ 24 \left( \frac{c+d}{2} \right)^4 + 6c^4 + 6d^4 \right\}}{36}.
 \end{aligned}$$

Using these exact and approximate evaluations in (35), we finally get: the local error term of the CNCS13 cubature scheme for the double integrals as:

$$R_{CNCS13}[f] = -\frac{(b-a)^5(d-c)}{2880} f_{xxxx}(\xi, \eta) - \frac{(b-a)(d-c)^5}{2880} f_{yyyy}(\xi, \eta) \quad (36)$$

where  $\xi \in (a, b)$  and  $\eta \in (c, d)$   $\square$

**Theorem 4.** Let  $a, b, c, d$  be finite real numbers, and  $f(x, y)$  is along with its fourth-order partial derivatives exist and are continuous in  $[a, b] \times [c, d]$ , then the CNCS38 double integral scheme in basic form with the local error term is defined as:

$$\begin{aligned}
 \int_c^d \int_a^b f(x, y) dx dy = & \frac{(b-a)(d-c)}{64} \left[ \begin{aligned} & f(a, c) + f(a, d) + f(b, c) + f(b, d) \\ & + 3 \left\{ f\left(a, \frac{2c+d}{3}\right) + f\left(b, \frac{2c+d}{3}\right) + f\left(a, \frac{c+2d}{3}\right) + f\left(b, \frac{c+2d}{3}\right) \right\} \\ & + 9 \left\{ f\left(\frac{2a+b}{3}, c\right) + f\left(\frac{2a+b}{3}, d\right) + f\left(\frac{a+2b}{3}, c\right) + f\left(\frac{a+2b}{3}, d\right) \right\} \\ & - \frac{(b-a)^5(d-c)}{6480} f_{xxxx}(\xi, \eta) - \frac{(b-a)(d-c)^5}{6480} f_{yyyy}(\xi, \eta) \end{aligned} \right]
 \end{aligned}$$

where  $\xi \in (a, b)$  and  $\eta \in (c, d)$ .

**Proof of Theorem 4.**

Using (34), the local error term of CNCS38 double integral scheme (10) can be represented as:

$$\begin{aligned} R_{CNCS38}[f] = & \frac{1}{4!} \left[ \int_c^d \int_a^b x^4 dx dy - CNCS38(x^4) \right] f_{xxxx}(\xi, \eta) \\ & + \frac{1}{6} \left[ \int_c^d \int_a^b x^3 y dx dy - CNCS38(x^3 y) \right] f_{yxxx}(\xi, \eta) \\ & + \frac{1}{4} \left[ \int_c^d \int_a^b x^2 y^2 dx dy - CNCS38(x^2 y^2) \right] f_{yyxx}(\xi, \eta) \\ & + \frac{1}{6} \left[ \int_c^d \int_a^b x y^3 dx dy - CNCS38(x y^3) \right] f_{yyyx}(\xi, \eta) \\ & + \frac{1}{4!} \left[ \int_c^d \int_a^b y^4 dx dy - CNCS38(y^4) \right] f_{yyyy}(\xi, \eta) \end{aligned} \quad (37)$$

The exact results in (37) are same as those used in (35), whereas the approximate CNCS38 cubature scheme results for the integrals in (37) are:

$$\begin{aligned} CNCS38(x^4) &= \frac{(a-b)(c-d) \left\{ 3 \left( \frac{a+2b}{3} \right)^4 + 3 \left( \frac{2a+b}{3} \right)^4 + a^4 + b^4 \right\}}{8}, CNCS38(x^3 y) = \frac{(b^4 - a^4)(d^2 - c^2)}{8} \\ CNCS38(x^2 y^2) &= \frac{(b^3 - a^3)(d^3 - c^3)}{9}, CNCS38(x y^3) = \frac{(b^2 - a^2)(d^4 - c^4)}{8}, \\ CNCS38(y^4) &= \frac{(a-b)(c-d) \left\{ 3 \left( \frac{c+2d}{3} \right)^4 + 3 \left( \frac{2c+d}{3} \right)^4 + c^4 + d^4 \right\}}{8}. \end{aligned}$$

Using these in (37), we have the local error term of the CNCS38 cubature scheme for double integrals, which is:

$$R_{CNCS38}[f] = -\frac{(b-a)^5(d-c)}{6480} f_{xxxx}(\xi, \eta) - \frac{(b-a)(d-c)^5}{6480} f_{yyyy}(\xi, \eta) \quad (38)$$

where  $\xi \in (a, b)$  and  $\eta \in (c, d)$ .  $\square$

For improved approximations, it is imperative to use the CNCT, CNCS13 and CNCS38 schemes in composite form by sub-dividing the original rectangle into sub-rectangles of area  $h^2$  and  $k^2$  along  $x$  and  $y$  axes leading to CNCT-Cn, CNCS13-Cn and CNCS38-Cn composite schemes for  $n^2$  elements, say.

Let  $a, b, c, d$  be finite real numbers, and  $f(x, y)$  along with its higher order partial derivatives ( $2^{nd}$ ,  $4^{th}$ ) exist and are continuous in  $[a, b] \times [c, d]$ . Let  $\{x_i = a + ih, i = 0, 1, \dots, n\}$  and  $\{y_j = c + jk, j = 0, 1, \dots, n\}$  form uniformly spaced partitions of  $[a, b]$  and  $[c, d]$  such that  $b-a = nh$  and  $d-c = nk$ , then the CNCT-Cn, CNCS13-Cn and CNCS38-Cn composite schemes for  $n^2$  elements with the global error terms are defined as in (39)-(41), respectively.

$$\begin{aligned} \int_c^d \int_a^b f(x, y) dx dy &= \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} f(x, y) dx dy \\ &= \text{CNCT-Cn} = \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \frac{hk}{4} [f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})] \\ &\quad - \frac{h^2(b-a)(d-c)}{12} f_{xx}(\underline{\xi}, \underline{\eta}) - \frac{k^2(b-a)(d-c)}{12} f_{yy}(\underline{\xi}, \underline{\eta}) \end{aligned} \quad (39)$$

where  $\underline{\xi} \in (a, b)$  and  $\underline{\eta} \in (c, d)$



$$\begin{aligned} \text{CNCS13-Cn} = & \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \left[ \frac{hk}{36} \left[ \begin{aligned} & f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1}) \\ & + 4 \left\{ f\left(x_i, \frac{y_j+y_{j+1}}{2}\right) + f\left(x_{i+1}, \frac{y_j+y_{j+1}}{2}\right) \right\} \\ & + f\left(\frac{x_i+x_{i+1}}{2}, y_j\right) + f\left(\frac{x_i+x_{i+1}}{2}, y_{j+1}\right) \\ & + 16f\left(\frac{x_i+x_{i+1}}{2}, \frac{y_j+y_{j+1}}{2}\right) \end{aligned} \right] \right. \\ & \left. - \frac{h^4(b-a)(d-c)}{2880} f_{xxxx}(\underline{\xi}, \underline{\eta}) - \frac{k^4(b-a)(d-c)}{2880} f_{yyyy}(\underline{\xi}, \underline{\eta}) \right] \end{aligned} \quad (40)$$

where  $\underline{\xi} \in (a, b)$  and  $\underline{\eta} \in (c, d)$

$$\begin{aligned} \text{CNCS38-Cn} = & \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \left[ \frac{hk}{64} \left[ \begin{aligned} & f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1}) \\ & + 3 \left\{ f\left(x_i, \frac{2y_j+y_{j+1}}{3}\right) + f\left(x_{i+1}, \frac{2y_j+y_{j+1}}{3}\right) + f\left(x_i, \frac{y_j+2y_{j+1}}{3}\right) + f\left(x_{i+1}, \frac{y_j+2y_{j+1}}{3}\right) \right\} \\ & + f\left(\frac{2x_i+x_{i+1}}{3}, y_j\right) + f\left(\frac{2x_i+x_{i+1}}{3}, y_{j+1}\right) + f\left(\frac{x_i+2x_{i+1}}{3}, y_j\right) + f\left(\frac{x_i+2x_{i+1}}{3}, y_{j+1}\right) \end{aligned} \right] \\ & + 9 \left\{ f\left(\frac{2x_i+x_{i+1}}{3}, \frac{2y_j+y_{j+1}}{3}\right) + f\left(\frac{2x_i+x_{i+1}}{3}, \frac{y_j+2y_{j+1}}{3}\right) \right. \\ & \left. + f\left(\frac{x_i+2x_{i+1}}{3}, \frac{2y_j+y_{j+1}}{3}\right) + f\left(\frac{x_i+2x_{i+1}}{3}, \frac{y_j+2y_{j+1}}{3}\right) \right\} \\ & \left. - \frac{h^4(b-a)(d-c)}{6480} f_{xxxx}(\underline{\xi}, \underline{\eta}) - \frac{k^4(b-a)(d-c)}{6480} f_{yyyy}(\underline{\xi}, \underline{\eta}) \right] \end{aligned} \quad (41)$$

where  $\underline{\xi} \in (a, b)$  and  $\underline{\eta} \in (c, d)$

## V. Numerical Experiments, Results and Discussion

We apply the CNCT, CNCS13 and CNCS38 composite schemes on two test double integrals from [XIII] under similar conditions for instant verification of the error bounds. Figs 1-2 show the decreasing error distributions for Examples 1 and 2, and the trends show the convergence of the schemes, and the order of accuracy of the rules are consistent with the derived error terms. The total computational cost to achieve an absolute percentage error of atmost 1E-08 in case of CNCT and 1E-12 for others in terms of function evaluations for Examples 1 and 2, respectively, of the CNCT-Cn scheme is 160801 and 333573696 evaluations, for the CNCS13-Cn is 16641 and 2002225, and for the CNCS38-Cn is 24649 and 3041536. Tables 1 and 2 show the observed computational order of accuracy [XIII],[XVIII]. From Tables 1-2, the theoretical error bounds are verified in terms of observed orders of accuracy.

## VI. Conclusion

In this paper, a pioneering investigation of the error terms and computational performance of the closed Newton-Cotes double integral cubature schemes was performed theoretically and then verified using numerical experiments. The theorems concerning the degree of precision and local error terms of three schemes: CNCT, CNCS13 and CNCS38 were derived in the basic form. The successful extension to the composite forms was established and the global error terms were reported. Numerical features, like the computational costs, observed orders of accuracy and

absolute error drops were also used for the verification of theoretical results proved in this work.

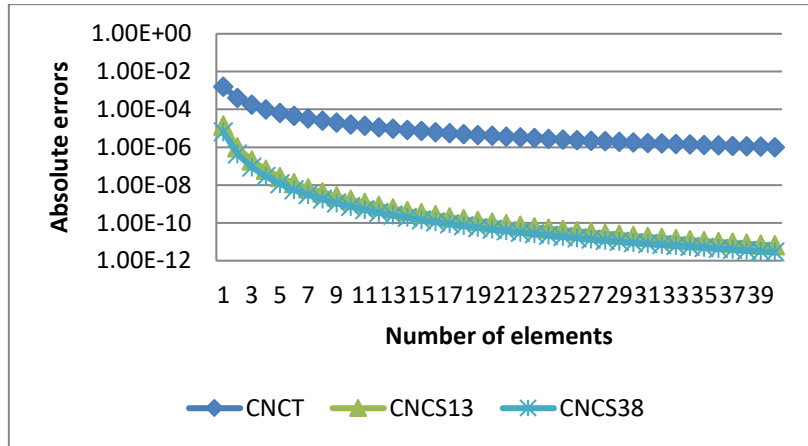


Figure 1: Absolute error distributions versus the number of elements of CNCT, CNCS13, CNCS38 schemes for Example 1

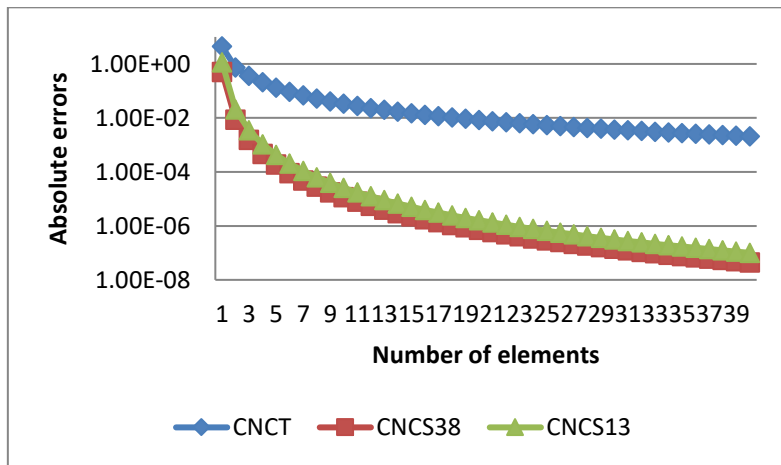


Figure 2: Absolute error distributions versus the number of elements of CNCT, CNCS13, CNCS38 schemes for Example 2

**Table 1: Observed order of accuracy for Example 1**

m Strips	Observed order of accuracy		
	CNCT	CNCS13	CNCS38
1	NA	NA	NA
2	1.973624	3.844743	3.86227
4	1.991894	3.959843	3.964333
8	1.997856	3.989914	3.991037
16	1.999467	3.997479	3.99776
32	1.999884	3.999481	3.999467
64	1.999944	4.000828	4.00327
128	2	3.925672	3.763397

**Table 2: Observed order of accuracy for Example 2**

m Strips	Observed order of accuracy		
	CNCT	CNCS13	CNCS38
1	NA	NA	NA
2	2.583861	5.803283	5.839356
4	1.826244	4.275194	4.220091
8	1.99035	4.052966	4.045168
16	1.998781	4.01214	4.010679
32	1.999778	4.00297	4.002634
64	1.99992	4.000739	4.000656
128	2.000018	4.000169	4.000158

## VII. Acknowledgements

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## Conflict of Interest:

There is no conflict of interest regarding this article

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