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JORDAN RIGHT DERIVATIONS ON SEMIPRIME Γ-RING

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Abstract

In this paper, we have analyzed the basic properties and related theorems of Jordan's right derivations on semiprime Γ-rings with their mathematical simulation. We mainly focused on the characterizations of 2 and 3-torsion-free semiprime Γ-ring by using Jordan Right Derivations. Important lemmas and theorems related to Jordan derivation on semiprime Γ-ring have been derived here with sufficient calculations. Our main objective is to prove that if S is a 2,3-torsion free semiprime Γ-ring and d, M be the Jordan right derivations on S provided that $d^2(S) = M(S)$ then $d(S) = 0$.

Keywords: Γ-Ring, Semiprime Γ-Ring, Derivation, Jordan Right Derivation.

I. Introduction

The concept of Γ-ring was established as a broad generalization of the classical ring concept. From its earlier introduction, the enhancement and generalization of different important issues of the classical ring theory of algebra have attracted wide attention to modern algebra as an emerging sector of research in algebra. Many prominent mathematicians around the world have worked with this fascinating field of research sector for determining many basic features of Γ-rings, and over the past few decades have produced more productive and creative results for Γ-rings.

N. Nobusawa [VII] stated Γ-ring as a generalization of the ternary ring. After generalizing the perception of N. Nobusawa's Γ-ring, Barnes [XIII] gave a precise definition of Γ-ring. Barnes, Luh, and Kyuno analyzed the structure of Γ-rings and found similar generalizations to parts related to ring theory. Since then, some papers have been published on the topic of Γ-rings. M. Soyturk [X] analyzed the commutativity of prime Γ-rings for both left and right derivations as well as retrieved some vital results on the commutativity of 2,3-torsion-free prime Γ-rings. Y. Ceven [XII] thoroughly worked on the presence of nonzero Jordan left derivation over completely prime Γ-ring and found that a Γ-ring M will be commutative if the condition $a\alpha b\beta c = a\beta b\alpha c$, is satisfied where $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. He also proved by using the same assumption that each Jordan left derivation over a Γ- ring which is completely prime is a left derivation on it. Mustafa Asci and Sahin Ceran [VI] investigated on nonzero left derivation named d on a prime Γ-ring M and stated that M is commutative if the condition $d(U\ ) \subseteq U$ and $d^2(U\ ) \subseteq Z$ is satisfied, where U and Z are respectively the ideal and center of M. Sapanci and A. Nakajima [XI] worked on derivation and Jordan derivation of Γ-rings and proved that each Jordan derivation which is from a specific type of Γ-ring which is completely prime will must be a derivation. They provided some specific examples of derivations and Jordan derivations of Γ-rings. In recent times, Md. Mizanor Rahman and A.C. Paul [I] worked on Jordan's left derivations of 2 and 3-torsion-free semiprime Γ-ring.

II. Preliminaries

In this section, some definitions have been discussed which are important for representing our main objective in the later sections.

II.i. Γ-ring

Let S and Γ be two abelian groups. Then if we have a mapping $S \times \Gamma \times S \to S$ such that the conditions

1. $a\alpha b \in S$
2. $(a + b)\alpha c = a\alpha c + b\alpha c, a(\alpha + \beta)b = a\alpha b + a\beta b,\ a\alpha(b + c) = a\alpha b + a\alpha c$
3. $(a\alpha b)\beta c = a\alpha(b\beta c)$

are satisfied for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$, then S is called a Γ-ring.

Example: Let R be a ring that is of characteristic 2 and has the unity element 1. If $S = S_{\ 1,2}(R)$ and $\Gamma = \left\{\binom{m.1}{0} : m \in Z\right\}$, then S is a Γ-ring. Again, if we assume $M = \{(x, x) : x \in R\} \subseteq S$, then M is a Γ-ring of S.

II.ii. Prime Γ-ring

If S is a Γ-ring, then S is called a prime Γ-ring if $a\Gamma S\Gamma b = 0\ (a, b \in S)$ where $a = 0$ or $b = 0$. Similarly, S is called prime if the zero ideal is also prime.

II.iii. Semiprime Γ-ring

If S is a Γ-ring, then S is said to be a prime Γ-ring if $a\Gamma S\Gamma a = 0\ \ (x \in S)$ where $a = 0$.

II.iv. Commutative Γ-ring

A Γ-ring S is called a commutative Γ-ring if it satisfies the condition $a\alpha b = b\alpha a$ for all $a, b \in S$ and $\alpha \in \Gamma$. Again, $[a, b]_\alpha = a\alpha b - b\alpha a$ is called a commutator of a and b concerning α, where $a, b \in S$ and $\alpha \in \Gamma$.

II.v. Nilpotent Ideal

If A is an ideal of a Γ-ring M, then A is called nilpotent if $(A\Gamma)^n A = (A\Gamma A\Gamma \ldots\ldots\ldots \Gamma A\Gamma)A = 0$, where n is defined as the least positive integer.

II.vi. n-torsion free or characteristic not equal to n

If S is a Γ-ring, then S is called n- torsion-free or characteristic not equal to n which is denoted as char. $S \neq m$, if $ma = 0$ implies $a = 0$ for all $a \in S$. S is called 2-torsion free when $2a = 0$ implies $a = 0$ for all $a \in S$.

II.vii. Derivation

If S is a Γ-ring and $d: M \to M$ is an additive mapping. Then d is said to be a derivation if

$$d(a\alpha b) = d(a)\alpha b + a\alpha d(b) \qquad where\ \ a, b \in S, \alpha \in \Gamma.$$

Example: Let the mapping $d: S \to S$ is defined by $d(A) = d\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ then d is a derivation on the Γ-ring S.

II.viii. Right Derivation

If S is a Γ-ring and $d: S \to S$ is defined as an additive mapping. Then d is said to be a right derivation if

$$d(a\alpha b) = d(a)\alpha b + d(b)\alpha a \qquad where\ \ a, b \in S, \alpha \in \Gamma$$

II.ix. Jordan Derivation

For a Γ-ring S and an additive mapping $d: S \to S$, d is said to be a Jordan derivation if

$$d(a\alpha a) = d(a)\alpha a + a\alpha d(a) \quad where\ \ a \in S, \alpha \in \Gamma.$$

Example: Let the mapping $D: S \to S$ is defined by $D\big((a, a)\big) = \big(d(a), d(a)\big)$ then D is a Jordan derivation on the Γ-ring S.

II.x. Jordan Right Derivation

For a Γ-ring S and an additive mapping $d: S \to S$, d is said to be a Jordan right derivation if

$$d(a\alpha a) = 2d(a)\alpha a \quad where\ \ a \in S, \alpha \in \Gamma.$$

III. Propositions and Theorems

III.i. Proposition

If S is a Γ-ring of 2-torsion free and d is a Jordan right derivation over S. Then for all $a, b, c \in S$ and $\alpha \in \Gamma$

(i) $d(a\alpha b + b\alpha a) = 2d(a)\alpha b + 2d(b)\alpha a$

(ii) $d(a\alpha b\alpha a) = d(b)\alpha a\alpha a + 3d(a)\alpha b\alpha a - d(a)\alpha a\alpha b$

(iii) $d(a\alpha b\alpha c + c\alpha b\alpha a) = d(b)\alpha a\alpha c + d(b)\alpha c\alpha a + 3d(a)\alpha b\alpha c + 3d(c)\alpha b\alpha a - d(a)\alpha c\alpha b - d(c)\alpha a\alpha b$

Proof: (**i**) $d\big((a+b)\alpha(a+b)\big) = d(a+b)\alpha(a+b) + d(a+b)\alpha(a+b)$

$\Rightarrow d\big((a+b)\alpha a + (a+b)\alpha b\big) = 2d(a+b)\alpha(a+b)$

$\Rightarrow d(a\alpha a + b\alpha a + a\alpha b + b\alpha b) = 2d(a+b)\alpha a + 2d(a+b)\alpha b$

$\Rightarrow d(a\alpha a) + d(b\alpha b) + d(a\alpha b + b\alpha a)$
$= 2d(a)\alpha a + 2d(b)\alpha a + 2d(a)\alpha b + 2d(b)\alpha b$

$\Rightarrow 2d(a)\alpha a + 2d(b)\alpha b + d(a\alpha b + b\alpha a)$
$= 2d(a)\alpha a + 2d(b)\alpha a + 2d(a)\alpha b + 2d(b)\alpha b$

$\therefore d(a\alpha b + b\alpha a) = 2d(a)\alpha b + 2d(b)\alpha a$

(**ii**) Replacing $a\alpha b + b\alpha a$ for b in (i), we obtain

$d(a\alpha(a\alpha b + b\alpha a) + (a\alpha b + b\alpha a)\alpha a)$
$= 2d(a)\alpha(a\alpha b + b\alpha a) + 2d(a\alpha b + b\alpha a)\alpha a$

$\Rightarrow d(a\alpha a\alpha b + a\alpha b\alpha a) + d(a\alpha b\alpha a + b\alpha a\alpha a)$
$= 2d(a)\alpha(a\alpha b) + 2d(a)\alpha(b\alpha a) + 4d(a)\alpha b\alpha a + 4d(b)\alpha a\alpha a$

$\Rightarrow d(a\alpha a\alpha b + b\alpha a\alpha a) + 2d(a\alpha b\alpha a) = 4d(b)\alpha a\alpha a + 6d(a)\alpha b\alpha a + 2d(a)\alpha a\alpha b$

$\Rightarrow d(a\alpha a)\alpha b + 2d(b)\alpha a\alpha a + 2d(a\alpha b\alpha a)$
$= 4d(b)\alpha a\alpha a + 6d(a)\alpha b\alpha a + 2d(a)\alpha a\alpha b$

$\Rightarrow 2d(a\alpha b\alpha a) = 2d(b)\alpha a\alpha a + 6d(a)\alpha b\alpha a + 2d(a)\alpha a\alpha b - 2d(a\alpha a)\alpha b$

$\Rightarrow 2d(a\alpha b\alpha a) = 2d(b)\alpha a\alpha a + 6d(a)\alpha b\alpha a + 2d(a)\alpha a\alpha b - 4d(a)\alpha a\alpha b$

$\Rightarrow 2d(a\alpha b\alpha a) = 2d(b)\alpha a\alpha a + 6d(a)\alpha b\alpha a - 2d(a)\alpha a\alpha b$

$\therefore d(a\alpha b\alpha a) = d(b)\alpha a\alpha a + 3d(a)\alpha b\alpha a - d(a)\alpha a\alpha b$

(**iii**) Replacing $(a+c)$ in the place of a in (ii), we obtain

$d((a+c)\alpha b\alpha(a+c))$
$= d(b)\alpha(a+c)\alpha(a+c) + 3d(a+c)\alpha b\alpha(a+c) - d(a+c)\alpha(a$
$+ c)\alpha b$

$\Rightarrow d(a\alpha b\alpha a) + d(a\alpha b\alpha c) + d(c\alpha b\alpha a) + d(c\alpha b\alpha c)$
$= d(b)\alpha(a+c)\alpha a + d(b)\alpha(a+c)\alpha c + 3d(a+c)\alpha(b\alpha a + b\alpha c)$
$- d(a+c)\alpha(a\alpha b + c\alpha b)$

$\Rightarrow d(a\alpha b\alpha a) + d(c\alpha b\alpha c) + d(a\alpha b\alpha c + c\alpha b\alpha a)$
$= d(b)\alpha a\alpha a + d(b)\alpha c\alpha c + d(b)\alpha(a\alpha c + c\alpha a) + 3d(a)\alpha b\alpha a$
$+ 3d(a)\alpha b\alpha c + 3d(c)\alpha b\alpha a + 3d(c)\alpha b\alpha c - d(a)\alpha a\alpha b$
$- d(a)\alpha c\alpha b - d(c)\alpha a\alpha b - d(c)\alpha c\alpha b$

$\Rightarrow d(a\alpha b\alpha a) + d(c\alpha b\alpha c) + d(a\alpha b\alpha c + c\alpha b\alpha a)$
$= d(b)\alpha a\alpha a + 3d(a)\alpha b\alpha a - d(a)\alpha a\alpha b + d(b)\alpha c\alpha c$
$+ 3d(c)\alpha b\alpha c - d(c)\alpha c\alpha b + d(b)\alpha a\alpha c + d(b)\alpha c\alpha a + 3d(a)\alpha b\alpha c$
$+ 3d(c)\alpha b\alpha a - d(a)\alpha c\alpha b - d(c)\alpha a\alpha b$

$$\Rightarrow d(a\alpha b\alpha a) + d(c\alpha b\alpha c) + d(a\alpha b\alpha c + c\alpha b\alpha a)$$
$$= d(a\alpha b\alpha a) + d(c\alpha b\alpha c) + d(b)\alpha a\alpha c + d(b)\alpha c\alpha a + 3d(a)\alpha b\alpha c + 3d(c)\alpha b\alpha a - d(a)\alpha c\alpha b - d(c)\alpha a\alpha b \quad [By\ (ii)]$$

$$\therefore d(a\alpha b\alpha c + c\alpha b\alpha a) = d(b)\alpha a\alpha c + d(b)\alpha c\alpha a + 3d(a)\alpha b\alpha c + 3d(c)\alpha b\alpha a - d(a)\alpha c\alpha b - d(c)\alpha a\alpha b$$

III.ii. Proposition

Let S be a Γ-ring of 2,3- torsion-free and $d: S \to S$ be a Jordan right derivation. If $d([a,[a,d(a)]_\alpha]_\alpha) = 0$ satisfies for all $a \in S$ and $\alpha \in \Gamma$, then $d(a)\alpha[a,d(a)]_\alpha = 0$ is satisfied for all $a \in S$ and $\alpha \in \Gamma$.

Proof: $d([a,[a,d(a)]_\alpha]_\alpha) = 0$

$$\Rightarrow d\left(\left[a,[a\alpha d(a) - d(a)\alpha a]\right]_\alpha\right) = 0$$

$$\Rightarrow d(a\alpha a\alpha d(a) - a\alpha d(a)\alpha a - a\alpha d(a)\alpha a + d(a)\alpha a\alpha a) = 0$$

$$\Rightarrow d(a\alpha a\alpha d(a) + d(a)\alpha a\alpha a) - 2d(a\alpha d(a)\alpha a) = 0$$

By using III.i(ii), (iii) we get

$$d(a)\alpha a\alpha d(a) + d(a)\alpha d(a)\alpha a + 3d(a)\alpha a\alpha d(a) + 3d^2(a)\alpha a\alpha a - d(a)\alpha d(a)\alpha a - d^2(a)\alpha a\alpha a - 2\,[d^2(a)\alpha a\alpha a + 3d(a)\alpha d(a)\alpha a - d(a)\alpha a\alpha d(a)] = 0$$

$$\Rightarrow 4d(a)\alpha a\alpha d(a) + 2d^2(a)\alpha a\alpha a - 2d^2(a)\alpha a\alpha a - 6d(a)\alpha d(a)\alpha a + 2d(a)\alpha a\alpha d(a) = 0$$

$$\Rightarrow 6d(a)\alpha a\alpha d(a) - 6d(a)\alpha d(a)\alpha a = 0$$

$$\Rightarrow 6[d(a)\alpha a\alpha d(a) - d(a)\alpha d(a)\alpha a] = 0$$

$$\Rightarrow 6d(a)\alpha[a,d(a)]_\alpha = 0$$

Since S is 2 and 3-torsion-free, hence we get

$d(x)\alpha[x,d(x)]_\alpha = 0$.

III.iii Theorem

Let M be a 2-torsion free semi-prime Γ-ring satisfying $a\alpha b\beta c = a\beta b\alpha c$, for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$ and $d: M \to M$ a Jordan right derivation. If there exists a positive integer n such that $d(x)\big(\alpha d(x)\big)^n = 0$, for all $x \in M$ and $\alpha \in \Gamma$, then $d = 0$.

Proof: Since M is a semiprime, $\cap P = (0)$, where the intersection runs over all prime ideals P of M. We need to show that $d(P) \subseteq P$, for every prime ideal P of M. Let $a \in P$, $x \in M$. Then by proposition III.i(i), we have

$$0 = d(a\alpha\, x + x\alpha a)\alpha d(a\alpha x + x\alpha a)$$

$$= (2d(a)\alpha x + 2d(x)\alpha a)\alpha(2d(a)\alpha x + 2d(x)\alpha a)$$

$$= 2^2(d(a)\alpha x\alpha d(a)\alpha x + d(a)\alpha x\alpha d(x)\alpha a + d(x)\alpha a\alpha d(a)\alpha x + d(x)\alpha a\alpha d(x)\alpha a)$$

Since M is 2-torsion free, $d(x)\alpha a \in P$ and $d(a)\alpha x \in M$, hence

$$(d(a)\alpha x)(\alpha d(a)\alpha x) \equiv 0(mod)P, \text{ for all } \alpha \in \Gamma.$$

Again,

$$0 = d(a\alpha x + x\alpha a)\alpha d(a\alpha x + x\alpha a)\alpha d(a\alpha x + x\alpha a)$$

$$= (2d(a)\alpha x + 2d(x)\alpha a)\alpha(2d(a)\alpha x + 2d(x)\alpha a)\alpha(2d(a)\alpha x + 2d(x)\alpha a)$$

$$= 2^3(d(a)\alpha x\alpha d(a)\alpha x\alpha d(a)\alpha x + d(a)\alpha x\alpha d(x)\alpha a\alpha d(x)\alpha a + d(x)\alpha a\alpha d(a)\alpha x\alpha d(x)\alpha a + d(a)\alpha x\alpha d(a)\alpha x\alpha d(x)\alpha a + d(x)\alpha a\alpha d(x)\alpha a\alpha d(a)\alpha x + d(a)\alpha x\alpha d(x)\alpha a\alpha d(a)\alpha x + d(x)\alpha a\alpha d(a)\alpha x\alpha d(a)\alpha x + d(a)\alpha x\alpha d(a)\alpha x\alpha d(a)\alpha x).$$

Since M is 2-torsion free, $d(x)\alpha a \in P$ and $d(a)\alpha x \in M$, hence

$$(d(a)\alpha x)(\alpha d(a)\alpha x)^2 \equiv 0(mod)P, \text{ for all } \alpha \in \Gamma.$$

Proceeding in this way, we have

$$(d(a)\alpha x)(\alpha d(a)\alpha x)^n \equiv 0(mod)P, \text{ for all } \alpha \in \Gamma.$$

Thus, in the prime Γ-ring, $M' = M/P$, we have $(d(a)'\alpha x')(\alpha d(a)'\alpha x')^n$, for all $x' \in M'$ and $\alpha \in \Gamma$. Again, we have $d(a)'\alpha x' = 0$, for all $x' \in M'$ and $\alpha \in \Gamma$. Since M' is prime, $d(a) = 0$. This gives $d(a) \in P$ and so $d(P) \subseteq P$. Therefore, $d(P) \subseteq P$ for all prime ideals P of M, and so d induces a Jordan right derivation d' on the prime Γ-ring, $M' = M/P$.

Let us first assume that M' is commutative. In this case, d' is a derivation and we also have $d'(x')\big(\alpha d'(x')\big)^n = 0$, which follows that $d' = 0$. In case, M' is non - commutative, it follows that $d' = 0$. Thus, in any case, $d'(M') = 0$, that is, $d(M) \subseteq P$ for all prime ideals P of M. Since $\cap P = (0)$, we obtain $d(M) = 0$ and hence $d = 0$.

III.iv. Theorem

Let M be 2-torsion free and 3-torsion free semiprime Γ-ring. If d and G of M into M are Jordan's right derivations such that $d^2(M) = G(M)$ then $d = 0$.

Proof: Let $x \in M$ then $x\alpha x \in M$, putting $x\alpha x$ for x in $d^2(x) = G(x)$ we obtain

$$d(d(x)\alpha x) = G(x)\alpha x \quad (1)$$

Let us prove that for all $x \in M$ and $\alpha \in \Gamma$

$$d\big(x\alpha d(x)\big) = 2d(x)\alpha d(x) + G(x)\alpha x \quad (2)$$

Using proposition III.i, we have

$$d(x\alpha d(x) + d(x)\alpha x) = 2d(x)\alpha d(x) + 2d^2(x)\alpha x$$

$$\Rightarrow d\big(x\alpha d(x)\big) = 2d(x)\alpha d(x) + 2d^2(x)\alpha x - d(d(x)\alpha x)$$

$\Rightarrow d\big(x\alpha d(x)\big) = 2d(x)\alpha d(x) + 2d^2(x)\alpha x - d(d(x)\alpha x)$

$\Rightarrow d\big(x\alpha d(x)\big) = 2d(x)\alpha d(x) + 2G(x)\alpha x - G(x)\alpha x$

$\Rightarrow d\big(x\alpha d(x)\big) = 2d(x)\alpha d(x) + G(x)\alpha x$

$\Rightarrow d\big(x\alpha d(x)\big) - G(x)\alpha x = 2d(x)\alpha d(x)$

$\Rightarrow d\big(x\alpha d(x)\big) - d(d(x)\alpha x) = 2d(x)\alpha d(x)$

$\Rightarrow d(x\alpha d(x) - d(x)\alpha x) = 2d(x)\alpha d(x)$

$\Rightarrow d([x, d(x)]_\alpha) = 2d(x)\alpha d(x)$ (3)

Linearing (3), we have

$d([x + y, d(x + y)]_\alpha) = 2d(x + y)\alpha d(x + y)$

$\Rightarrow d([x + y, d(x) + d(y)]_\alpha) = 2\big(d(x) + d(y)\big)\alpha\big(d(x) + d(y)\big)$

$$\Rightarrow d([x, d(x)]_\alpha + [y, d(x)]_\alpha + [x, d(y)]_\alpha + [y, d(y)]_\alpha) = 2[d(x)\alpha d(x) + d(x)\alpha d(y) + d(y)\alpha d(x) + d(y)\alpha d(y)]$$

Using (3), we get

$$d([x, d(y)]_\alpha + [y, d(x)]_\alpha) = 2d(x)\alpha d(y) + 2d(y)\alpha d(x)$$

Putting in the above relation $y = x\alpha x$, using proposition III.i and (3), we obtain

$d([x, d(x\alpha x)]_\alpha + [x\alpha x, d(x)]_\alpha) - 2d(x)\alpha d(x\alpha x) - 2d(x\alpha x)\alpha d(x) = 0$

$$\Rightarrow d([x, 2d(x)\alpha x]_\alpha + [x\alpha x\alpha d(x) - d(x)\alpha x\alpha x]) - 2d(x)\alpha 2d(x)\alpha x - 4d(x)\alpha x\alpha d(x) = 0$$

$$\Rightarrow d\big((2x\alpha d(x)\alpha x - 2d(x)\alpha x\alpha x) + [x\alpha x\alpha d(x) - d(x)\alpha x\alpha x]\big) - 4d(x)\alpha d(x)\alpha x - 4d(x)\alpha x\alpha d(x) = 0$$

$$\Rightarrow 2d([x, d(x)]_\alpha\,\alpha x) + d([x\alpha x\alpha d(x) - x\alpha d(x)\alpha x + x\alpha d(x)\alpha x - d(x)\alpha x\alpha x]) - 4d(x)\alpha d(x)\alpha x - 4d(x)\alpha x\alpha d(x) = 0$$

$$\Rightarrow 2d([x, d(x)]_\alpha\,\alpha x) + d(x\alpha[x, d(x)]_\alpha + [x, d(x)]_\alpha\alpha x) - 4d(x)\alpha d(x)\alpha x - 4d(x)\alpha x\alpha d(x) = 0$$

$$\Rightarrow 2d([x, d(x)]_\alpha\,\alpha x) + 2d(x)\alpha[x, d(x)]_\alpha + 2d([x, d(x)]_\alpha)\alpha x - 4d(x)\alpha d(x)\alpha x - 4d(x)\alpha x\alpha d(x) = 0$$

$$\Rightarrow 2d([x, d(x)]_\alpha\,\alpha x) + 2d(x)\alpha x\alpha d(x) - 2d(x)\alpha d(x)\alpha x + 4d(x)\alpha d(x)\alpha x - 4d(x)\alpha d(x)\alpha x - 4d(x)\alpha x\alpha d(x) = 0$$

$\Rightarrow 2d([x, d(x)]_\alpha\,\alpha x) - 2d(x)\alpha d(x)\alpha x - 2d(x)\alpha x\alpha d(x) = 0$

Thus, we have

$d([x, d(x)]_\alpha\,\alpha x) = d(x)\alpha d(x)\alpha x + d(x)\alpha x\alpha d(x)$, for $x \in M$ and $\alpha \in \Gamma$ (4)

Let us prove the identity

$$d(x\alpha[x,d(x)]_\alpha) = d(x)\alpha d(x)\alpha x + d(x)\alpha x\alpha d(x), \text{for } x \in M \text{ and } \alpha \in \Gamma \qquad (5)$$

Using proposition III.i and III.ii, we have

$$d([x,d(x)]_\alpha\, \alpha x + x\alpha[x,d(x)]_\alpha) = 2d([x,d(x)]_\alpha)\alpha x + 2d(x)\alpha[x,d(x)]_\alpha$$
$$= 4d(x)\alpha d(x)\alpha x + 2d(x)\alpha[x,d(x)]_\alpha$$

Now applying (4), we obtain

$$d(x\alpha[x,d(x)]_\alpha) = 4d(x)\alpha d(x)\alpha x + 2d(x)\alpha[x,d(x)]_\alpha - d([x,d(x)]_\alpha\, \alpha x)$$
$$= 4d(x)\alpha d(x)\alpha x + 2d(x)\alpha[x,d(x)]_\alpha - d(x)\alpha d(x)\alpha x - d(x)\alpha x\alpha d(x)$$
$$= 3d(x)\alpha d(x)\alpha x + 2d(x)\alpha x\alpha d(x) - 2d(x)\alpha d(x)\alpha x - d(x)\alpha x\alpha d(x)$$
$$= d(x)\alpha d(x)\alpha x + d(x)\alpha x\alpha d(x)$$

which completes the proof (5).

From (4) and (5), we obtain

$$d([x\alpha[x,d(x)]_\alpha]_\alpha) = 0, x \in M \qquad (6)$$

From (6) and proposition III.ii it follows

$$d(x)\alpha[x,d(x)]_\alpha = 0, x \in M \qquad (7)$$

Using (7), (3) and proposition III.i, we obtain

$$d\big([x,d(x)]_\alpha \alpha d(x)\big) = d([x,d(x)]_\alpha\, \alpha d(x) + d(x)\alpha[x,d(x)]_\alpha)$$
$$= 2d([x,d(x)]_\alpha)\alpha d(x) + 2d^2(x)\alpha[x,d(x)]_\alpha$$
$$= 2d([x,d(x)]_\alpha)\alpha d(x) + 2G(x)\alpha[x,d(x)]_\alpha$$

Thus, we have

$$d\big([x,d(x)]_\alpha \alpha d(x)\big) = 4d(x)\alpha d(x)\alpha d(x) + 2G(x)\alpha[x,d(x)]_\alpha, x \in M \text{ and } \alpha \in \Gamma \quad (8)$$

Let us prove the relation

$$d\big([x,d(x)]_\alpha \alpha d(x)\big) = -6G(x)\alpha[x,d(x)]_\alpha, \; x \in M \text{ and } \alpha \in \Gamma \qquad (9)$$

Using (7) and proposition III.i, we obtain

$d(d(x)\alpha[x,d(x)]_\alpha) = 0$

$\Rightarrow d(d(x)\alpha x\alpha d(x) - d(x)\alpha d(x)\alpha x) = 0$

$\Rightarrow d(d(x)\alpha x\alpha d(x)) - d(d(x)\alpha d(x)\alpha x) = 0$

$\Rightarrow d(x)\alpha d(x)\alpha d(x) + 3d^2(x)\alpha x\alpha d(x) - d^2(x)\alpha d(x)\alpha x - d(d(x)\alpha d(x)\alpha x) = 0$

$\Rightarrow d(x)\alpha d(x)\alpha d(x) + 3G(x)\alpha x\alpha d(x) - G(x)\alpha d(x)\alpha x - d(d(x)\alpha d(x)\alpha x) = 0$

Thus, we have

$d(d(x)\alpha d(x)\alpha x) = d(x)\alpha d(x)\alpha d(x) + 3G(x)\alpha x\alpha d(x) - G(x)\alpha d(x)\alpha x$, $x \in M$ and $\alpha \in \Gamma$.

Now we have

$$\begin{aligned} d(x\alpha d(x)\alpha d(x) + d(x)\alpha d(x)\alpha x) &= d^2(x)\alpha x\alpha d(x) + d^2(x)\alpha d(x)\alpha x + 3d(x)\alpha d(x)\alpha d(x) \\ &\quad + 3d^2(x)\alpha d(x)\alpha x - d(x)\alpha d(x)\alpha d(x) - d^2(x)\alpha x\alpha d(x) \\ &= 4d^2(x)\alpha d(x)\alpha x + 2d(x)\alpha d(x)\alpha d(x) \end{aligned}$$

$$\therefore d(x\alpha d(x)\alpha d(x) + d(x)\alpha d(x)\alpha x) = 4G(x)\alpha d(x)\alpha x + 2d(x)\alpha d(x)\alpha d(x), x \in M \text{ and } \alpha \in \Gamma \qquad (10)$$

From the above relation and (10) it follows

$$d\big(x\alpha d(x)\alpha d(x)\big) = d(x)\alpha d(x)\alpha d(x) + 5G(x)\alpha d(x)\alpha x - 3G(x)\alpha x\alpha d(x), x \in M \text{ and } \alpha \in \Gamma \qquad (11)$$

From (10) and (11), we obtain

$$d([x, d(x)\alpha d(x)]_\alpha) = 6G(x)\alpha[d(x), x]_\alpha$$

Thus according to (7), we have

$$\begin{aligned} 6G(x)\alpha[d(x), x]_\alpha &= d([x, d(x)\alpha d(x)]_\alpha) \\ &= d(d(x)\alpha[x, d(x)]_\alpha + [x, d(x)]_\alpha \alpha d(x)) \\ &= d([x, d(x)]_\alpha \alpha d(x)) \end{aligned}$$

which completes the proof of (9). Combining (8) with (9), we arrive at

$$4d(x)\alpha d(x)\alpha d(x) + 2G(x)\alpha[x, d(x)]_\alpha = -6G(x)\alpha[x, d(x)]_\alpha$$

$$\Rightarrow d(x)\alpha d(x)\alpha d(x) + 2G(x)\alpha[x, d(x)]_\alpha = 0, for\ x \in M \text{ and } \alpha \in \Gamma \qquad (12)$$

Now starting from (7) and using proposition III.i, we have

$$d\big(d(x)\alpha[x, d(x)]_\alpha \alpha d(x)\big) = 0$$

$$\Rightarrow d([x, d(x)]_\alpha)\alpha d(x)\alpha d(x) + 3d^2(x)\alpha[x, d(x)]_\alpha \alpha d(x) - d^2(x)\alpha d(x)\alpha[x, d(x)]_\alpha = 0$$

$$\Rightarrow 2d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 3G(x)\alpha[x, d(x)]_\alpha \alpha d(x) = 0$$

Thus, we have

$$2d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 3G(x)\alpha[x, d(x)]_\alpha \alpha d(x) = 0, \text{for } x \in M \text{ and } \alpha \in \Gamma$$

$$\Rightarrow 4d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 6G(x)\alpha[x, d(x)]_\alpha \alpha d(x) = 0$$

$$\Rightarrow d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 3d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 6G(x)\alpha[x, d(x)]_\alpha \alpha d(x) = 0$$

$$\Rightarrow d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 3\big(d(x)\alpha d(x)\alpha d(x) + 2G(x)\alpha[x, d(x)]_\alpha \alpha d(x)\big)\,\alpha d(x) = 0$$

$$\Rightarrow d(x)\big(\alpha d(x)\big)^3 = 0 \qquad \text{[By using 12]}$$

We have seen that $d(x)$ is a nilpotent element of M. But since semiprime Γ-ring contains no non-zero nilpotent elements then by theorem 3.3 we have $d(x) = 0$ for all $x \in M$, which completes the proof.

V. Conclusions

Γ-Ring is one of the most important and modern branches of algebra in mathematics nowadays. There are huge properties of Γ-ring for the researchers to work by which they can enrich the world of algebra. Many modern algebraists are now engaged to analyze and characterize various properties of Γ-ring by different methods. In this thesis paper, we have tried to characterize some special properties of semiprime Γ-ring on Jordan right derivation which may help future researchers to proceed further.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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