



LOCAL STABILITY ANALYSIS OF A PREDATOR-PREY DYNAMICS INCORPORATING BOTH SPECIES DENSITY INCREASING FUNCTIONAL RESPONSE

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<https://doi.org/10.26782/jmcms.2022.01.00006>

(Received: November 2, 2021; Accepted: January 6, 2022)

Abstract

Most of the functional responses which have been incorporated to formulate mathematical biology consider individual contact or predator cooperation. In this study, we have introduced a different functional response that describes the prey-predator system when predators form a line and cooperatively attack a group of predators. We have also described the effect of prey on this system. Additionally, we find all the equilibrium points and their local stability behaviour.

Keywords: Predator cooperation, Prey predator system, Equilibrium points, Local stability behaviour

1. Introduction

Prey predator interaction is becoming a very interesting topic in mathematical biology. In the present scenario, the conservation of biodiversity is an urgent need. Since statistical data provides a fraction of dynamical understanding and future outcomes, the mathematical formulation of any system becomes more essential. The traditional structure of a prey-predator model is

$$\frac{dX}{dt} = rX \left(1 - \frac{X}{K}\right) - g(X, Y)Y$$

$$\frac{dY}{dt} = E g(X, Y)Y - \mu Y$$

$$X(0) > 0, Y(0) > 0;$$

where X and Y represent the density of prey species and predator species at any time T . r is the intrinsic growth rate and K is the carrying capacity of prey. $g(x, y)$ is the rate of prey consumption and μ is the death rate of a predator. There are various interesting functional responses introduced by various researchers. Some of them are Lotka-Volterra type $d(xy)$, Holling type II $\left(\frac{dx}{a+bx}\right)$, Holling type III $\left(\frac{dx^2}{a+bx^2}\right)$ which are dependent

Shilpa Samaddar et al

on x only. Ratio dependent type ($\frac{dx}{x+ay}$) and Beddington type ($\frac{dx}{a+bx+cy}$) responses depend on both prey and predator species. The biological meaning of a, b, c, d are described in the studies ref. in [II – IX].

In this study, we introduce a different functional response that reflects the cooperation effects described by Conser et al. [I], and we analyse the dynamics of the system mathematically. This kind of response happens when the preys are in a cluster and predators cannot consume them entirely in a course of time. Holling [II] and Conser et al. [I] represented the functional response as $g(X, Y) = \frac{CbXY}{1+hCbXY}$. Here C represents the density of prey captured by a predator in one counter, b is the total encounter coefficient, and h is the handling time. Contrasting the conventional functional responses, this response is monotonic concerning both the prey and predator species. In any prey species, there are always some preys that take refuge to hide from a predator. Incorporating the above response with prey refuge, the prey-predator system turns into

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right) - \frac{Cb(1-m)XY^2}{1+hCb(1-m)XY} \quad (1.1a)$$

$$\frac{dY}{dT} = \frac{ECb(1-m)XY^2}{1+hCb(1-m)XY} - \mu Y \quad (1.1b)$$

$$X(0) > 0, Y(0) > 0;$$

All the parameters r, C, b, h, E, μ, m are positive. m denotes the refuge coefficient and $m \in [0, 1]$. By taking the transformation $t = rT, x = \frac{X}{K}, y = hCbKY$, we non dimensionalize the system. The system transforms to

$$\frac{dx}{dt} = rx(1 - x) - \frac{\alpha(1-m)xy^2}{1+(1-m)xy} \quad (1.2a)$$

$$\frac{dy}{dt} = \frac{\beta(1-m)xy^2}{1+(1-m)xy} - \mu y \quad (1.2b)$$

$$x(0) > 0, y(0) > 0;$$

where,

$$\alpha = \frac{1}{Cb(hK)2r}, \beta = \frac{E}{rh}E, \gamma = \frac{\mu}{r}.$$

II. Mathematical Analysis

II.i. Uniform Boundedness

Theorem 2.1. The solutions of the system (1.2) are uniformly bounded if $\gamma < 1$.

Proof. From the firsts equation of the system (1.2), it is clear that $x(t) \leq 1$. We define, $\omega(t) = x(t) + \frac{\alpha}{\beta}y$. Then $\frac{d\omega(t)}{dt} = x(1 - x) - \frac{\alpha}{\beta}\gamma y$. Hence, $\frac{d\omega(t)}{dt} +$

$\gamma\omega(t) = x(1 + \gamma - x) \leq \frac{(1+\gamma)^2}{4}$ if $\gamma < 1$. The standard comparison displays $\limsup_{t \rightarrow \infty} (x(t) + \alpha \beta y) \leq \frac{(1+\gamma)^2}{4}$, which implies $\limsup_{t \rightarrow \infty} y(t) \leq \frac{\beta}{\alpha} \frac{(1+\gamma)^2}{4}$. Hence, all the solutions are uniformly bounded.

II.ii. Existence of Equilibria

The zero growth rate isoclines $F(x, y) = G(x, y) = 0$ provide three ecologically meaningful equilibria as:

1. Species less equilibrium $E_0(0,0)$.
2. The only prey species equilibrium $E_1(1,0)$, here prey population reaches its maximum.
3. The coexisting equilibrium $E_3(x_*, y_*)$, is an intersection point of the following nullclines

$$1 - x = \frac{\alpha(1-m)y^2}{1 + (1-m)xy}, \quad (2.1a)$$

$$\frac{\beta(1-m)xy}{1 + (1-m)xy} = \gamma; \quad (2.1b)$$

Here, nullcline 2.1b. provides $y_* = \frac{\gamma}{(\beta-\gamma)(1-m)x}$. The substitution of y_* in nullcline 2.1a. gives x_* as a positive solution of

$$x^3 - x^2 + \frac{\alpha\gamma^2}{\beta(1-m)(\beta-\gamma)} = 0 \quad (2.2)$$

Remark: y_* exists if $\beta > \gamma$. By Descartes's rule of signs, Eqn 2.2 has two positive solutions or no solutions. Let $\varphi(x) = x^3 - x^2 + \frac{\alpha\gamma^2}{\beta(1-m)(\beta-\gamma)}$. Then $\varphi(x)$ has its minimum at $x = \frac{2}{3}$.

- (i) The system has no solution if $\alpha > \alpha_* := \frac{4}{27} \frac{\beta(1-m)(\beta-\gamma)}{\gamma^2}$.
- (ii) The system has a unique solution $(x_0, y_0) = (\frac{2}{3}, \frac{3\gamma}{2(\beta-\gamma)(1-m)})$ that means both the positive solutions of Eqn 2.2 coincide if $\alpha = \alpha_*$.
- (iii) Two distinct positive solutions (x_1, y_1) and (x_2, y_2) where $x_1 < \frac{2}{3} < x_2$ if $\alpha < \alpha_*$. The Jacobian matrix of the system at any equilibrium point (x, y) is

$$J = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

where

$$P_{11} = 1 - x - \frac{\alpha(1-m)y^2}{1 + (1-m)xy} + x \left(-1 + \frac{\alpha(1-m)^2 y^3}{[1 + (1-m)xy]^2} \right)$$

$$P_{12} = - \frac{\alpha(1-m)xy[2 + (1-m)xy]}{[1 + (1-m)xy]^2} < 0$$

$$P_{21} = \frac{\beta(1-m)y^2}{[1+(1-m)xy]^2} > 0$$

$$P_{22} = -\gamma + \frac{\beta(1-m)xy}{1+(1-m)xy} + \frac{\beta(1-m)xy}{[1+(1-m)xy]^2}.$$

II.iii. Local stability analysis

At the equilibrium point (0,0), the Jacobian matrix becomes

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -\gamma \end{pmatrix}$$

The system has one positive and one negative eigenvalue at the equilibrium point (0,0). Hence the system is the unstable saddle.

At the equilibrium point (1,0), the Jacobian matrix becomes

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -\gamma \end{pmatrix}$$

The system has two negative eigenvalues at the equilibrium point (1,0). Hence the system is locally asymptotically stable.

At the interior equilibrium point $(x_j, y_j)_{(j=0,1,2)}$, the Jacobian matrix is

$$J = \begin{pmatrix} \frac{\gamma(1-m)}{\beta} - \frac{[\beta+\gamma(1-m)]x_j}{\beta} & -\frac{\alpha\gamma(2\beta-\gamma)}{\beta^2} \\ \frac{\beta-\gamma}{\alpha}(1-x_j) & \frac{\gamma(\beta-\gamma)}{\beta} \end{pmatrix}$$

Now,

$$Trace(J) = \frac{\beta+\gamma(1-m)}{\beta} \left(\frac{\gamma[\beta-\gamma+(1-m)]}{\beta+\gamma(1-m)} - x_j \right)$$

$$Det(J) = \frac{\gamma(\beta-\gamma)}{\beta^2(3\beta-m\gamma)} \left(\frac{2\beta-m\gamma}{3\beta-m\gamma} - x_j \right)$$

Let $A = \frac{2\beta-m\gamma}{3\beta-m\gamma}$. At the equilibrium point (x_0, y_0) , $Det(J) < 0$ since $A - \frac{2}{3} = -\frac{m\gamma}{3(3\beta-m\gamma)} < 0$ as $\beta > \gamma$ and $m < 1$.

Case1: $A < x_1 < \frac{2}{3} < x_2$. Here, $Det(J) < 0$ for both the equilibrium points.

Hence both the equilibrium points are unstable saddle.

Case2: $x_1 < A < \frac{2}{3} < x_2$. Here, $Det(J) > 0$ at (x_1, y_1) and $Det(J) < 0$ for (x_2, y_2) . Hence (x_1, y_1) is anti-saddle and (x_2, y_2) is the unstable saddle. The equilibrium point (x_1, y_1) will be stable if $x_1 > \gamma[\beta - \gamma + (1 - m)] / \beta + \gamma(1 - m)$.

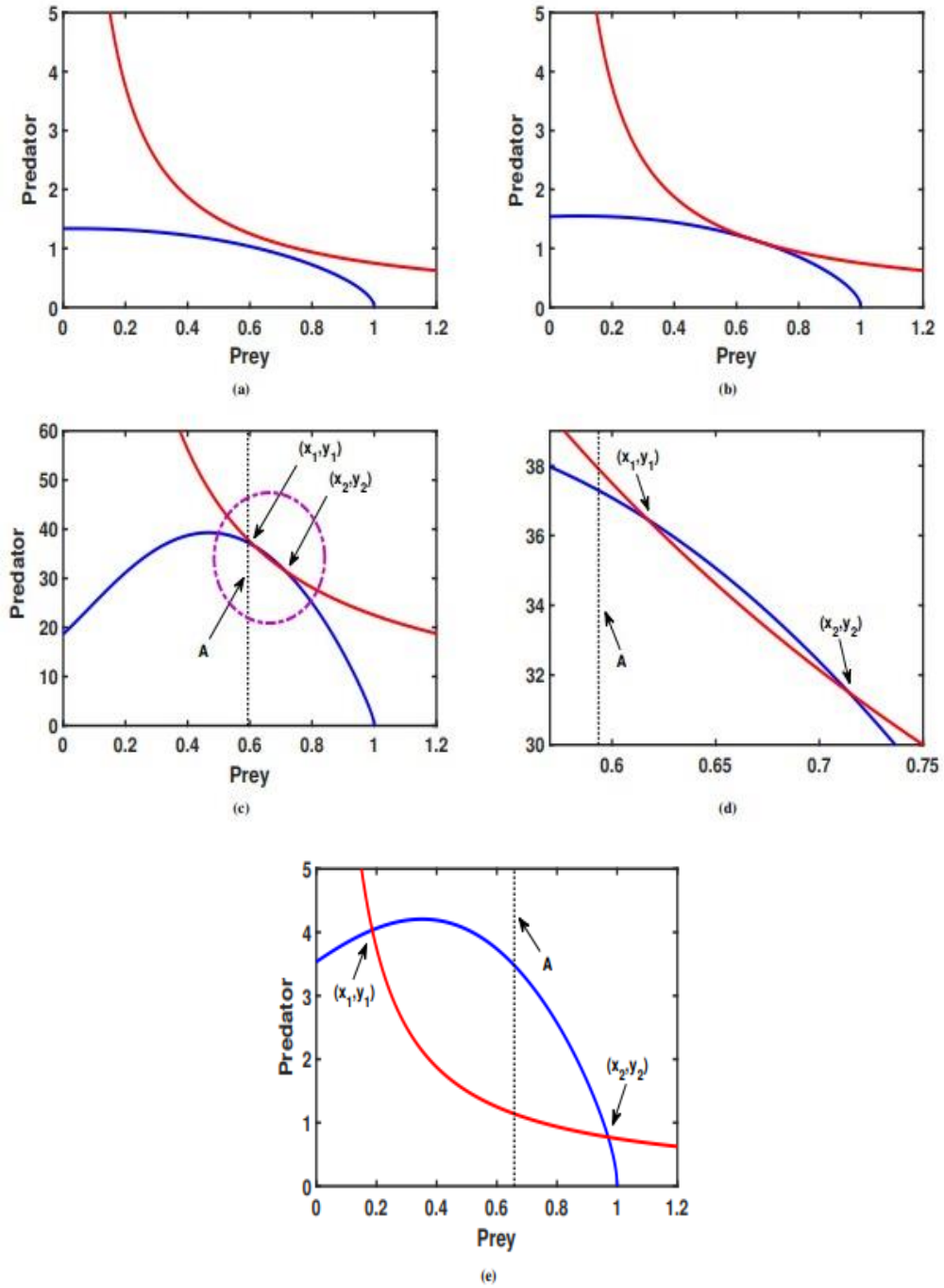


Figure 1: Schematic diagram of Prey and Predator nullclines at different parameter value: (a) $\alpha > \alpha_s$, (b) $\alpha = \alpha_s$, (c) $A < x_1 < \frac{2}{3} < x_2$, (d) Zoom view of the purple circle in figure (c), (e) $x_1 < A < \frac{2}{3} < x_2$

III. Hopf Bifurcation Analysis

Theorem 3.1. The system performs Hopf-bifurcation at $m^{[h]} = \frac{1-\gamma}{\gamma} \frac{2+\beta x_1-\beta \gamma}{1-x}$ around the interior equilibrium point (x_1, y_1) .

Proof. At the equilibrium point (x_1, y_1) , the Jacobian matrix always has positive determinants. When the trace of the Jacobian becomes positive, the equilibrium becomes unstable. Hence, there is a chance of the occurrence of Hopf bifurcation at the bifurcating point. At $m = m^{[h]}$, $\text{Trace}(J) = 0$. Also $\frac{d}{dm} \text{Trace}(J)|_{m^{[h]}} = -\frac{\beta+\gamma(1-m)}{\beta} \frac{\gamma[\beta(1-\gamma)+\gamma^2]}{[\beta+\gamma(1-m)]^2} - \frac{\alpha\gamma^2}{3x_1(\frac{2}{3}-x_1)} \neq 0$. Hence, the system undergoes Hopf bifurcation at $m = m^{[h]}$. Taking m as a bifurcating parameter and fixing other parameter values as $\alpha = 2.245, \beta = 0.4, \gamma = 0.15$, the system crosses Hopf bifurcation at $m^{[h]} = 0.225$. Figure 2 represents the Hopf bifurcation diagram.

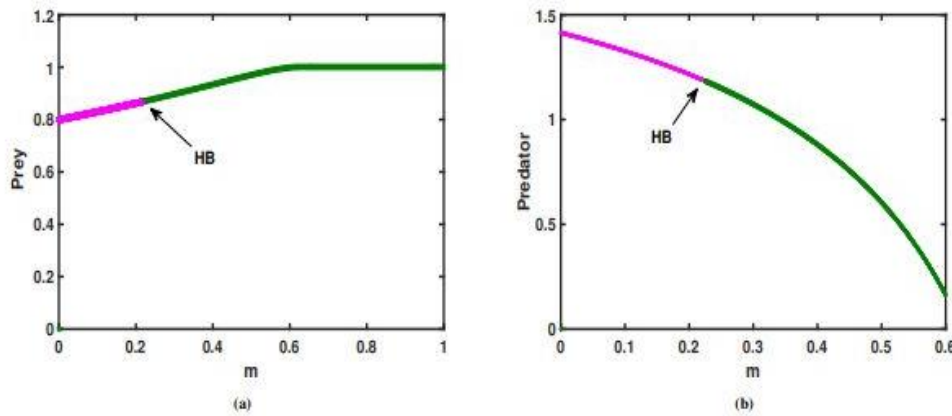


Fig 2. Hopf bifurcation diagram of (a) Prey and (b) Predator concerning prey refuge (m), where pink marks represent unstable periodic orbits and green marks represent stable periodic orbits.

IV. Discussion and Conclusion

In this paper, we analysed a prey-predator system concerning a newly suggested functional response by Conser et al. [9]. This kind of functional response is derived from the influence of a group of predators performing in a line and attacking a school of prey. We explained the criterion for the existence of all ecological equilibria as well as their local stability. It is shown that there might exist two interior equilibria, one will be always unstable but another will go through stability, instability and Hopf bifurcation depending on parameter values.

Conflicts of Interest:

There is no conflict of interest regarding this article

Shilpa Samaddar et al

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