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FIXATION OF THE RELATION BETWEEN FREQUENCY AND AMPLITUDE FOR NONLINEAR OSCILLATOR HAVING FRACTIONAL TERM APPLYING MODIFIED MICKENS' EXTENDED ITERATION METHOD

M. M. Ayub Hossain[1], B M Ikramul Haque[2]

[1, 2] Department of Mathematics, Khulna University of Engineering & Technology, Khulna 9203, Bangladesh.

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Abstract

A modified extended iteration procedure is applied to compute the analytical periodic solutions of the nonlinear oscillator having fractional terms. A nonlinear oscillator with $u^{1/3}$ force is given to demonstrate the effectiveness and expediency of the iteration scheme. Mickens' extended iteration method is a well-established method for studying random oscillations. The method is also simple and straightforward to accomplish approximate frequency and the corresponding periodic solution of the strongly nonlinear oscillator. The method gives high validity for both small and large initial amplitudes of oscillations. We have used an appropriate truncation of the obtained Fourier cosine series in each step of iterations to determine the approximate analytic solution of the oscillators. The second, third, and fourth approximate frequencies of the truly nonlinear oscillator with $u^{1/3}$ force show a good agreement with their exact values. Also, we have compared the calculated results with some of the existing results. We have shown that the method performs reasonably better.

Keywords: Mickens' Extended iteration procedure; Nonlinear oscillator with the fractional term; Nonlinearity; Fourier series.

I. Introduction

Nonlinear oscillations are a significant expression of dynamic behavior met in various fields. Every dynamic system displays oscillations of some kind. The most intuitional and evident nonlinear oscillations are in the field of engineering, but they are also regularly met in other fields, such as electromagnetism, logistics (stock), economy (business cycles), biology (population cycles), etc. The exact mathematical description and characterization of nonlinear oscillations are therefore of great importance in science, engineering, medical science, economics, etc. So studies on nonlinear oscillations system are most attractive of researchers. Although research on non-linear systems is a lot complex and sensitive because the characteristic of the non-linear system unexpectedly changes due to some small deviation of existing parameters as well as time. A lot of analytical methods are established to solve nonlinear oscillators such as Perturbation Method [XXX, XXXI]; Homotopy Perturbation Method [III]; He's Homotopy Perturbation Method [IV, V]; Harmonic Balance Method [VI, XXIV-XXVI]; Iterative Method [I, XI-XXII, XXVII-XXIX]; Cubication Method [IX]; He's Max-Min Method [II], Rational Energy Balance Method [VII]; Energy Balance Method [XXIII], He's Energy Balance Method [VIII, X], etc. Among them, Mickens' extended iteration method is one of the most extensively utilized methods in which the nonlinear term is strong. Mickens has developed an extended iteration technique and further effort has been through by Lim, Hu, Wu, and Haque.

The core point of this paper is to solve approximately the nonlinear oscillator with the fractional term by using Mickens' extended iteration method and to compare the output obtained with the exact one and with the result obtained by He's energy balance method (HEBM) [X] to the oscillator. As we can see, the results presented in this paper reveal that the Mickens extended iteration method is very efficient and competent for the nonlinear oscillator with the fractional term.

II. The methodology

There are three steps in the iterative process:

(i) Considering a second-order ordinary differential equation

(ii) Constructing it into standard form

(iii) Taking iterative scheme

(i) Consider the second-order nonlinear differential equation of the form

$$F(\ddot{u},u)=0 \text{ with } u(0)=a\ ,\ \dot{u}(0)=0 \tag{1}$$

Equation (1) rewritten of the form

$$\ddot{u}+f(u)=0 \tag{2}$$

(ii) Now the iteration form of equation (2) is

$$\ddot{u}+\omega^2u=\omega^2u-f(u)=H(u,\omega) \tag{3}$$

where ω is the natural frequency and currently ω^2 as well as ω is unknown.

(iii) The Iterative scheme of equation (3) is of the form

$$\ddot{u}_{k+1} + \omega_k^2 u_{k+1} = H(u_k, \omega_k)\,; k = 0,1,2,\cdots \tag{4}$$

$$u(t) = a\cos(\omega t)\,, \tag{5}$$

And

$$u_{k+1}(0) = a\ ,\ \dot{u}_{k+1}(0) = 0\,, \tag{6}$$

where a is the amplitude of the oscillator.

The extended iteration scheme is of the form

$$\ddot{u}_{k+1} + \omega_k^2 u_{k+1} = H(u_k, \ddot{u}_k) + H_u(u_0, \omega_k)(u_k - u_0) \tag{7}$$

where $H_u = \frac{\partial H}{\partial u}$ And u_{k+1} satisfies the conditions (6)

$u_1(t)$, $u_2(t)$, $u_3(t)$ …. and ω_0 , ω_1 , ω_2 …. are the first, second, third, …… approximate roots and corresponding frequencies of the oscillators respectively obtained by avoiding the secular terms in each step of iteration.

III. Solution Procedure

We consider an $u^{1/3}$ force nonlinear oscillator as

$$\ddot{u} + \varepsilon u^{\frac{1}{3}} = 0. \tag{8}$$

Adding $\omega^2 u$ on both sides of equation (8), we get

$$\ddot{u} + \omega^2 \mathrm{u} = \omega^2 \mathrm{u} - \varepsilon u^{\frac{1}{3}} = H(u, \omega^2) \tag{9}$$

where $H(u, \omega^2) = \omega^2 \mathrm{u} - \varepsilon u^{\frac{1}{3}}$ (10)

Therefore $H_u = \omega^2 - \frac{1}{3}\varepsilon u^{-\frac{2}{3}}$ (11)

According to our consideration, the extended scheme of equation (7) will be

$$\ddot{u}_{k+1} + u_k^2 x_{k+1} = (\omega_k^2 u_0 - \varepsilon u_0^{\frac{1}{3}}) + (\omega_k{}^2 - \frac{1}{3}\varepsilon u_0{}^{-\frac{2}{3}})(u_k - u_0) \tag{12}$$

To obtain first iterated result, we have

$$\ddot{u}_1 + \omega_0^2 u_1 = \omega_0^2 a\cos\theta - \varepsilon(a\cos\theta)^{\frac{1}{3}} \tag{13}$$

After expanding the equation (13) reduces to

$$\ddot{u}_1 + \omega_0^2 u_1 = (\omega_0^2 a - 1.1595952670\varepsilon a^{\frac{1}{3}})\cos\theta + 0.2319190534\varepsilon a^{\frac{1}{3}}\cos 3\theta - 0.1159595267\varepsilon a^{\frac{1}{3}}\cos 5\theta \tag{14}$$

To avoid secular terms from equation (14), we obtain

$$\Omega_0 = \frac{1.0768450525\varepsilon^{\frac{1}{2}}}{a^{\frac{1}{3}}}. \tag{15}$$

Ω_0 indicates the first approximate frequency of the oscillator.

After removing secular terms, the equation (14) changes to

$$\ddot{u}_1 + \omega_0^2 u_1 = 0.2319190534\varepsilon a^{\frac{1}{3}}\cos 3\theta - 0.1159595267\varepsilon a^{\frac{1}{3}}\cos 5\theta \tag{16}$$

The solution of equation (16) is

$$u_1(t) = C\cos\theta + a(-0.0250000000\cos 3\theta + 0.0041666667\cos 5\theta) \tag{17}$$

Using $u_1(0) = a$, we have $C = 1.0208333333a$

Therefore,

$$u_1(t) = a(1.0208333333\cos\theta - 0.0250000000\cos 3\theta + 0.0041666667\cos 5\theta) \tag{18}$$

This $u_1(t)$ represents the first approximate analytical solution of the oscillator.

For the second level, we have

$$\ddot{u}_2 + \omega_1^2 u_2 = (\omega_1^2 u_0 - u_0^{\frac{1}{3}}) + (\omega_1^2 - \frac{1}{3}u_0^{\frac{-2}{3}})(u_1 - u_0). \tag{19}$$

That is $\ddot{u}_2 + \omega_1^2 u_2 = \omega_1^2 u_1 - \frac{1}{3}\varepsilon u_0^{\frac{-2}{3}} u_1 - \frac{2}{3}\varepsilon u_0^{\frac{1}{3}}$. (20)

After expanding the equation (20) reduces to

$$\ddot{u}_2 + \omega_1^2 u_2 = (1.0208333333 a\omega_1^{\ 2} - 1.1697417255 \varepsilon a^{\frac{1}{3}}) \cos\theta$$
$$+ (-0.0250000000\, a\omega_1^{\ 2} + 0.2466775386\, \varepsilon\, a^{\frac{1}{3}}) \cos 3\theta \qquad (21)$$
$$+ (0.0041666667 a\omega_1^{\ 2} - 0.1225481362\, \varepsilon\, a^{\frac{1}{3}}) \cos 5\theta$$

To avoid secular terms from equation (21), we obtain

$$\text{i.e. } \omega_1 = \frac{1.0704529160 \varepsilon^{\frac{1}{2}}}{a^{\frac{1}{3}}} \qquad (22)$$

Ω_1 indicates the second approximate frequency of the oscillator.

Then equation (21) yield

$$\ddot{u}_2 + \omega_1^2 u_2 = 0.2180308025 \varepsilon\, a^{\frac{1}{3}} \cos 3\theta - 0.1177736801 \varepsilon\, a^{\frac{1}{3}} \cos 5\theta \qquad (23)$$

This $u_1(t)$ represents the first approximate analytical solution of the oscillator.

The solution of (23) is

$$u_2(t) = C_1 \cos\theta + a\,(-0.0237844289 \cos 3\theta + 0.0042825443 \cos 5\theta) . \qquad (24)$$

Using $u_2(0) = a$, we have $C_1 = 1.0195018846 a$

Therefore,

$$u_2(t) = a\,(1.0195018846 \cos\theta - 0.0237844289 \cos 3\theta + 0.0042825443 \cos 5\theta) \qquad (25)$$

This is the second approximate analytical solution of the oscillator.

For the third level, we have

$$\ddot{u}_3 + \omega_2^2 u_2 = \omega_2^{\ 2} u_2 - \frac{1}{3} \varepsilon\, u_0^{\frac{-2}{3}} u_2 - \frac{2}{3} \varepsilon\, u_0^{\frac{1}{3}} . \qquad (26)$$

By expanding, we have

$$\ddot{u}_3 + \omega_2^2 u_3 = (1.0195018846 a\omega_2^{\ 2} - 1.1691375861 \varepsilon\, a^{\frac{1}{3}}) \cos\theta$$
$$- (0.0237844289 a\omega_2^{\ 2} - 0.2459800828 \varepsilon\, a^{\frac{1}{3}}) \cos 3\theta \qquad (27)$$
$$+ (0.0042825443 a\omega_2^{\ 2} - 0.1223889282 \varepsilon\, a^{\frac{1}{3}}) \cos 5\theta$$

To avoid secular terms from equation (27), we obtain

$$\omega_2 = \frac{1.0708750369\varepsilon^{\frac{1}{2}}}{a^{\frac{1}{3}}} \tag{28}$$

This is the third approximate frequency of the oscillator.

Then equation (27) becomes

$$\ddot{u}_3 + \omega_2^2 u_3 = 0.2187047337\,\varepsilon a^{\frac{1}{3}} \cos 3\theta - 0.1174778205\,\varepsilon a^{\frac{1}{3}} \cos 5\theta \tag{29}$$

The solution of equation (29) is

$$u_3(t) = C_2 \cos\theta + a(-0.0238391412 \cos 3\theta + 0.0042684190 \cos 5\theta). \tag{30}$$

Using $u_3(0) = a$, we have $C_2 = 1.0195707222a$

Therefore,

$$u_3(t) = a(1.0195707222\cos\theta - 0.0238391412\cos 3\theta + 0.0042684190\cos 5\theta) \tag{31}$$

This $u_3(\text{t})$ indicates the third approximate analytical solution of the oscillator.

Proceeding to the fourth level $x_4(t)$ satisfies the equation

$$\ddot{u}_4 + \omega_3^2 u_4 = \omega_3{}^2 u_3 - \frac{1}{3}\varepsilon u_0^{\frac{-2}{3}} u_3 - \frac{2}{3}\varepsilon u_0^{\frac{1}{3}}. \tag{32}$$

By expanding, the equation (33) reduces to

$$\begin{aligned}\ddot{u}_4 + \omega_3^2 x_4 &= (1.0195707222a\omega_3{}^2 - 1.1691676263\varepsilon a^{\frac{1}{3}})\cos\theta \\ &- (0.0238391412a\omega_3{}^2 - 0.2460109113\varepsilon a^{\frac{1}{3}})\cos 3\theta \\ &+ (0.0042684190a\omega_3{}^2 - 0.1223915856\varepsilon a^{\frac{1}{3}})\cos 5\theta\end{aligned} \tag{33}$$

To avoid secular terms from equation (33), we obtain

$$\omega_3 = \frac{1.0708526428\varepsilon^{\frac{1}{2}}}{a^{\frac{1}{3}}} \tag{34}$$

Which is the fourth approximate frequency of the oscillator.

IV. Results and discussions

To obtain approximate solutions of a nonlinear oscillator with $u^{1/3}$ force, we have applied an extended iteration method. Here we have calculated first, second, third, and fourth approximate frequencies ω_0, ω_1, ω_2 and ω_3. The frequency-amplitude relationships are given in Table-1, Table-2, Table-3, and Table-4. To show the validity of the obtained solutions we have compared the results with the existing results determined by He's energy balance method [10]. The comparison between the third-order approximate solution of equation (8) for $\varepsilon = 0.1$& $a = 10$, $\varepsilon = 1.0$& $a = 10$, $\varepsilon = 0.1$& $a = 10$ and $\varepsilon = 0.1$& $a = 10$ together with corresponding exact solutions are presented in Figure-1**,** Figure-2, Figure-3, and Figure-4**.** Analyzing the results, we see that in our proposed method, the relative errors are less for all values of amplitude for the existing He's energy balance method and the values of frequencies are very proximate to exact values.

Table 1: Comparison of the approximate frequencies with exact frequency ω_e of $\ddot{u} + \varepsilon u^{\frac{1}{3}} = 0$ when $\varepsilon = 0.1$

$\varepsilon = 0.1$						
a	ω_0 Er(%)	ω_1 Er(%)	ω_2 Er(%)	ω_3 Er(%)	$\omega_{[X]}^{HEBM}$ Er(%)	$\omega_{[XXIX]}^{ex}$
0.1	0.7336459938 0.5973	0.7292910820 0.0002	0.7295786696 0.0396	0.7295634126 0.0375	0.71782402 35 1.572	0.72928977 65
0.5	0.4290387799 0.5973	0.4264920120 0.0002	0.4266601943 0.0396	0.4266512719 0.0375	0.41978603 55 1.572	0.42649124 87
1	0.3405283053 0.5973	0.3385069343 0.0002	0.3386404206 0.0396	0.3386333390 0.0375	0.33318439 72 1.572	0.33850632 83
5	0.1991421610 0.5973	0.1979600560 0.0002	0.1980381193 0.0396	0.1980339778 0.0375	0.19484741 74 1.572	0.19795970 17
10	0.1580592379 0.5973	0.1571210006 0.0002	0.1571829595 0.0396	0.1571796725 0.0375	0.15712071 94 1.572	0.15805923 79
50	0.0924336030 8 0.5973	0.0918844918 6 0.0002	0.0919211523 4 0.0396	0.09191923009 0.0375	0.09188475 413 1.572	0.09188475 41
100	0.0733645993 0.5973	0.0729291082 0.0002	0.0729578669 0.0396	0.0729563412 0.0375	0.07292897 76 1.572	0.07292897 77

500	0.0429038779 0.5973	0.0426492012 0.0002	0.0426660194 0.0396	0.0426651272 0.0375	0.04264912 48 1.572	0.04264912 49
1000	0.0340528305 0.5973	0.0338506934 0.0002	0.0338640421 0.0396	0.033863333 0.0375	0.03385063 28 1.572	0.03385063 28

Table 2: Comparison of the approximate frequencies with exact frequency ω_e of $\ddot{u}+\varepsilon u^{\frac{1}{3}}=0$ when $\varepsilon=1.0$

$\varepsilon=1.0$						
a	ω_0 Er(%)	ω_1 Er(%)	ω_2 Er(%)	ω_3 Er(%)	$\omega_{[X]}^{HEBM}$ Er(%)	$\omega_{[XXIX]}^{ex}$
0.1	2.31999233 0.59732	2.30622089 0.00018	2.30713032 0.03961	2.30708208 0.03752	2.26995887 1.572	2.30621676
0.5	1.35673974 0.59732	1.34868616 0.00018	1.349218 0.03961	1.34918978 0.03752	1.32748 1.572	1.34868374
1	1.07684505 0.59732	1.07045291 0.00018	1.07087503 0.03961	1.07085264 0.03752	1.05362157 1.572	1.070451
5	0.6297428 0.59732	0.62600466 0.00018	0.62625152 0.03961	0.62623842 0.03752	0.616161635 1.572	0.62600354
10	0.49982719 0.59732	0.49686023 0.00018	0.49705616 0.03961	0.49704576 0.03752	0.489047814 1.572	0.49685934
50	0.29230071 0.59732	0.29056562 0.00018	0.2906802 0.03961	0.29067412 0.03752	0.285996896 1.572	0.2905651
100	0.23199923 0.59732	0.23062208 0.00018	0.23071303 0.03961	0.2307082 0.03752	0.226995887 1.572	0.23062167
500	0.13567397 0.59732	0.13486861 0.00018	0.1349218 0.03961	0.13491897 0.03752	0.132748 1.572	0.13486837
1000	0.1076845 0.59732	0.10704529 0.00018	0.1070875 0.03961	0.10708526 0.03752	0.105362157 1.572	0.1070451

Table 3: Comparison of the approximate frequencies with exact frequency ω_e of $\ddot{u}+\varepsilon u^{\frac{1}{3}}=0$ when $\varepsilon=10.0$

$\varepsilon=10.0$						
a	ω_0 Er(%)	ω_1 Er(%)	ω_2 Er(%)	ω_3 Er(%)	$\omega_{[X]}^{HEBM}$ Er(%)	$\omega_{[XXIX]}^{ex}$
0.1	7.336459938 0.5973	7.292910819 0.0002	7.295786695 0.0396	7.295634126 0.0375	7.178240235 1.572	7.292897765
0.5	4.290387799 0.5973	4.264920120 0.0002	4.266601942 0.0396	4.266512719 0.0375	4.197860355 1.572	4.264912487
1	3.405283052 0.5973	3.385069342 0.0002	3.386404206 0.0396	3.386333389 0.0375	3.331843972 1.572	3.385063283
5	1.991421610 0.5973	1.979600560 0.0002	1.980381193 0.0396	1.980339779 0.0375	1.948474174 1.572	1.979597017
10	1.580592379 0.5973	1.571210006 0.0002	1.571829594 0.0396	1.571796724 0.0375	1.546504978 1.572	1.571207194
50	0.924336030 0.5973	0.918849185 0.0002	0.919211523 0.0396	0.919192300 0.0375	0.9044015966 1.572	0.918847541
100	0.733645993 0.5973	0.729291081 0.0002	0.729578669 0.0396	0.729563412 0.0375	0.7178240235 1.572	0.729289776
500	0.429038779 0.5973	0.426492012 0.0002	0.426660194 0.0396	0.426651271 0.0375	0.4197860355 1.572	0.426491248
1000	0.340528305 0.5973	0.338506934 0.0002	0.338640420 0.0396	0.338633338 0.0375	0.3331843972 1.572	0.338506328

Table-4: Comparison of the approximate frequencies with exact frequency

ω_e **of** $\ddot{u}+\varepsilon u^{\frac{1}{3}}=0$ **when** $\varepsilon=100.0$

$\varepsilon=100.0$						
a	ω_0 Er(%)	ω_1 Er(%)	ω_2 Er(%)	ω_3 Er(%)	$\omega_{[X]}^{HEBM}$ Er(%)	$\omega_{[XXIX]}^{ex}$
0.1	23.199923368 0.5973	23.062208962 0.0002	23.071303281 0.0396	23.0708207 98 0.0375	22.699588 74 1.572	23.06216 768
0.5	13.567397491 0.5973	13.486861617 0.0002	13.492180007 0.0396	13.4918978 49 0.0375	13.274800 02 1.572	13.48683 748
1	10.768450525 0.5973	10.704529160 0.0002	10.708750369 0.0396	10.7085264 20 0.0375	10.536215 76 1.572	10.70451 000
5	6.297428069 0.5973	6.260046628 0.0002	6.262515206 0.0396	6.26238424 0 0.0375	6.1616163 53 1.572	6.260035 423
10	4.998271971 0.5973	4.968602301 0.0002	4.970561613 0.0396	4.97045766 5 0.0375	4.8904781 41 1.572	4.968593 409
50	2.923007180 0.5973	2.905656252 0.0002	2.906802065 0.0396	2.90674127 6 0.0375	2.8599689 65 1.572	2.905651 053
100	2.319992336 0.5973	2.306220896 0.0002	2.307130328 0.0396	2.30708207 9 0.0375	2.2699588 74 1.572	2.306216 768
500	1.356739749 0.5973	1.348686161 0.0002	1.349218000 0.0396	1.34918978 4 0.0375	1.3274800 02 1.572	1.348683 748
1000	1.076845052 0.5973	1.070452916 0.0002	1.070875036 0.0396	1.07085264 2 0.0375	1.0536215 76 1.572	1.070451 000

Note: Here $\omega_{[X]}^{HEBM}$ represents the approximation frequency obtained by Ganji S. S. *et al* [X].

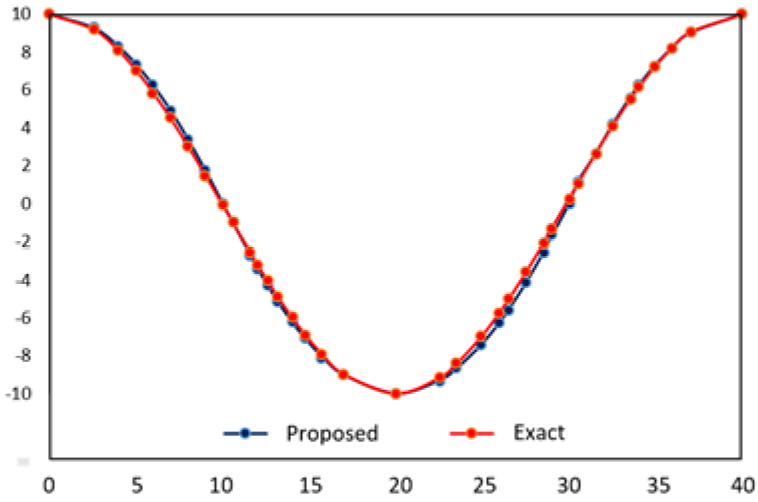

Fig1. A comparison between the third-order approximate solutions of $\ddot{u}+\varepsilon u^{\frac{1}{3}}=0$ when $\varepsilon=0.1$ and $a=10$ together with the corresponding exact solution.

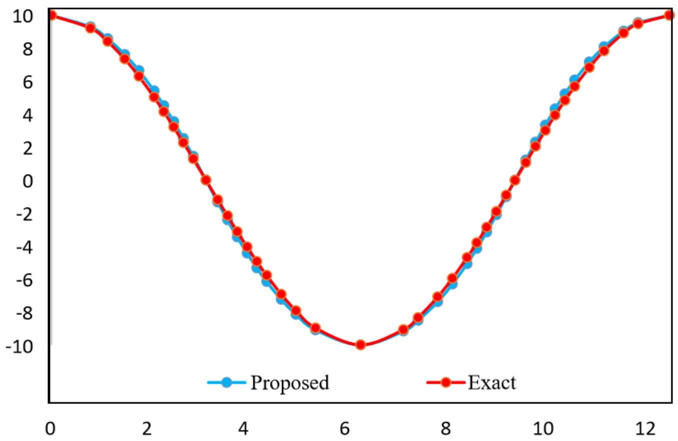

Fig2. A comparison between the third-order approximate solutions of $\ddot{u}+\varepsilon u^{\frac{1}{3}}=0$ when $\varepsilon=1.0$ and $a=10$ together with the corresponding exact solution.

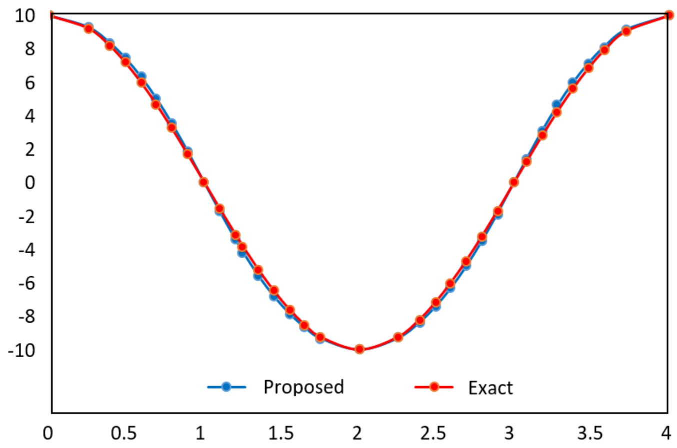

Fig 3. A comparison between the third-order approximate solution of $\ddot{u}+\varepsilon u^{\frac{1}{3}}=0$ when $\varepsilon=10$ and $a=10$ together with the corresponding exact solution.

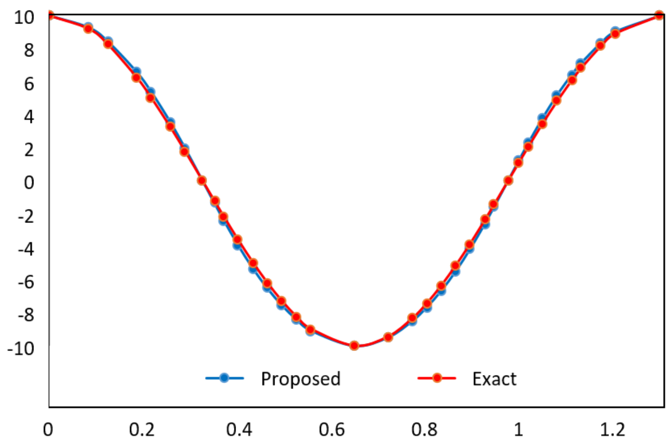

Fig 4. A comparison between the third-order approximate solution of $\ddot{u}+\varepsilon u^{\frac{1}{3}}=0$ when $\varepsilon=100$ and $a=10$ together with the corresponding exact solution.

V. Conclusion

We used a very easy but efficient method for the nonlinear oscillator. It can be observed that the second, third, and fourth approximations of the nonlinear oscillator provide almost the exact result. The method can be easily extended to any nonlinear oscillator without any difficulty. Moreover, the present work can be used as an illustration for many other applications in searching for periodic solutions of nonlinear oscillations and so can be found widely applicable in engineering and science.

Conflicts of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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