



## ON ORTHOGONALIZATION OF BOUBAKER POLYNOMIALS

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### Abstract

*In this work, we explore some unknown properties of the Boubaker polynomials. The orthogonalization of the Boubaker polynomials has not been discussed in the literature. Since most of the application areas of such polynomial sequences demand orthogonal polynomials, the orthogonality of the Boubaker polynomials will help extend its theareas of application. We investigate orthogonality of classical Boubaker polynomials using Sturm-Liouville form and then apply the Gram-Schmidt orthogonalization process to develop modified Boubaker polynomials which are also orthogonal. Some classical properties, like orthogonality and orthonormality relation and zeros, of the modified Boubaker polynomials, have been proved. The contributions from this study have an impact on the further application of modified Boubaker polynomials to not only the cases where classical polynomials could be used but also in cases where the classical ones could not be used due to orthogonality issue.*

**Keywords:** Orthogonalization, Boubaker polynomials, zeros, Recurrence relation, Gram-Schmidt process, Sturm-Liouville form.

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### I. Introduction

A sequence of polynomial indexed by the nonnegative integers 0, 1, 2, 3, ..., in which each index is equal to the degree of the corresponding polynomial [I]. Polynomial sequences are applied in physics, approximation theory, and ordinary differential equations. For example, Legendre, Hermite, Jacobi, Laguerre, Bessel's, and Chebyshev polynomials can be used in these applications because of orthogonality property [VI].

The most recent polynomials sequences are the Boubaker polynomials which were given by Prof Dr. Karim Boubaker in 2007 while solving heat equation. The

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Boubaker polynomials are components of a polynomial sequence [VII] which were proposed in a physics study that yielded a thermal model of the pyrolysis spray device, They represent a mathematical tool for solving heat transfer equation inside a given domain  $[0, 2]$  [IX], The Boubaker polynomials have been used to develop a new numerical method for computation of the number of complex zeros of real polynomials in the open unit disk by Shaikh and Boubaker [XVIII]. These polynomials have also been used to solve nonlinear Volterra-Fredholm integral equations and for solving inhomogeneous second order differential equations using Boubaker-Turki polynomials [XII]. The well-known BPES (Boubaker polynomial expansion scheme) and related Boubaker theorem have also been used widely for the optimization of Copper tin sulphide ternary materials precursor's ratio [XXI].

Several works in literature have already explored the application of Boubaker polynomials in many aspects. To mention a few, we briefly review here. In [VII], the solution to the heat equation in particular cases of uniform and the non-uniform fluid deposit was yielded as a guide to geometrical and temporal parameters control. On the other hand, the morphological and optical properties of as-grown copper tin sulfide layered materials have been investigated using an original analytical protocol [III]. The Boubaker polynomial scheme was also applied to determine the neutron angular flux profile in a particular nuclear system [VIII]. Some new properties of the Boubaker polynomials and applications to the approximate analytical solution of Love's integral equation were presented in [XV]. In [XII], the Chebyshev-dependent inhomogeneous second-order differential equation for the m-Boubaker polynomials was formulated. The application of Boubaker polynomials on the Lotka–Volterra equation and the stability with Routh–Hurwitz criterion were discussed in [XIII],[XVI]. Shaikh and Boubaker [XVIII] developed an efficient numerical method for computation of the number of complex zeros of real polynomials inside the open unit disk. In [XI], an asymptotic expression was presented calculating two oppositely charged discs' capacitance. Labiadh and Boubaker [XIV] presented a Sturm-Liouville shaped characteristic differential equation to the Boubaker polynomials as a supply to further efforts for proposing different analytic expression.

A dynamical nonlinear model of the spatial time-dependent evolution of A3 point during a particular sequence of resistance spot welding in steel material was treated using Boubaker polynomials [XIX]. The solution to heat equation in particular cases of uniform and the non-uniform fluid deposit was yielded as a guide to geometrical and temporal parameters control [XVII]. An analytical method was introduced for the identification of predator-prey population's time-dependent evolution in a Lotka–Volterra predator-prey model which takes into account the concept of accelerated-predator-satiety using Boubaker polynomials [XVIII]. Yücel discussed some new applications of the Boubaker polynomials expansion scheme (BPES) in the field of fluids motion and wave dynamics [XX]. A computational method for solving the optimization problem was presented using the indirect method (spectral method technique) based on Boubaker polynomials [XVII]. The numerical solution of boundary-value problems was acquired recently using an enhanced version of Boubaker polynomials [II].

The main features of the present work are to provide a means for the development of some classical properties, like the Sturm–Liouville form and orthogonality of Boubaker polynomials, as these have not been established until now, so, it is important here to investigate these properties. If these properties are proved, then these will open-up ways for new research in this field. We present characteristic differential equations and other general properties of Boubaker polynomials known till now to discuss orthogonalization and derive orthogonality relationship, orthonormality relationship, zeros of modified Boubaker polynomials.

## II. Classical Properties of Orthogonal Polynomials

The classical properties which are followed by a conventional polynomial family are: characteristics differential equation, series solution, Rodriguez's formula, the three-term recurrence relation, generating function, orthogonality relation, self-adjoint form or Sturm-Liouville form, weight function, eigenvalues and eigenfunction, norm, zeros, etc,

For example, we first discuss the classical properties of the Legendre's polynomials [I]. For the Legendre's polynomials (LPs), the second-order characteristic differential equation is:

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0 \quad (1)$$

The power series solution of (1) can be expressed as:

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \quad (2)$$

$$\text{where, } y_1(x) = 1 - \frac{n(n+1)x^2}{2!} + \frac{(n-1)n(n+1)(n+2)x^4}{4!} - \dots \quad (3)$$

$$\text{and, } y_2(x) = x - \frac{(n-1)(n+2)x^3}{2!} + \frac{(n-3)(n-1)(n+2)(n+4)x^5}{4!} - \dots \quad (4)$$

Rodriguez' formula for the LPs is:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (1 - x^2)^n, n=0,1,2,\dots \quad (5)$$

The three-term recurrence relation for LPs is given as:

$$(2n+1)xP_{n+1}(x) = (n+1)P_n(x) - nP_{n-1}(x), n = 1, 2, 3, \dots \quad (6)$$

The LPs are generated using the following generating function:

$$G(x, t) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \quad (7)$$

The orthogonality relation of LPs in  $[-1, 1]$  is expressed as:

$$\int_{-1}^1 p_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{nm} \text{ with } \delta_{nm} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \quad (8)$$

Similarly, the Laguerre polynomials [V], Chebyshev polynomials [VI], Jacobi polynomials [IV], Hermite and Bessel polynomials [X] also follow similar classical properties like LPs.

### III. Basic Properties of Original Boubaker Polynomials

A characteristic differential equation, series solution, the three-term recurrence relation, and generating function of original Boubaker polynomials (BPs) are already well-known. The characteristic differential equation of BPs is:

$$(x^2 - 1)(3nx^2 + n - 2)y'' + 3n(nx^2 + 3n - 2)y' - n(3n^2x^2 + n^2 - 6n + 8)y = 0 \quad (9)$$

The series solution of (9) can be represented as:

$$B_n(x) = \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n-4p}{n-p} \binom{n-p}{p} (-1)^p x^{n-2p} \quad (10)$$

The first ten BPs are:

$$B_0(x) = 1$$

$$B_1(x) = x$$

$$B_2(x) = x^2 + 2$$

$$B_3(x) = x^3 + x$$

$$B_4(x) = x^4 - 2$$

$$B_5(x) = x^5 - x^3 - 3x$$

$$B_6(x) = x^6 - 2x^4 - 3x^2 + 2$$

$$B_7(x) = x^7 - 3x^5 - 2x^3 + 5x$$

$$B_8(x) = x^8 - 4x^6 + 8x^2 - 2$$

$$B_9(x) = x^9 - 5x^7 + 3x^5 + 10x^3 - 7x$$

The three-term recurrence relation for original BPs can be given as:

$$B_m(x) = xB_{m-1}(x) - B_{m-2}(x) \quad \text{for } m > 2 \quad (11)$$

The generating function for BPs is:

$$G(x, t) = \frac{1+3t^2}{1+t(t-x)} \quad (12)$$

The original BPs are defined in  $[0, 2]$  but are not orthogonal.

### IV. Present Contributions and Main Results

First, we attempt to discuss orthogonalization of the original BPs using the Sturm-Liouville form, and we will show that the outcome of such an approach on original BPs is not viable. The procedure is followed from [I] as presented in the following Lemma 1.

**Lemma 1.** Consider a second-order homogeneous differential equation

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \quad (13)$$

with boundary conditions (14) and (15) as

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$$\alpha_1 y(x_0) + \alpha_2 y'(x_0) = 0 \quad (14)$$

$$\beta_1 y(x_{01}) + \beta_2 y'(x_{01}) = 0 \quad (15)$$

With the following functions, (13) can be converted to the Strum-Liouville equation

$$p(x) = \exp \left( \int_{x_0}^{x_1} \frac{b(s)}{a(s)} ds \right) \quad (16)$$

$$r(x) = \frac{1}{a(s)} \exp \left( \int_{x_0}^{x_1} \frac{b(s)}{a(s)} ds \right) \quad (17)$$

$$q(x) = \frac{c(x)}{a(s)} \exp \left( \int_{x_0}^{x_1} \frac{b(s)}{a(s)} ds \right) \quad (18)$$

With these definitions above differential equation can be transformed to the known Strum- Liouville equation

$$\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + (q(x) + \lambda r(x))y(x) = 0 \quad (19)$$

$$\frac{1}{r(x)} \left( \frac{\partial}{\partial x} (p(y) \frac{\partial y}{\partial x}) + q(x) \right) y(x) = -\lambda y(x) \quad (20)$$

$$\text{Or, } [r(x)y']' + [q(x) + \lambda p(x)]y = 0 \quad (21)$$

(20) and (21) are both Strum-Liouville forms of (13). The function  $p$ ,  $q$  and  $r$  must be independent of  $n$ .

Now, we apply Lemma 1 on the characteristic differential equation of original BPs to attempt to derive Strum-Liouville form and hence discuss the orthogonality. The BPs characteristic differential equation is:

$$(x^2-1)(3nx^2+n-2)y'' + 3x(nx^2+3n-2)y' - n(3n^2x^2+n^2-6x+8)y = 0 \quad (22)$$

Using coefficients of (22) we have  $q(x) = 0$ , and

$$p(x) = e^{\int \frac{3x(nx^2+3n-2)}{(x^2-1)(3nx^2+n-2)} dx} \quad (23)$$

$$r(x) = \frac{1}{(x^2-1)(3nx^2+n-2)} e^{\int_0^2 \frac{3x(nx^2+3n-2)}{(x^2-1)(3nx^2+n-2)} dx} \quad (24)$$

Solving (23) by integration partial fractions, we have:

$$p(x) = \exp \left[ \frac{1}{3n} \int_0^2 \left[ \frac{9n}{2(x-1)} + \frac{9n(2n-1)}{2(n+1)(x+1)} + \frac{(3n-6n^2)x}{(x^2+\frac{n-2}{3n})} + \frac{9n(2n+1)}{(n+1)(x^2+\frac{n-2}{3n})} \right] dx \right] \quad (25)$$

Solving integration in (25), we have:

$$p(x) = \frac{3}{2} \ln(x-1) + \frac{3(2n-1)}{2(n+1)} \ln(x+1) + \frac{(1-2n)}{2} \ln \left( x^2 + \frac{n-2}{3n} \right) + \frac{9n(2n-1)}{(n+1)(n-1)} \tan^{-1} \left\{ \frac{3n(n-2)(x+2)}{(n-2)^2 - 18n^2x} \right\} \quad (26)$$

Putting these in Strum- Liouville form, we have

$$\left[ \frac{y'}{(x^2-1)(3nx^2+n-2)} \left\{ \frac{3}{2} \ln(x-1) + \frac{3(2n-1)}{2(n+1)} \ln(x+1) + \frac{(1-2n)}{2} \ln \left( x^2 + \frac{n-2}{3n} \right) \right\} + \frac{9n(2n-1)}{(n+1)(n-1)} \tan^{-1} \left( \frac{3n(n-2)(x+2)}{(n-2)^2-18n^2x} \right) \right] + \lambda \left[ \left\{ \frac{3}{2} \ln(x-1) + \frac{3(2n-1)}{2(n+1)} \ln(x+1) + \frac{(1-2n)}{2} \ln \left( x^2 + \frac{n-2}{3n} \right) \right\} + \frac{9n(2n-1)}{(n+1)(n-1)} \tan^{-1} \left( \frac{3n(n-2)(x+2)}{(n-2)^2-18n^2x} \right) \right] y = 0 \quad (27)$$

We observe that with every coefficient, there exists “n” which does not allow making the Sturm-Liouville form of the original Boubaker differential equation. Therefore, orthogonality cannot be proved for original Boubaker polynomials.

It is necessary to modify original Boubaker polynomials, so that orthogonality, Sturm-Liouville and other classical properties can be established.

Another famous method to prove orthogonalization of Boubaker polynomials that is the Gram–Schmidt process is used ahead [I],[II], as quoted in Lemma 2.

**Lemma 2.** Let  $V$  (,) be the inner product vector space of polynomials and  $\{u_1, u_2, u_3, \dots, u_n\}$  be the arbitrary basis of  $V$  then an orthogonal basis of  $V$  is given by the vectors  $\{v_1, v_2, v_3, \dots, v_n\}$  as

$$\begin{aligned} v_1(x) &= u_1(x) \\ v_2 &= u_2 - \frac{\langle u_1, v_1 \rangle v_1}{\|v_1\|^2} \\ v_3 &= u_3 - \frac{\langle u_3, v_1 \rangle v_1}{\|v_1\|^2} - \frac{\langle u_3, v_2 \rangle v_2}{\|v_2\|^2} \\ v_n &= u_n - \sum_{i=1}^{n-1} \frac{\langle u_n, v_i \rangle v_i}{\|v_i\|^2} \end{aligned} \quad (28)$$

where inner product from and norm are defined as:

$$\langle f(x), g(x) \rangle = \int_0^2 f(x)g(x)dx \quad (29)$$

$$\text{And, } \|f(x)g(x)\|^2 = \int_0^2 (f(x)g(x))^2 dx \quad (30)$$

We now apply the process of Lemma 2 to derive modified orthogonal Boubaker polynomials.

Let  $B_0(x) = 1$ ,  $B_1(x) = x$  and  $B_2(x) = x^2+2$ ,  $B_3(x) = x^3+x$ ,  $B_4(x) = x^4-4$ ,  $\dots B_n(x)$  are original BPs, and  $\mathbf{X} = \{1, x, x^2+2, x^3+x, x^4-2, \dots B_n(x)\}$  be finite arbitrary Basis. Let  $\mathbf{Y} = \{P_0(x), P_1(x), P_2(x), \dots P_n(x)\}$  be the required orthogonal basis for the modified orthogonal Boubaker polynomials (MOBPs).

The inner product space used here is defined as:

$$\langle f(x), g(x) \rangle = \int_0^2 w(x)f(x)g(x)dx \quad (31)$$

Here, we consider  $w(x) = 1$  as the weight function of MOBPs and the norm space as defined as:

$$\|f(x)g(x)\|^2 = \int_0^2 (f(x)g(x))^2 dx \quad (32)$$

Now,

$$p_0(x) = 1, \text{ and}$$

$$p_1(x) = B_1(x) - \frac{\langle B_1(x), P_0(x) \rangle}{\langle P_0(x), P_0(x) \rangle} P_0(x)$$

$$p_1(x) = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} = x - 1$$

$$\begin{aligned} p_2(x) &= B_2(x) - \frac{\langle B_2(x), P_0(x) \rangle}{\langle P_0(x), P_0(x) \rangle} P_0(x) - \frac{\langle B_2(x), P_1(x) \rangle}{\langle P_1(x), P_1(x) \rangle} P_1(x) \\ &= x^2 + 2 - \frac{\langle x^2+2, 1 \rangle}{\langle 1, 1 \rangle} - \frac{\langle x^2+2, x-1 \rangle}{\langle x-1, x-1 \rangle} (x-1) = x^2 - 2x + \frac{2}{3} \end{aligned}$$

Similarly

$$p_0(x) = 1$$

$$p_1(x) = (x-1)$$

$$p_2(x) = (x^2 - 2x + \frac{2}{3})$$

$$p_3(x) = (x^3 - 3x^2 + \frac{12}{5}x - \frac{2}{5})$$

$$p_4(x) = (x^4 - 4x^3 + \frac{36}{7}x^2 - \frac{16}{7}x + \frac{8}{35})$$

$$p_5(x) = (x^5 - 5x^4 + \frac{80}{9}x^3 - \frac{20}{3}x^2 + \frac{40}{21}x - \frac{8}{63})$$

$$p_6(x) = (x^6 - 6x^5 + \frac{150}{11}x^4 - \frac{160}{11}x^3 + \frac{80}{11}x^2 - \frac{16}{11}x + \frac{16}{231})$$

$$p_7(x) = (x^7 - 7x^6 + \frac{252}{13}x^5 - \frac{350}{13}x^4 + \frac{2800}{143}x^3 - \frac{1008}{143}x^2 + \frac{448}{429}x - \frac{16}{429})$$

$$p_8(x) = (x^8 - 8x^7 + \frac{392}{15}x^6 - \frac{224}{5}x^5 + \frac{560}{13}x^4 - \frac{896}{39}x^3 + \frac{896}{143}x^2 - \frac{512}{715}x + \frac{128}{6435})$$

$$\begin{aligned} p_9(x) &= (x^9 - 9x^8 + \frac{7616}{225}x^7 - \frac{15512}{225}x^6 + \frac{26768}{325}x^5 - \frac{6832}{117}x^4 + \frac{30464}{1287}x^3 - \\ &\frac{17792}{3575}x^2 + \frac{42368}{96525}x - \frac{896}{96525}) \end{aligned}$$

$$P_n(x) = B_n(x) - \sum_{i=0}^{n-1} \frac{\langle B_n(x), P_i(x) \rangle}{\langle P_i(x), P_i(x) \rangle} P_i(x), \quad (33)$$

$$\text{Where } B_n(x) = \sum_{p=0}^n \frac{n!}{p!} \frac{n-4p}{n-p} \binom{n-p}{p} (-1)^p x^{(n-2p)}.$$

Hence, the orthogonal basis for the MOBPs is:

$$Y = \{P_0(x), P_1(x), P_2(x), \dots, P_n(x)\}$$

$$Y = \{1, x-1, \frac{1}{3}(3x^2-6x+2), \frac{1}{35}(35x^4-140x^3+180x^2-80x+8),$$

$$\frac{1}{126}(21x^6-126x^5-35720x^4+143720x^3-172680x^2+57584x)\dots P_n(x)\} \quad (34)$$

Figs. 1 and 2, respectively, show the proposed orthogonal Boubaker polynomials  $P_0$ - $P_4$  and  $P_5$ - $P_9$ , whereas Fig. 3 shows the first ten MOBPs together. It is evident from Fig. 1-3 that all zeros of the proposed MOBPs are real and lie in  $[0, 2]$ , thus MOBPs is a new family of orthogonal polynomials defined in  $[0, 2]$  with all real zeros contained in the same interval. Table 1 lists all the zeros of the first ten proposed MOBPs in decimal form, whereas in Table 2 zeros of polynomials up to  $n = 5$  are listed in closed form.

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The proposed modified orthogonal Boubaker polynomials (MOBPs) as defined in (33)-(34) are orthogonal in the interval  $[0, 2]$ , and the corresponding orthogonality relation is given as:

$$\int_0^2 [P_m(x)P_n(x)] dx = 0 \text{ if } m \neq n \quad (35)$$

For instant verification, one can see that for  $p_0$  and  $p_1$ , and for  $p_0$  and  $p_2$ , the inner products are zero.

$$\langle P_0(x), p_1(x) \rangle = \int_0^2 1(x-1) dx = 0$$

$$\text{Now } \langle P_0(x), P_2(x) \rangle = \int_0^2 1\left\{\frac{1}{3}(3x^2 - 6x + 2)\right\}dx$$

The orthonormal set of polynomials corresponding to the orthogonal set (34) can also be defined for the MOBPs as

Let  $\mathbf{Z} = \{e_0, e_1, e_2, e_3, \dots, e_{n-1}\}$  be orthonormal set

From (33), we have

$$\|P_0(x)\|^2 = \int_0^2 (P_0(x))^2 dx = \int_0^2 (1)^2 dx = 2$$

$$\|P_0(x)\| = \sqrt{2}.$$

Now, since

$$e_{n-1} = \frac{P_{n-1}(x)}{\|P_{n-1}(x)\|}, \quad n = 1, 2, 3, 4 \quad (36)$$

So, we have:

$$e_0 = \frac{1}{\sqrt{2}}$$

$$e_1 = \sqrt{\frac{2}{3}}(x-1)$$

$$e_2 = \sqrt{\frac{8}{45}} \left(x^2 - 2x + \frac{2}{3}\right)$$

$$e_3 = \sqrt{\frac{8}{175}} \left(x^3 - 3x^2 + \frac{12}{5}x - \frac{2}{5}\right), \dots$$

Hence  $\mathbf{Z} = \{e_0, e_1, e_2, e_3, \dots, e_{n-1}\}$  is an orthonormal set of  $\mathbf{Y}$ .

This is the fundamental set of results regarding the orthogonality and orthonormality of the Boubaker polynomials in  $[0, 2]$  as presented in this work in comparison to the existing literature on this subject. The MOBPs may be used in future applications in the fields of boundary-value problems, numerical integration and integral equations.

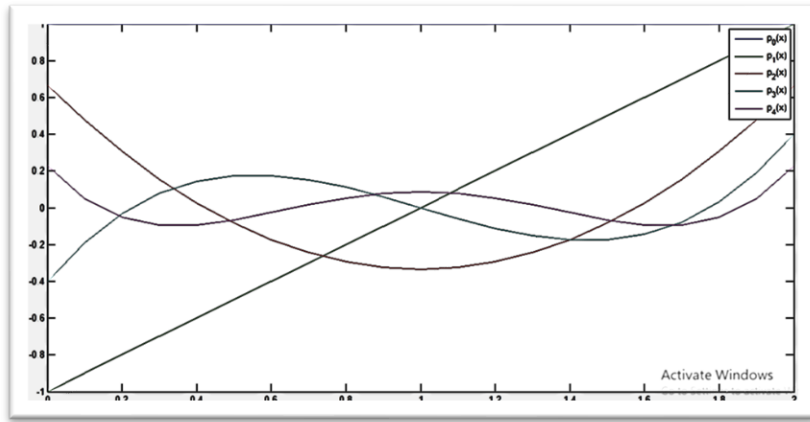


Figure 1: Proposed modified orthogonal BPs from  $P_0(x)$  to  $P_4(x)$

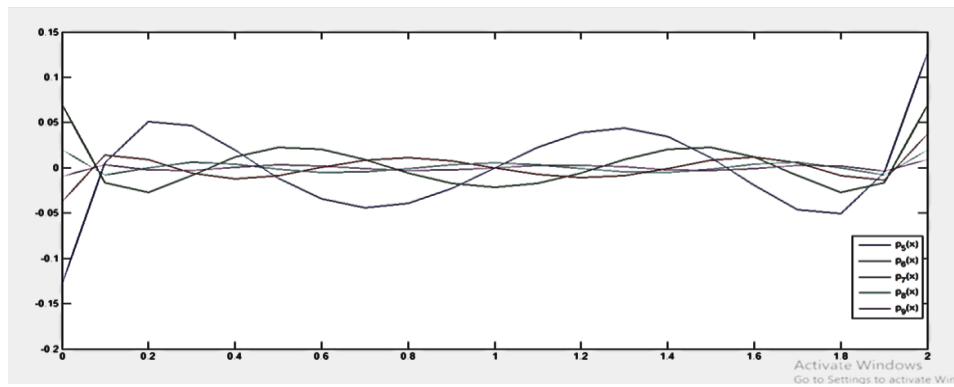


Figure 2: Proposed modified orthogonal BPs from  $P_5(x)$  to  $P_9(x)$

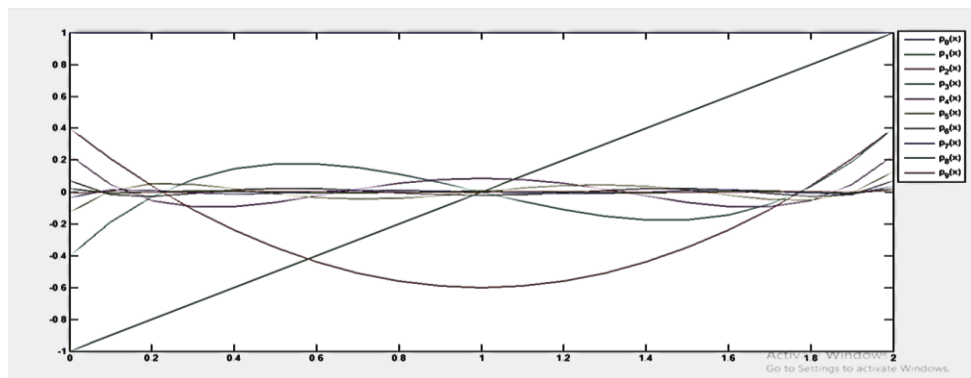


Figure 3: Proposed modified orthogonal BPs from  $P_0(x)$  to  $P_9(x)$

**Table 1:** Zeros or roots of orthogonality polynomials

Serial number	Value of n	$P_N(x)$	Zeros
01	0	$P_0(x)$	NA
02	1	$P_1(x)$	1
03	2	$P_2(x)$	1.577350269189626 0.422649730810374
04	3	$P_3(x)$	1.774596669241485 1.000000000000000 0.225403330758517
05	4	$P_4(x)$	1.861136311594047 1.339981043584861 0.660018956415143 0.138863688405947
06	5	$P_5(x)$	1.932469514203213 1.661209386466132 1.238619186083275 0.761380813916789 0.338790613533737 0.067530485796848
07	6	$P_6(x)$	1.932469514203213 1.661209386466132 1.238619186083275 0.761380813916789 0.338790613533737 0.067530485796848
08	7	$P_7(x)$	1.949107912343022 1.741531185598892 1.405845151377761 0.999999999999842 0.594154848622642 0.258468814400603 0.050892087657241
09	8	$P_8(x)$	1.960289856497143 1.796666477414708 1.525532409915025 1.183434642496488 0.816565357504091 0.474467590083699 0.203333522586372 0.039710143502464
10	9	$P_9(x)$	1.970000361771120 1.843285948669672 1.623725031434496 1.331732276411801 1.000000000000182 0.668267723586097 0.376274968571590 0.156714051322658 0.029999638232384

**Table 2:** First five polynomials Zeros in square root form

Serial number	Value of n	$P_N(x)$	Zeros
01	0	$P_0(x)$	
02	1	$P_1(x)$	1
03	2	$P_2(x)$	$\frac{1}{3}(3 \pm \sqrt{15})$
04	3	$P_3(x)$	$1, \frac{1}{5}(5 \pm \sqrt{15})$
05	4	$P_4(x)$	$1 \pm \sqrt{\frac{1}{35}(15 - 2\sqrt{30})},$  $\frac{1}{2}[2 \pm \sqrt{4 + \frac{8}{35}(\sqrt{30} - 10)}]$
06	5	$P_5(x)$	$1, \frac{1}{2}[2 \pm \sqrt{4 + \frac{8}{63}(\sqrt{70} - 14)}],$  $\frac{1}{3}[3 \pm \sqrt{\frac{1}{7}(35 - 2\sqrt{70})}]$

## V. Conclusion

The conventional Boubaker polynomials were considered in this study, and an attempt was made to discuss the orthogonalization of original BPs so that these can also be used in the future for applications limited to orthogonal polynomial families only. We outlined issues related to the Sturm-Liouville form of the original BPs and limitations of the original BPs for orthogonalization. Using the Gram-Schmidt orthogonalization process, we proposed a new family of modified orthogonal Boubaker polynomials (MOBPs) in the interval  $[0, 2]$ . The zeros of MOBPs, the relations of orthogonality were discussed, and an orthonormal basis was also formulated for the proposed MOBPs. The contributions of this research will help researchers apply the new polynomials without any limitation of the orthogonality.

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## Conflict of Interest:

No conflict of interest regarding this article

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