# A NEW AND EFFICIENT SIMPSON'S 1/3-TYPE QUADRATURE RULE FOR RIEMANN-STIELTJES INTEGRAL

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## Abstract

*In this research paper, a new derivative-free Simpson 1/3-type quadrature scheme has been proposed for the approximation of the Riemann-Stieltjes integral (RSI). The composite form of the proposed scheme on the RSI has been derived using the concept of precision. The theorems concerning basic form, composite form, local and global errors of the new scheme have been proved theoretically. For the trivial case of the integrator in the proposed RS scheme, successful reduction to the corresponding Riemann scheme is proved. The performance of the proposed scheme has been tested by numerical experiments using MATLAB on some test problems of RS integrals from literature against some existing schemes. The computational cost, the order of accuracy and average CPU times (in seconds) of the discussed rules have been computed to demonstrate cost-effectiveness, time-efficiency and rapid convergence of the proposed scheme under similar conditions.*

## I.  Introduction

In numerical integration, the approximate computation of a definite integral is the basic problem and this approximate value is known as the area of function under the curve which is used in engineering applications. Definite integral $I(f) = \int_a^b f(x)dx$ cannot be solved analytically, e.g. for the integrands $f(x) = -e^{x^2}$ and $f(x) = \sin x^2$. Numerical evaluation of definite integral $I(f)$ is known as numerical integration or quadrature. Quadrature rules are commonly based on polynomial interpolation. Most of the authors studied and used quadrature rules to approximate the classical Riemann

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integral. However, few authors used quadrature rules to approximate the RSI. Riemann-Stieltjes integral is an extension of Riemann integral. Suppose that $f(x)$ is a real-valued function and bounded on $[a, b]$ and $\alpha$ is a monotonically increasing function on $[a, b]$, then RSI is defined [IV] as

$$RS(f(x): \alpha; a, b) = \int_a^b f(x)d\alpha(x) \tag{1}$$

where $f(x)$ is integrand and $\alpha(x)$ is integrator.

Several applications have been used in numerical evaluation of RSI such as Statistics and probability theory, Complex analysis, Functional analysis, Operator theory and others. Much work in the literature has been devoted to improving quadrature approximations of the Riemann integral, to mention a few [XIV]-[II] and related extension for integral equations [XIV]. Meanwhile, recently some efficient numerical integration schemes for Riemann integral with standard error analysis were promoted for numerical cubature and simulation of switched reluctance machines [V], [VII].

In this regard, the quadrature rules for the RSI are focused on a few works in literature. In 2008, Mercer [VIII] proposed a trapezoidal rule for the RSI without numerical test and derived Hadamard integral inequality. In 2012, Mercer [IX] used the idea of relative convexity and just discussed the inequalities in the midpoint scheme and Simpson's scheme on the RSI. In 2013, Zhao *et al* [XVIII] introduced a new family of closed Newton-Cotes quadrature with a Midpoint derivative.

In 2014, Zhao *et al*. [XIX] presented the midpoint derivative-based trapezoid-type scheme for the RSI without numerical verification in which midpoint is used for the evaluation of function derivative. In 2015, Zhao *et al*. [XVIII] presented a composite trapezoidal rule for the RSI without numerical verification. Memon et al. [VI] proposed a derivative-based trapezoidal rule for the RSI with some numerical examples.

In this research, a new Simpson's 1/3-type scheme for the RSI is derived in basic and composite forms. The theoretical error analysis and consequent verification on some test problems are presented to demonstrate cost-effectiveness, time efficiency and rapid convergence of the proposed scheme.

## II. General Formulation of Quadrature Rules for the Riemann Integral

The general integration formula to evaluate a definite integral over a finite interval $[a, b]$ is described in [II],[III] as follows:

$$I(f; a, b) = \int_a^b f(x)dx \approx \sum_{i=0}^n w_i f(x_i) \tag{2}$$

Where $n + 1$ has different integration points at $x_0, x_1,\ldots, x_n$ within the interval $[a, b]$ and $n+1$ weights $w_i$, i = 0, 1, 2,..., $n$. If the integration points are divided equally over the interval, then

$x_i$ = a+ih, where $h = \frac{b-a}{n}$.

The closed quadrature formula of Newton-Cotes for evaluating a definite integral over $[a, b]$ is defined as:

$$I(f; a, b) = \int_{x_0}^{x_n} f(x)dx \approx \sum_{i=0}^n w_i f(x_0 + ih) \tag{3}$$

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There are several ways to determine the weighting coefficients $w_i$'s in equation (3). A common approach depends on the precision of a quadrature formula where the constants $w_0, w_1, ..., w_n$ are chosen so that the approximation error in equation (3)

becomes zero, i.e.

$$E(f) = \int_a^b f(x)dx - \sum_{i=0}^n w_i f(x_i) = 0 \tag{4}$$

where $f(x) = x^j$, $j = 0, 1, 2, ..., n$.

**Definition 1.** The largest positive integer $n$ for which quadrature rule has exact value for all polynomials of degree less than or equal to $n$ is known as the degree of precision.

**Definition 2.** The order of leading error terms in a method is known as the order of accuracy of a quadrature rule.

**Definition 3.** The local error term of a quadrature rule with precision $p$ is defined as

$$R[f] = C(b-a)^{p+2} f^{(p+1)}(\xi), \text{ where } \xi \in (a, b). \tag{5}$$

where $C$ is constant and $p + 2$ is the order of accuracy of a quadrature rule.

**Definition 4.** The global error term of a quadrature rule with precision $p$ is defined as

$$R[f] = D(b-a)h^{p+1} f^{(p+1)}(\mu), \text{ where } \mu \in (a, b). \tag{6}$$

where $D$ is constant and $p + 1$ is the order of accuracy of a quadrature rule.

Some well-known closed Newton-Cotes quadrature formulas for the classical Riemann integral in the basic form [XIV],[II] are:

Trapezoidal rule :

$$\int_a^b f(x)dx = T = \frac{b-a}{2}(f(a) + f(b)) - \frac{(b-a)^3}{12} f''(\xi), \tag{7}$$

where $\xi \in (a, b)$.

The Trapezoidal rule provides an exact answer for all polynomials whose degree one or less, so its precision is 1, and the order of accuracy is 3.

Simpson's 1/3 rule:

$$\int_a^b f(x)dx = S13 = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^4(\xi), \tag{8}$$

where $\xi \in (a, b)$.

The Simpson's 1/3 rule provides an exact answer for all polynomials whose degree three or less, so its precision is 3 and order of accuracy is 5.

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Simpson's 3/8 rule:

$$\int_a^b f(x)dx = S38 = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480}f^4(\xi), \qquad (9)$$

where $\xi \in (a, b)$.

The Simpson's 3/8 rule provides an exact answer for all polynomials whose degree three or less, so its precision is 3 and order of accuracy is 5.

Some well-known closed Newton-Cotes quadrature formulas for the classical Riemann integral in the composite form [XIV],[XVIII] are:

Trapezoidal rule:

$$\int_a^b f(x)dx = CT = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right] - \frac{(b-a)h^2}{12}f^{(2)}(\mu), \quad (10)$$

where $\mu \in (a, b)$.

The Trapezoidal rule provides an exact answer for all polynomials whose degree one or less, so its precision is 1, and the order of accuracy is 2.

Simpson's 1/3 rule:

$$\int_a^b f(x)dx = CS13 = \frac{h}{3}\left[\begin{array}{c} f(a) + 2\sum_{i=1}^{\left(\frac{n}{2}\right)-1} f(x_{2i}) \\ +4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + f(b) \end{array}\right] - \frac{(b-a)h^4}{180}f^{(4)}(\mu), \quad (11)$$

where $\mu \in (a, b)$.

The Simpson's 1/3 rule provides an exact answer for all polynomials whose degree three or less, so its precision is 3 and order of accuracy is 4.

Simpson's 3/8 rule:

$$\int_a^b f(x)dx = CS38 = \frac{3h}{8}\left[\begin{array}{c} f(a) + 3\sum_{i=1}^{n-2}\big(f(x_i) + f(x_{i+1})\big) \\ +2\sum_{i=3}^{n-3} f(x_i) + f(b) \end{array}\right] - \frac{(b-a)h^4}{80}f^{(4)}(\mu), \qquad (12)$$

where $\mu \in (a, b)$.

The Simpson's 3/8 rule provides an exact answer for all polynomials whose degree three or less, so its precision is 3 and order of accuracy is 4.

## III.  Proposed Simpson's 1/3 Scheme for the Riemann-Stieltjes Integral

The proposed Simpson's 1/3 scheme for the RSI in basic form is given in Theorem 1.

**Theorem 1.** Let f(t) and g(t) be continuous on [a, b] and g(t) be increasing there. The Simpson's 1/3 scheme for the RSI can be described as

$$\int_a^b f(t)dg \approx S13 = \left(\frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)f(a)$$

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$$+ \left( \frac{4}{b-a} \int_a^b g(t)dt - \frac{8}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt \right) f\left( \frac{a+b}{2} \right)$$

$$+ \left( g(b) - \frac{3}{b-a} \int_a^b g(t)dt + \frac{4}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt \right) f(b). \tag{13}$$

**Proof of Theorem 1.**

Looking for the Simpson's 1/3 scheme for the RSI, we find $a_0, b_0, c_0$ such that:

$$\int_a^b f(t)dg \approx a_0 f(a) + b_0 f\left( \frac{a+b}{2} \right) + c_0 f(b) \tag{A}$$

is exact for $f(t) = 1, t, t^2, t^3$. That is,

$$\int_a^b 1 dg = a_0 + b_0 + c_0$$

$$\int_a^b t dg = a_0 a + b_0 \left( \frac{a+b}{2} \right) + c_0 b$$

$$\int_a^b t^2 dg = a_0 a^2 + b_0 \left( \frac{a+b}{2} \right)^2 + c_0 b^2$$

$$\int_a^b t^3 dg = a_0 a^3 + b_0 \left( \frac{a+b}{2} \right)^3 + c_0 b^3$$

By using integration by parts of the RS integral, as in [XIX], we have the following system of equations (B)-(E).

$$a_0 + b_0 + c_0 = g(b) - g(a) \tag{B}$$

$$a_0 a + b_0 \left( \frac{a+b}{2} \right) + c_0 b = bg(b) - ag(a) - \int_a^b g(t)dt \tag{C}$$

$$a_0 a^2 + b_0 \left( \frac{a+b}{2} \right)^2 + c_0 b^2 = b^2 g(b) - a^2 g(a) - 2b \int_a^b g(t)dt + 2 \int_a^b \int_a^t g(x)dx \, dt \tag{D}$$

$$a_0 a^3 + b_0 \left( \frac{a+b}{2} \right)^3 + c_0 b^3 = b^3 g(b) - a^3 g(a) - 3b^2 \int_a^b g(t)dt$$

$$+ 6b \int_a^b \int_a^t g(x)dxdt - 6 \int_a^b \int_a^t \int_a^y g(x)dxdt \tag{E}$$

The coefficient matrix of the system of linear equations (B)-(E) can be written as M and $M_R$ is its reduced echelon form.

$$M = \begin{bmatrix} 1 & 1 & 1 \\ a & \frac{a+b}{2} & b \\ a^2 & \left( \frac{a+b}{2} \right)^2 & b^2 \\ a^3 & \left( \frac{a+b}{2} \right)^3 & b^3 \end{bmatrix}, \quad M \overset{R}{\approx} M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

As in $M_R$, the first three rows are nonzero rows, so $M$ has three linearly independent rows, and rank($M$) = 3. To find the coefficients $a_0$, $b_0$ and $c_0$, we solve equations (B), (C) and (D) simultaneously, to have:

$$a_0 = \frac{4}{(b-a)^2} \int_b^a \int_b^t g(x)dxdt - \frac{1}{b-a} \int_b^a g(t)dt - g(a),$$

$$b_0 = \frac{4}{b-a} \int_b^a g(t)dt - \frac{8}{(b-a)^2} \int_b^a \int_b^t g(x)dxdt,$$

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$$c_0 = g(b) - \frac{3}{b-a}\int_b^a g(t)dt + \frac{4}{(b-a)^2}\int_b^a \int_b^t g(x)dxdt.$$

Putting the values of coefficients $a_0$, $b_0$ and $c_0$ in $(A)$, we have:

$$\int_a^b f(t)dg \approx S13 = \left(\frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)f(a)$$

$$+ \left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt\right)f\left(\frac{a+b}{2}\right)$$

$$+ \left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt\right)f(b).$$

which is the same as (13), and the precision of this scheme is 3.

The local error term of proposed Simpson's 1/3 scheme using the concept of precision is derived in Theorem 2 and the reduction for the classical Riemann integral is discussed in Theorem 3.

**Theorem 2.** Let f(t) and g(t) be continuous on [a, b] and g(t) be increasing there. The Simpson's 1/3 scheme for the RSI with local error term can be described as

$$\int_a^b f(t)dg = S13 + R_{S13}[f] = \left(\begin{array}{c}\frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt \\ -\frac{1}{b-a}\int_a^b g(t)dt - g(a)\end{array}\right)f(a)$$

$$+ \left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt\right)f\left(\frac{a+b}{2}\right)$$

$$+ \left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt\right)f(b)$$

$$+ \left(\frac{-(a-b)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b \int_a^t g(x)dxdt\right.$$

$$\left. -b\int_a^b \int_a^t \int_a^y g(x)dxdy\,dt + \int_a^b \int_a^t \int_a^z \int_a^y g(x)dxdy\,dz\,dt\right)f^{(4)}(\mu)g'(\eta) \qquad (14)$$

$$where \mu, \eta \in (a, b).$$

**Proof of Theorem 2.**

The precision of Simpson's 3/8 scheme for RSI is 3. Let $f(t) = \frac{t^4}{4!}$.

So the local error term of Simpson's 1/3 scheme (13) for RSI is determined by

$$R_{S13}[f] = \frac{1}{4!}\int_a^b t^4 dg - S13 \qquad (15)$$

From [8], we know that

$$\frac{1}{4!}\int_a^b t^4 dg = \frac{1}{24}(b^4 g(b) - a^4 g(a)) - \frac{b^3}{6}\int_a^b g(t)dt + \frac{b^2}{2}\int_a^b \int_a^t g(x)dxdt$$

$$-b\int_a^b \int_a^t \int_a^y g(x)dxdydt + \int_a^b \int_a^t \int_a^z \int_a^y g(x)dxdydzdt \qquad (16)$$

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By Theorem 1 and scheme (13), we have:

$$S13(t^4; g; a, b) = \left( \frac{4}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt - \frac{1}{b-a} \int_a^b g(t)dt - g(a) \right) \frac{a^4}{4!}$$

$$+ \left( \frac{4}{b-a} \int_a^b g(t)dt - \frac{8}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt \right) \frac{(a+b)^4}{2^4 \cdot 4!}$$

$$+ \left( g(b) - \frac{3}{b-a} \int_a^b g(t)dt + \frac{4}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt \right) \frac{b^4}{4!} \qquad (17)$$

Using (16) and (17) in (15), we have:

$$R_{S13}[f] = \left( \frac{-(a-b)^2(3a+5b)}{96} \int_a^b g(t)dt + \frac{17b^2 - 10ab - 7a^2}{48} \int_a^b \int_a^t g(x)dxdt \right.$$

$$\left. -b \int_a^b \int_a^t \int_a^y g(x)dxdy\, dt + \int_a^b \int_a^t \int_a^z \int_a^y g(x)dxdy\, dz\, dt \right) f^{(4)}(\mu)g'(\eta) \qquad (18)$$

This is the required local error term of the proposed S13 scheme for RS-integral.

**Theorem 3.** *With g(t) = t, the proposed S13 scheme with the error term (14) for the RS-integral reduces to the corresponding S13 scheme (see [II]) i.e. (8) for the classical Riemann integral.*

**Proof of Theorem 3.**

By Theorem 2, we have

$$\int_a^b f(t)dg = \int_a^b f(t)dt = \left( \frac{4}{(b-a)^2} \int_a^b \int_a^t x dx dt - \frac{1}{b-a} \int_a^b t dt - g(a) \right) f(a)$$

$$+ \left( \frac{4}{b-a} \int_a^b t dt - \frac{8}{(b-a)^2} \int_a^b \int_a^t x dx dt \right) f\left( \frac{a+b}{2} \right)$$

$$+ \left( g(b) - \frac{3}{b-a} \int_a^b t dt + \frac{4}{(b-a)^2} \int_a^b \int_a^t x dx dt \right) f(b)$$

$$+ \left( \frac{-(a-b)^2(3a+5b)}{96} \int_a^b t dt + \frac{17b^2 - 10ab - 7a^2}{48} \int_a^b \int_a^t x dx dt \right.$$

$$\left. -b \int_a^b \int_a^t \int_a^y x dx dy\, dt + \int_a^b \int_a^t \int_a^z \int_a^y x dx dy\, dz\, dt \right) f^{(4)}(\mu)g'(\eta) \qquad (19)$$

It is easy to obtain:

$$\int_a^b t dt = \frac{b^2 - a^2}{2},$$

$$\int_a^b \int_a^t x dx dt = \frac{b^3}{6} - \frac{a^2 b}{2} + \frac{a^3}{3},$$

$$\int_a^b \int_a^t \int_a^y x dx dy\, dt = \frac{b^4}{24} - \frac{a^2 b^2}{4} + \frac{a^3 b}{3} - \frac{a^4}{8},$$

$$\int_a^b \int_a^t \int_a^z \int_a^y x dx dy\, dz\, dt = \frac{b^5}{120} - \frac{a^2 b^3}{12} + \frac{a^3 b^2}{6} - \frac{a^4 b}{8} + \frac{a^5}{30},$$

and, using these in (19), we finally get:

$$\int_a^b f(x)dx = \frac{b-a}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi), \qquad (20)$$

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where $\xi \in (a, b)$, which shows the reducibility of the proposed S13 scheme to the classical Riemann integral form (8).

Now, the proposed composite Simpson's 1/3 scheme for the RSI integral is derived by dividing the interval into small subintervals and applying the integration rule to each subinterval, and the results are showcased in Theorem 4.

**Theorem 4.** *Let f(t) and g(t) be continuous on [a, b] and g(t) be increasing there. Let the interval [a, b] be subdivided into 2n subintervals $[x_k, x_{k+1}]$ with width $h = \frac{b-a}{n}$ by using the equally spaced nodes $x_k = a+kh$, where $k = 0, 1, ..., n$. The composite Simpson's 1/3 scheme to 2n subintervals for the RSI can be described as*

$$\int_a^b f(t)dg \approx CS13 = \left[ \frac{4n^2}{(b-a)^2} \int_a^{x_1} \int_a^t g(x)dxdt - \frac{n}{b-a} \int_a^{x_1} g(t)dt - g(a) \right] f(a)$$

$$+ \frac{4n}{b-a} \sum_{k=1}^n \left[ \int_{x_{k-1}}^{x_k} g(t)dt - \frac{2n}{b-a} \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^t g(x)dxdt \right] f\left( \frac{x_{k-1}+x_k}{2} \right)$$

$$+ \frac{n}{b-a} \sum_{k=1}^{n-1} \begin{bmatrix} \frac{4n}{b-a} \left\{ \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^t g(x)dxdt + \int_{x_k}^{x_{k+1}} \int_{x_k}^t g(x)dxdt \right\} \\ - \left\{ 3\int_{x_{k-1}}^{x_k} g(t)dt + \int_{x_k}^{x_{k+1}} g(t)dt \right\} \end{bmatrix} f(x_k)$$

$$+ \left[ g(b) - \frac{3n}{b-a} \int_{x_{n-1}}^b g(t)dt + \frac{4n^2}{(b-a)^2} \int_{x_{n-1}}^b \int_{x_{n-1}}^t g(x)dxdt \right] f(b) \qquad (21)$$

**Proof of Theorem 4.**

By (13), the Simpson's scheme 1/3 for RSI integral is

$$\int_a^b f(t)dg \approx \left( \frac{4}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt - \frac{1}{b-a} \int_a^b g(t)dt - g(a) \right) f(a)$$

$$+ \left( \frac{4}{b-a} \int_a^b g(t)dt - \frac{8}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt \right) f\left( \frac{a+b}{2} \right)$$

$$+ \left( g(b) - \frac{3}{b-a} \int_a^b g(t)dt + \frac{4}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt \right) f(b). \qquad (22)$$

Applying rule on (22) over each subinterval, we have

$$\int_a^b f(t)dg \approx \left[ \frac{4}{\left( \frac{b-a}{n} \right)^2} \int_a^{x_1} \int_a^t g(x)dxdt - \frac{1}{\frac{b-a}{n}} \int_a^{x_1} g(t)dt - g(a) \right] f(a)$$

$$+ \left[ \frac{4}{\frac{b-a}{n}} \int_a^{x_1} g(t)dt - \frac{8}{\left( \frac{b-a}{n} \right)^2} \int_a^{x_1} \int_a^t g(x)dxdt \right] f\left( \frac{a+x_1}{2} \right)$$

$$+ \left[ g(x_1) - \frac{3}{\frac{b-a}{n}} \int_a^{x_1} g(t)dt + \frac{4}{\left( \frac{b-a}{n} \right)^2} \int_a^{x_1} \int_a^t g(x)dxdt \right] f(x_1)$$

$$+ ... + \left[ \frac{4}{\left( \frac{b-a}{n} \right)^2} \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^t g(x)dxdt - \frac{1}{\frac{b-a}{n}} \int_{x_{k-1}}^{x_k} g(t)dt - g(x_{k-1}) \right] f(x_{k-1})$$

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$$+\left[\frac{4}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k}g(t)dt-\frac{8}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\left[g(x_k)-\frac{3}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k}g(t)dt+\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt\right]f(x_k)$$

$$+\ldots+\left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt-\frac{1}{\frac{b-a}{n}}\int_{x_{n-1}}^{b}g(t)dt-g(x_{n-1})\right]f(x_{n-1})$$

$$+\left[\frac{4}{\frac{b-a}{n}}\int_{x_{n-1}}^{b}g(t)dt-\frac{8}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\right]f\left(\frac{x_{n-1}+b}{2}\right)$$

$$+\left[g(b)-\frac{3}{\frac{b-a}{n}}\int_{x_{n-1}}^{b}g(t)dt+\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\right]f(b)$$

$$=\left[\frac{4n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t}g(x)dxdt-\frac{n}{b-a}\int_{a}^{x_1}g(t)dt-g(a)\right]f(a)$$

$$+\left[\frac{4n}{b-a}\int_{a}^{x_1}g(t)dt-\frac{8n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t}g(x)dxdt\right]f\left(\frac{a+x_1}{2}\right)$$

$$+\left[\frac{4n}{b-a}\int_{x_1}^{x_2}g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_1}^{x_2}\int_{x_1}^{t}g(x)dxdt\right]f\left(\frac{x_1+x_2}{2}\right)$$

$$+\left[\frac{4n}{b-a}\int_{x_{k-1}}^{x_k}g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\left[\frac{4n}{b-a}\int_{x_{n-1}}^{b}g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\right]f\left(\frac{x_{n-1}+b}{2}\right)$$

$$+\left[\begin{array}{c}\frac{4n^2}{(b-a)^2}\left(\int_{a}^{x_1}\int_{a}^{t}g(x)dxdt+\int_{x_1}^{x_2}\int_{x_1}^{t}g(x)dxdt\right)\\-\frac{n}{b-a}\left(3\int_{a}^{x_1}g(t)dt+\int_{x_1}^{x_2}g(t)dt\right)\end{array}\right]f(x_1)$$

$$+\left[\begin{array}{c}\frac{4n^2}{(b-a)^2}\left(\int_{x_1}^{x_2}\int_{x_1}^{t}g(x)dxdt+\int_{x_2}^{x_3}\int_{x_2}^{t}g(x)dxdt\right)\\-\frac{n}{b-a}\left(3\int_{x_1}^{x_2}g(t)dt+\int_{x_2}^{x_3}g(t)dt\right)\end{array}\right]f(x_2)$$

$$+\ldots$$

$$+\left[\begin{array}{c}\frac{4n^2}{(b-a)^2}\left(\int_{x_{n-1}}^{x_n}\int_{x_{n-1}}^{t}g(x)dxdt+\int_{x_n}^{x_{n+1}}\int_{x_n}^{t}g(x)dxdt\right)\\-\frac{n}{b-a}\left(3\int_{x_{n-1}}^{x_n}g(t)dt+\int_{x_n}^{x_{n+1}}g(t)dt\right)\end{array}\right]f(x_n)$$

$$+\left[g(b)-\frac{3n}{b-a}\int_{x_{n-1}}^{b}g(t)dt+\frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\right]f(b)$$

$$=\left[\frac{4n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t}g(x)dxdt-\frac{n}{b-a}\int_{a}^{x_1}g(t)dt-g(a)\right]f(a)$$

$$+\frac{4n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}g(t)dt-\frac{2n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\begin{array}{c}\frac{4n}{b-a}\left(\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt+\int_{x_k}^{x_{k+1}}\int_{x_k}^{t}g(x)dxdt\right)\\-\left(3\int_{x_{k-1}}^{x_k}g(t)dt+\int_{x_k}^{x_{k+1}}g(t)dt\right)\end{array}\right]f(x_k)$$

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$$+ \left[ g(b) - \frac{3n}{b-a} \int_{x_{n-1}}^{b} g(t)dt + \frac{4n^2}{(b-a)^2} \int_{x_{n-1}}^{b} \int_{x_{n-1}}^{t} g(x)dxdt \right] f(b).$$

The proposed composite Simpson's 1/3 scheme for the RSI has been proved.

The global error term of the proposed S13 scheme for RSI is described in Theorem 5.

**Theorem 5.** Let f(t) and g(t) be continuous on [a, b] and g(t) be increasing there. Let the interval [a, b] be subdivided into 2n subintervals $[x_k, x_{k+1}]$ with width $h = \frac{b-a}{n}$ by using the equally spaced nodes $x_k = a+kh$, where k = 0, 1, …, n. The composite Simpson's 1/3 scheme with global error term RCS13[f] for the RSI can be described as

$$\int_a^b f(t)dg = CS13 + R_{CS13}[f]$$

$$= \left[ \frac{4n^2}{(b-a)^2} \int_a^{x_1} \int_a^t g(x)dxdt - \frac{n}{b-a} \int_a^{x_1} g(t)dt - g(a) \right] f(a)$$

$$+ \frac{4n}{b-a} \sum_{k=1}^{n} \left[ \int_{x_{k-1}}^{x_k} g(t)dt - \frac{2n}{b-a} \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{t} g(x)dxdt \right] f\left( \frac{x_{k-1}+x_k}{2} \right)$$

$$+ \frac{n}{b-a} \sum_{k=1}^{n-1} \left[ \begin{array}{c} \frac{4n}{b-a} \left\{ \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{t} g(x)dxdt + \int_{x_k}^{x_{k+1}} \int_{x_k}^{t} g(x)dxdt \right\} \\ - \left\{ 3 \int_{x_{k-1}}^{x_k} g(t)dt + \int_{x_k}^{x_{k+1}} g(t)dt \right\} \end{array} \right] f(x_k)$$

$$+ \left[ g(b) - \frac{3n}{b-a} \int_{x_{n-1}}^{b} g(t)dt + \frac{4n^2}{(b-a)^2} \int_{x_{n-1}}^{b} \int_{x_{n-1}}^{t} g(x)dxdt \right] f(b)$$

$$+ n \left[ \begin{array}{c} \frac{-(a-b)^2(3a+5b)}{96} \int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48} \int_a^b \int_b^t g(x)dxdt \\ -b \int_a^b \int_a^t \int_a^y g(x)dxdy\,dt + \int_a^b \int_a^t \int_a^z \int_a^y g(x)dxdy\,dz\,dt \end{array} \right] f^{(4)}(\mu)g'(\eta) \qquad (23)$$

$where \mu, \eta \in (a, b).$

Proof of Theorem 5.

From (19), the local error in a sub-interval, say $[x_{p-1}, x_p]$, of the S13 rule is

$$\left( \frac{-(x_{p-1}-x_p)^2(3x_{p-1}+5x_p)}{96} \int_{x_{p-1}}^{x_p} g(t)dt + \right.$$

$$\frac{17x_p^2-10x_{p-1}x_p-7x_{p-1}^2}{48} \int_{x_{p-1}}^{x_p} \int_{x_{p-1}}^{t} g(x)dxdt$$

$$\left. -x_p \int_{x_{p-1}}^{x_p} \int_{x_{p-1}}^{t} \int_{x_{p-1}}^{y} g(x)dxdy\,dt + \int_{x_{p-1}}^{x_p} \int_{x_{p-1}}^{t} \int_{x_{p-1}}^{z} \int_{x_{p-1}}^{y} g(x)dxdy\,dz\,dt \right) f^4(\mu)g'(\eta),$$

$where \mu_p, \eta \in (x_{p-1}, x_p).$

Hence, the global error is obtained by summing over *n* such terms:

$$\sum_{k=1}^{n}\begin{pmatrix}\frac{-(x_{k-1}-x_k)^2(3x_{k-1}+5x_k)}{96}\int_{x_{k-1}}^{x_k}g(t)dt \\ +\frac{17x_k{}^2-10x_{k-1}x_k-7x_{k-1}{}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt \\ -x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y}g(x)dxdy\,dt \\ +\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y}g(x)dxdy\,dz\,dt\end{pmatrix}f^{(4)}(\mu)g'(\eta),$$

$$=$$

$$n\begin{bmatrix}\frac{-(a-b)^2(3a+5b)}{96}\int_a^b g(t)dt+\frac{17b^2-10ab-7a^2}{48}\int_a^b\int_b^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdy\,dt+\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdy\,dz\,dt\end{bmatrix}\left[\frac{1}{n}\sum_{k=1}^{n}f^{(4)}(\mu_k)g'(\eta)\right].$$

Let $M=\frac{1}{n}\sum_{k=1}^{n}f^{(4)}(\mu_k)g'(\eta)$.

Clearly, $\min_{x\in[a,b]}\{f^{(4)}(x)g'(x)\}\le M\le\max_{x\in[a,b]}\{f^{(4)}(x)g'(x)\}$. Since f $^{(4)}$(t) and g'(t) are continuous in [a, b], then there exist two points $\mu$ and $\eta$ such that M $=f^{(4)}(\mu)g'(\eta)$.

This implies that the error term RS13[f] is

$$=$$

$$n\begin{bmatrix}\frac{-(a-b)^2(3a+5b)}{96}\int_a^b g(t)dt+\frac{17b^2-10ab-7a^2}{48}\int_a^b\int_b^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdy\,dt+\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdy\,dz\,dt\end{bmatrix}f^{(4)}(\mu)g'(\eta),$$

$where \mu,\eta\in(a,b)$ and h $=\frac{b-a}{n}$.

## IV. Results and Discussion

In previous studies [VIII], [IX], [XIX], focusing the quadrature schemes for RS- integral numerical experiments were not performed for confirming the theoretical accuracy. Here, in order to confirm the validity of theoretical results, the present study solved three numerical problems for each scheme taken from [VI], [I], [III], [XII] etc. All the results are computed using MATLAB in Intel (R) Core (TM) Laptop with RAM  8.00GB and processing speed of 1.00GHz-1.61GHz. For numerical results, double-precision arithmetic is used. The performance of the proposed CS13 scheme for the RSI is compared with other existing schemes: CT, ZCT and MZCT [XVIII],[XIX],[VI].

Example 1. $\int_{3.5}^{4.5}sin\,5\,xd\,cos\,x=0.227676016130689$

Example 2. $\int_{5}^{6}sin\,x\,dx^3=$ -59.655908136641912

Example 3. $\int_{5}^{6}e^x d\,sin\,x=187.4269314248657$

The absolute error is expressed in [XVI] as

$$\text{Absolute Error} = |\text{ Exact value} - \text{Approximate value }| \qquad (24)$$

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The computational order of accuracy as described in [XV] is defined as

$$p = \frac{ln\left(\frac{|N(2h)-N(0)|}{|N(h)-N(0)|}\right)}{ln\,2},\tag{25}$$

where $N(h)$ represents approximate value at step size $h$ and $N(0)$ represents the exact result. Table 1 compares all schemes of quadrature variants in terms of absolute errors versus the number of strips for all test problems. Line plots of the error of all schemes discussed above have been shown in Figs. 1-3 for all test problems.

It has been observed from the line plots in Figs. 1-3 that the error of existing schemes reduced slowly whereas the error of the proposed scheme reduced rapidly for all test problems. Tables 2, 3 and 4 confirm that the computational orders of accuracy are the same as those of theoretically proved previous section, of all schemes for all test problems. The proposed CS13 scheme exhibits an order of accuracy of 4 which is the same as the MZCT scheme but the error reduction is rapid for the former. The CT scheme exhibits an order of accuracy of 2, whereas for the ZCT scheme, due to the issues and mistakes highlighted in [VI], the order oscillates and doesn't converge to 4.

Tables 5 shows the quantity of function, derivative and integral estimations for all schemes for m strips. Figs. 4-6 show computational performance in terms of total computational cost and the average CPU usage in seconds for the three integrals mentioned in Examples 1-3 using CT, ZCT, MZCT and CS13 schemes. It is observed from numerical results that the proposed scheme took less cost to achieve the error $10^{-5}$ as compared to existing schemes for all test problems as also confirmed in Figs. 4-6 (a).

It is seen from Figs. 4-6 (b) that the proposed scheme has taken smaller average CPU time to achieve the error $10^{-5}$ as compared to existing schemes for Examples 1-3.

**Table 1:** Comparison of absolute errors by all schemes for Examples 1-3.

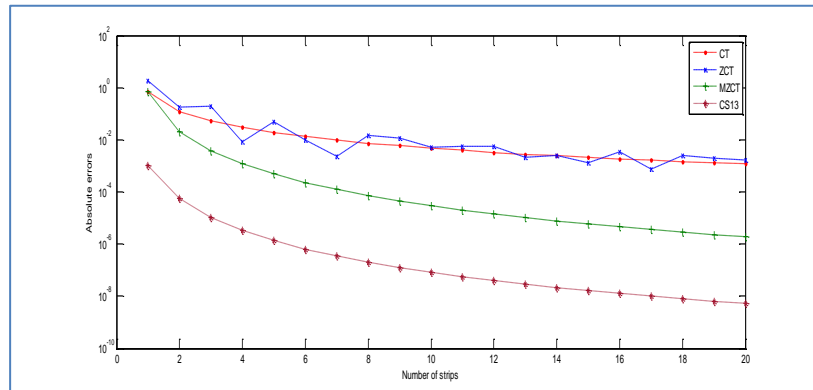| Quadrature variants | Example 1 (m=20) | Example 2 (m=100) | Example 3 (m=20) |
|---|---|---|---|
| CT | 1.1862E-03 | 4.9713E-04 | 3.9042E-02 |
| ZCT | 1.6698 E-03 | 5.5959 E-04 | 3.9042E-02 |
| MZCT | 1.8552 E-06 | 1,2428E-09 | 2.4399E-06 |
| CS13 | 5.2161E-09 | 3.2709E-11 | 1.1106E-07 |

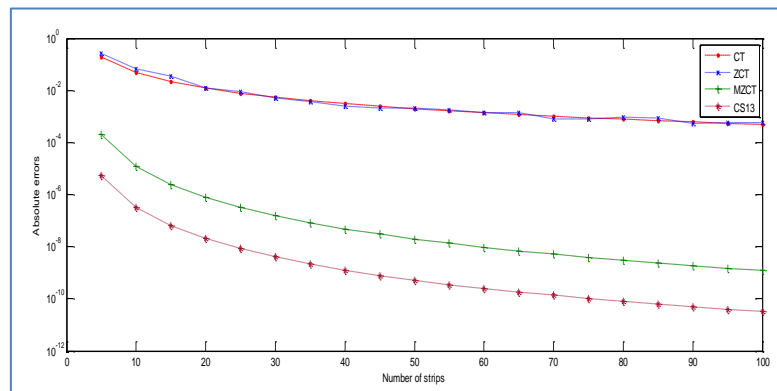Fig 1. Error Plots to approximate integral in Example 1



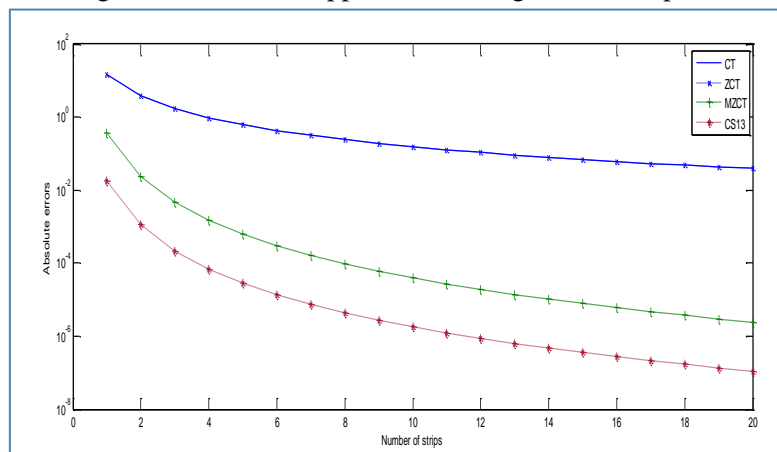Fig 2. Error Plots to approximate integral in Example 2



Fig 3. Error Plots to approximate integral in Example 3

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**Table 2:** Computational order of accuracy in all methods for Example 1

| Number of stripsm | CT | | ZCT | | MZCT | | CS13 | |
|---|---|---|---|---|---|---|---|---|
| | n | P | n | P | n | P | n | P |
| 1 | 1 | NA | 1 | NA | 1 | NA | 2 | NA |
| 2 | 2 | 2.5728 | 2 | 3.3788 | 2 | 5.0724 | 4 | 4.2504 |
| 4 | 4 | 2.0420 | 4 | 4.3106 | 4 | 4.1474 | 8 | 4.0593 |
| 8 | 8 | 2.0091 | 8 | -0.8154 | 8 | 4.0348 | 16 | 4.0147 |
| 16 | 16 | 2.0022 | 16 | 2.1713 | 16 | 4.0086 | 32 | 4.0037 |
| 32 | 32 | 2.0006 | 32 | 3.2118 | 32 | 4.0021 | 64 | 4.0009 |
| 64 | 64 | 2.0001 | 64 | 1.1185 | 64 | 4.0006 | 128 | 4.0002 |

**Table 3:** Computational order of accuracy in all methods for Example 2

| Number of stripsm | CT | | ZCT | | MZCT | | CS13 | |
|---|---|---|---|---|---|---|---|---|
| | n | P | n | P | n | P | n | P |
| 5 | 5 | NA | 5 | NA | 5 | NA | 10 | NA |
| 10 | 10 | 2.0018 | 10 | 1.9243 | 10 | 4.0025 | 20 | 4.0004 |
| 20 | 20 | 2.0004 | 20 | 2.4487 | 20 | 4.0007 | 40 | 4.0001 |
| 40 | 40 | 2.0001 | 40 | 2.3985 | 40 | 4.0001 | 80 | 4.0001 |
| 80 | 80 | 2.0001 | 80 | 2.3985 | 80 | 4.0000 | 160 | 4.0000 |
| 160 | 160 | 2.0000 | 160 | 2.4557 | 160 | 3.9998 | 320 | Exact |

**Table 4:** Computational order of accuracy in all methods for Example 3

| Number of stripsm | CT | | ZCT | | MZCT | | CS13 | |
|---|---|---|---|---|---|---|---|---|
| | n | P | n | P | n | P | n | P |
| 1 | 1 | NA | 1 | NA | 1 | NA | 2 | NA |
| 2 | 2 | 1.9319 | 2 | 1.9319 | 2 | 3.8984 | 4 | 4.0133 |
| 4 | 4 | 1.9844 | 4 | 1.9844 | 4 | 3.9761 | 8 | 4.0034 |
| 8 | 8 | 1.9962 | 8 | 1.9962 | 8 | 3.9941 | 16 | 4.0009 |
| 16 | 16 | 1.9990 | 16 | 1.9990 | 16 | 3.9985 | 32 | 4.0002 |
| 32 | 32 | 1.9998 | 32 | 1.9998 | 32 | 3.9985 | 64 | 4.0001 |

**Table 5:** Computational cost in quadrature variants for m strips.

| Quadrature Variants | Functional Evaluations | | Derivative Evaluations | | Integral evaluations of $g$ $(x)$ | | | Total |
|---|---|---|---|---|---|---|---|---|
| | $f$ | $g$ | $f^{(1)}$ | $f^{(2)}$ | Single | Double | Triple | |
| CT | m+1 | 2 | 0 | 0 | m | 0 | 0 | 2m+3 |
| ZCT | m+1 | 2 | 0 | M | m | m | m | 5m+3 |
| MZCT | m+1 | 2 | 0 | M | m | m | m | 5m+3 |
| CS13 | 2m+1 | 2 | 0 | 0 | m | m | 0 | 4m+3 |



| (a) Total computational cost | (b) Average CPU time (in seconds) |
|---|---|
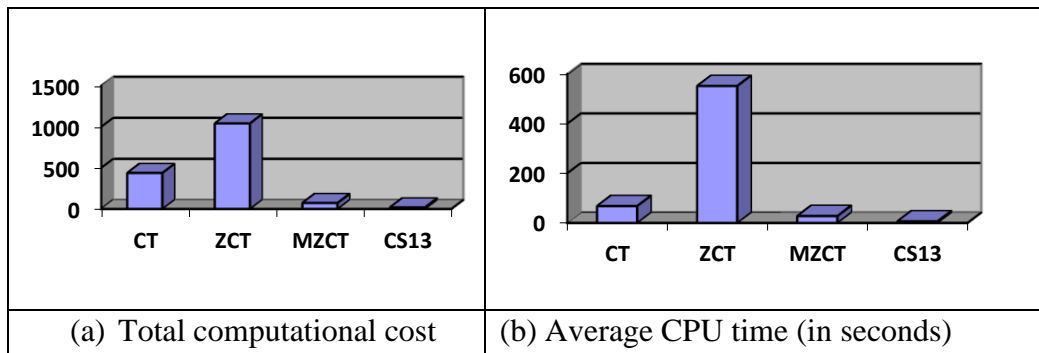
Fig 4. Performance to achieve absolute error of atmost 1E-05 for Example 1
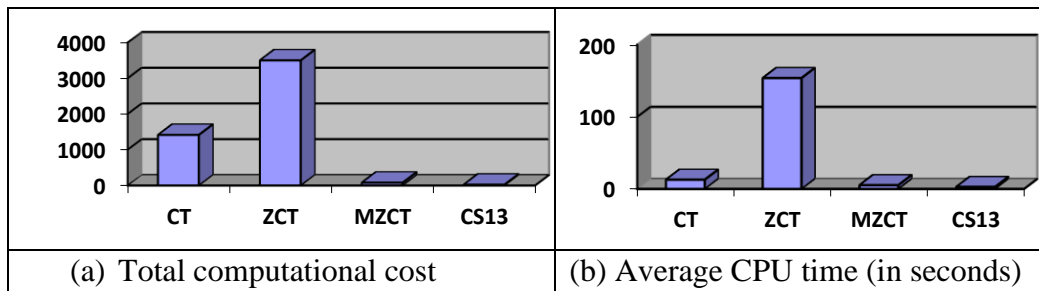


| (a) Total computational cost | (b) Average CPU time (in seconds) |
|---|---|

Fig 5. Performance to achieve an absolute error of atmost 1E-05 for Example 2



| (a) Total computational cost | (b) Average CPU time (in seconds) |
|---|---|
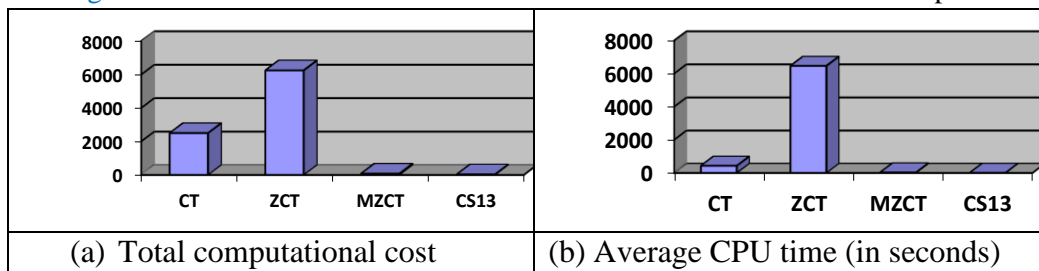
Fig 6. Performance to achieve an absolute error of atmost 1E-05 for Example 3

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## V. Conclusion

A new derivative-free Simpson's 1/3 scheme has been proposed for RSI with a local error term. The composite form of this scheme has been derived with a global error term. Three numerical problems have been tested to confirm the validity of the proposed scheme. The numerical results are discussed in terms of absolute error, the order of accuracy, computational cost and average CPU time. It is observed from numerical results that the proposed scheme is much efficient as compared to existing schemes for all test problems. In the future, the focus may be diverted to the derivative-based Simpson's 1/3 schemes for the RSI in basic and composite form with error terms.

## Conflict of Interest:

Authors declared : No conflict of interest  regarding this article

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