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NUMERICAL EXPERIMENTS FOR NONLINEAR BURGER'S PROBLEM

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Abstract

The article contains the results of computational experiments for the non-homogeneous Burger's problem and finding its solution by using the non-classical variational-Cole-Hopf transformation approach. On using exact linearization via Cole-Hopf transformation, as well as the application of the non-classical variational approach, then the non-homogeneous Burger's problem has been solved. The solution which is obtained by this approach is in a compact form so that the original nonlinear solution is easy to be approximated. The accuracy of this method is dependent on the types of selected basis and its number.

Keywords: Burger's problem; numerical solution; Cole-Hopf transformation; non-classical variational.

I. Introduction

Consider the following non-homogeneous one-dimensional nonlinear Burger's equation:

$$\frac{\partial}{\partial t}u(x, t) - \varepsilon\frac{\partial^2}{\partial x^2}u(x, t) + u(x, t)\frac{\partial}{\partial x}u(x, t) = J(x, t),\ a<x<b,\ t > 0 \qquad (1)$$

with the initial condition

$$u(x, 0) = h(x), \quad a < x < b \qquad (2)$$

and the boundary conditions:

$$\left.\begin{array}{l} u(a, t) = f(t) \\ \text{and} \qquad\qquad , t > 0 \\ u(b, t) = g(t) \end{array}\right\} \qquad (3)$$

where $\varepsilon > 0$ and a, b are real numbers.

First appeared in 1915 by Bateman [II], who used the equation as a model for the motion of a viscous fluid when the viscosity approaches zero and derived two types of steady-state solutions for the infinite domain problem [I, VII, XI, XIII, XVIII, XIX].

Several people have investigated Burger's equation, there have been a lot of studies and researches on the equation [IV, VI, VIII, XIV]. Lighthill formally derived the equation as a second-order approximation to the dimensional unsteady Navier-Stokes equations [III, XII, XVII]. A similar result and observation were obtained by Moran and Shen, [V,IX,X,XV,XVI].

This article aims to solve the problem (1)-(3) approximately by using Cole-Hopf transformation together with a non-classical variational approach.

II. Exact Linearization of Problem Formulation:

If the linear equation:

$$\frac{\partial}{\partial t}\psi(x,t)=\varepsilon\frac{\partial^2}{\partial x^2}\psi(x,t)-\frac{1}{2\varepsilon}\psi(x,t)V(x,t) \quad (4)$$

Where:

$$V(x,t)=\int_0^x J(x',t)\,dx' \quad (5)$$

Is subjected to the transformation:

$$\psi(x,t)=F(u) \quad (6)$$

Where, for the moment F is specified.

Now, finding the derivatives of $\psi(x, t)$ as follows:

$$\frac{\partial}{\partial t}\psi(x,t)=F'\frac{\partial}{\partial t}u(x,t) \quad (7)$$

$$\frac{\partial}{\partial x}\psi(x,t)=F'\frac{\partial}{\partial x}u(x,t)$$

$$\frac{\partial^2}{\partial x^2}\psi(x,t)=F'\frac{\partial^2}{\partial x^2}u(x,t)+F''\left(\frac{\partial}{\partial x}u(x,t)\right)^2 \quad (8)$$

where Prime denotes the derivative of F with respect to u.

Substituting (7)-(8) in equation (4), we get:

$$F'\frac{\partial}{\partial x}u(x,t)=\varepsilon\left[F'\frac{\partial^2}{\partial x^2}u(x,t)+F''\left(\frac{\partial}{\partial x}u(x,t)\right)^2\right]-\frac{1}{2\varepsilon}FV(x,t) \quad (9)$$

Dividing both sides of equation (9) by F′, we obtain:

$$\frac{\partial}{\partial x} u(x, t) = \varepsilon \left(\frac{\partial^2}{\partial x^2} u(x,t) \right) + \varepsilon \left(\frac{F''}{F'} \right) \left(\frac{\partial}{\partial x} u(x,t) \right)^2 - \frac{1}{2\varepsilon} \left(\frac{F}{F'} \right) V(x, t) \qquad (10)$$

To put equation (1) in the form of equation (10), modified equation (1) by the introduction of continuously differentiable function $\phi(x, t)$ defined by setting:

$$u(x, t) = \frac{\partial}{\partial x} \phi(x, t) \qquad (11)$$

then equation (1) becomes:

$$\frac{\partial^2}{\partial x \partial t} \phi(x, t) + \frac{\partial}{\partial x} \phi(x, t) \frac{\partial^2}{\partial x^2} \phi(x, t) = \varepsilon \frac{\partial^3}{\partial x^3} \phi(x, t) + G(x, t) \qquad (12)$$

which upon integrating with respect to x, then equation (12) becomes:

$$\frac{\partial}{\partial t} \phi(x, t) + \frac{1}{2} \left(\frac{\partial}{\partial x} \phi(x,t) \right)^2 = \varepsilon \frac{\partial^3}{\partial x^3} \phi(x, t) + V(x, t) \qquad (13)$$

where:

$$V(x, t) = \int_a^x G(x', t)\, dx'$$

i.e., equation (1) is equivalent to equation (10), if:

$$\varepsilon \left(\frac{F''}{F'} \right) = -\frac{1}{2}$$

$$F'' + \frac{1}{2\varepsilon} F' = 0$$

Then:

$$F = A \exp(-\phi/2\varepsilon) + B \qquad (14)$$

where A and B are arbitrary constants.

For simplicity, in case of the Burger's equation (with A = 1, B = 0), we have:

$$\phi(x, t) = -2\varepsilon \ln\psi(x, t) \qquad (15)$$

The following transformation has been developed which is defined by:

$$u(x,t) = -2\varepsilon \frac{\frac{\partial}{\partial x}\psi(x,t)}{\psi(x,t)}, \ \psi(x,t) \not\equiv 0, \ \forall\ x, t \in \Omega \tag{16}$$

where $\Omega = \{(x, t) \mid a < x < b,\ t > 0\}$.

Transformation (16) derived from (11) and (15). Then Burger's equation (1) reduces to the linear heat equation:

$$\frac{\partial}{\partial t}\psi(x,t) = \varepsilon \frac{\partial^2}{\partial x^2}\psi(x,t) - \frac{1}{2\varepsilon}\psi(x,t)V(x,t) \tag{17}$$

where:

$$V(x,t) = \int_a^x J(x', t)\, dx'$$

The transformation of the initial condition (2), becomes:

$$\psi(x,0;\varepsilon) = e^{-\frac{1}{2\varepsilon}\left(\int_a^x \phi(x')dx'\right)}, \ a < x < b \tag{18}$$

and boundary conditions becomes:

$$\left.\begin{aligned} &\psi(a,t;\varepsilon) = e^{-\frac{1}{2\varepsilon}\left(\int_0^t f(t')dt'\right)} \\ \text{and} \qquad\qquad &\qquad\qquad\qquad\qquad , t > 0 \\ &\psi(b,t;\varepsilon) = e^{-\frac{1}{2\varepsilon}\left(\int_a^b \phi(x')dx' + \int_0^t g(t')dt'\right)} \end{aligned}\right\} \tag{19}$$

III. Non-Classical Variational Formulation:

Now, using the non-classical variation technique to solve approximately the problem:

$$\frac{\partial}{\partial t}\psi(x,t) = \varepsilon \frac{\partial^2}{\partial x^2}\psi(x,t) - \frac{1}{2\varepsilon}\psi(x,t)V(x,t), \ a < x < b, \ t > 0 \tag{20}$$

where:

$$V(x,t) = \int_a^x J(x', t)\, dx'$$

with initial condition:

$$\psi(x, 0; \varepsilon) = e^{-\frac{1}{2\varepsilon}\left(\int_a^x \phi(x')\,dx'\right)}, \quad a < x < b \tag{21}$$

and boundary conditions:

$$\left.\begin{aligned} \psi(a, t; \varepsilon) &= e^{-\frac{1}{2\varepsilon}\left(\int_0^t f(t')\,dt'\right)} \\ \text{and} \qquad & \qquad , \; t > 0 \\ \psi(b, t; \varepsilon) &= e^{-\frac{1}{2\varepsilon}\left(\int_a^b \phi(x')\,dx' + \int_0^t g(t')\,dt'\right)} \end{aligned}\right\} \tag{22}$$

Let:

$$\frac{\partial}{\partial t}\psi(x, t) - \varepsilon\frac{\partial^2}{\partial x^2}\psi(x, t) + \frac{1}{2\varepsilon}\psi(x, t)V(x, t) = 0$$

Define the operator by:

$$\left[\frac{\partial}{\partial t}. - \varepsilon\frac{\partial^2}{\partial x^2}. + \frac{1}{2\varepsilon}V(x,t)\right]\psi(x, t) = 0$$

Then, we have:

$$L\psi(x, t; \varepsilon) = 0 \tag{23}$$

Where $\psi(x, t; \varepsilon) \in \Gamma$, $\Gamma = C^2([a, b]\times[0, T])$, for some $T > 0$, where the linear operator L is defined by:

$$L = \left[\frac{\partial}{\partial t}. - \varepsilon\frac{\partial^2}{\partial x^2}. + \frac{1}{2\varepsilon}V(x,t)\right] \tag{24}$$

The linear operator L is not symmetric relative to the classical bilinear form, then a non-classical variational approach can be defined as:

Step.1: Define arbitrary bilinear form as follows:

$$(\psi_1, \psi_2) = \int_\Omega \psi_1\psi_2\,d\Omega \tag{25}$$

where $\Omega = \{(x, t) \mid x \in [a, b], t \in [0, T], T > 0\}$.

Step.2: Define new bilinear form, as follows:

$$<\psi_1, \psi_2> = (\psi_1, L\psi_2) \tag{26}$$

Where L is defined by (24).

Step.3: Construct the functional as:

$$F[\psi] = \frac{1}{2} <L\psi, \psi> - <f, \psi>$$

$$= \frac{1}{2} (L\psi, L\psi) - (f, L\psi)$$

and since f(x, t) ≡ 0, then:

$$F[\psi] = \int_a^b \int_0^T [L\psi(x,t)]^2 \, dtdx, \quad \text{for some } T \in (0, t) \tag{27}$$

From (3.24) and (3.27), we have:

$$F[\psi] = \frac{1}{2} \int_a^b \int_0^T \left[\frac{\partial}{\partial t} \psi(x, t) - \varepsilon \frac{\partial^2}{\partial x^2} \psi(x, t) + \frac{1}{2\varepsilon} \psi(x, t)V(x, t) \right]^2 dtdx \tag{28}$$

Step.4: To find the approximate solution to the functional (28) and to overcome the problem of the calculus of variations, the Ritz method which is mentioned in chapter two have been applied to get a suitable approximation (as a critical point of the functional (28)) and as follows:

$$\psi(x, t; \varepsilon) = W(x, t; \varepsilon) + \sum_{i=1}^{n} a_i(\varepsilon)G_i(x, t; \varepsilon) \tag{29}$$

where W(x, t; ε) is a chosen function of indicated variables satisfying the given non-homogeneous boundary an initial conditions (21)-(22) and $G_i(x, t; \varepsilon)$, i = 1, 2, …, n, are the suitable basis (for example polynomial functions) of the independent variables x and t. in the relation (29), the a_i's are constants to be determined, such that the functional (28) is satisfied.

The derivatives computations are as follows:

$$\frac{\partial\psi(x,t;\varepsilon)}{\partial t} = \frac{\partial W(x,t;\varepsilon)}{\partial t} + \sum_{i=1}^{n} a_i(\varepsilon) \frac{\partial G_i(x,t;\varepsilon)}{\partial t} \tag{30}$$

$$\frac{\partial^2\psi(x,t;\varepsilon)}{\partial x^2} = \frac{\partial^2 W(x,t;\varepsilon)}{\partial x^2} + \sum_{i=1}^{n} a_i(\varepsilon) \frac{\partial^2 G_i(x,t;\varepsilon)}{\partial x^2} \tag{31}$$

Then back substitution of equations (29), (30) and (31) into the functional (28), yields:

$$F[\psi] = \frac{1}{2}\int_a^b\int_0^T \left[\left(\frac{\partial W(x,t;\varepsilon)}{\partial t} + \sum_{i=1}^{n} a_i \frac{\partial G_i(x,t;\varepsilon)}{\partial t}\right) - \varepsilon \left(\frac{\partial^2 W(x,t;\varepsilon)}{\partial x^2} + \sum_{i=1}^{n} a_i \frac{\partial^2 G_i(x,t;\varepsilon)}{\partial x^2}\right) + \frac{1}{2\varepsilon} V(x,t;\varepsilon)\left(W(x,t;\varepsilon) + \sum_{i=1}^{n} a_i G_i(x,t;\varepsilon)\right)\right]^2 dtdx \quad (32)$$

Step.5: Take the derivatives of the functional (32) with respect to a_j's, j = 1, 2, …, n; to find the critical points of the functional (32),

$$\frac{\partial F}{\partial a_j} = 0, \; j = 1, 2, \ldots, n$$

$$\frac{1}{2}\int_a^b\int_0^T \frac{\partial}{\partial a_j}\left[\left(\frac{\partial W(x,t;\varepsilon)}{\partial t} + \sum_{i=1}^{n} a_i \frac{\partial G_i(x,t;\varepsilon)}{\partial t}\right) - \varepsilon \left(\frac{\partial^2 W(x,t;\varepsilon)}{\partial x^2} + \sum_{i=1}^{n} a_i \frac{\partial^2 G_i(x,t;\varepsilon)}{\partial x^2}\right) + \frac{1}{2\varepsilon} V(x,t)\left(W(x,t;\varepsilon) + \sum_{i=1}^{n} a_i G_i(x,t;\varepsilon)\right)\right]^2 dtdx = 0$$

$$\int_a^b\int_0^T \left[\left(\frac{\partial W(x,t;\varepsilon)}{\partial t} + \sum_{i=1}^{n} a_i \frac{\partial G_i(x,t;\varepsilon)}{\partial t}\right) - \varepsilon \left(\frac{\partial^2 W(x,t;\varepsilon)}{\partial x^2} + \sum_{i=1}^{n} a_i \frac{\partial^2 G_i(x,t;\varepsilon)}{\partial x^2}\right) + \frac{1}{2\varepsilon} V(x,t)\left(W(x,t;\varepsilon) + \sum_{i=1}^{n} a_i G_i(x,t;\varepsilon)\right)\right]$$

$$\left[\frac{\partial}{\partial a_j}\sum_{i=1}^{n} a_i \frac{\partial G_i(x,t;\varepsilon)}{\partial t} - \varepsilon \frac{\partial}{\partial a_j}\sum_{i=1}^{n} a_i \frac{\partial^2 G_i(x,t;\varepsilon)}{\partial x^2} + \frac{1}{2\varepsilon} V(x,tx)\frac{\partial}{\partial a_j}\sum_{i=1}^{n} a_i G_i(x,t;\varepsilon)\right] dtdx = 0$$

$$\int_a^b\int_0^T \left[\left(\frac{\partial W(x,t;\varepsilon)}{\partial t} - \varepsilon\frac{\partial^2 W(x,t;\varepsilon)}{\partial x^2} + \frac{1}{2\varepsilon}V(x,t)W(x,t;\varepsilon)\right) + \sum_{i=1}^{n} a_i \left(\frac{\partial G_i(x,t;\varepsilon)}{\partial t} - \varepsilon\frac{\partial^2 G_i(x,t;\varepsilon)}{\partial x^2} + \frac{1}{2\varepsilon}V(x,t)G_i(x,t;\varepsilon)\right)\right]\left[\frac{\partial}{\partial a_j}\sum_{i=1}^{n} a_i \left(\frac{\partial G_i(x,t;\varepsilon)}{\partial t} - \varepsilon\frac{\partial^2 G_i(x,t;\varepsilon)}{\partial x^2} + \frac{1}{2\varepsilon}V(x,t)G_i(x,t;\varepsilon)\right)\right] dtdx = 0$$

Let:

$$f_{ij}(x, t; \varepsilon) = h_i(x, t; \varepsilon)h_j(x, t; \varepsilon),\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n$$

where:

$$h_i(x, t; \varepsilon) = \frac{\partial G_i(x,t;\varepsilon)}{\partial t} - \varepsilon\frac{\partial^2 G_i(x,t;\varepsilon)}{\partial x^2} + \frac{1}{2\varepsilon}V(x,t)G_i(x,t;\varepsilon),\ i = 1,\ldots,n$$

and

$$d_i(x, t; \varepsilon) = h_i(x, t; \varepsilon)M(x, t; \varepsilon),\ i = 1, 2, \ldots, n$$

such that:

$$M(x, t; \varepsilon) = \frac{\partial W(x,t;\varepsilon)}{\partial t} - \varepsilon\frac{\partial^2 W(x,t;\varepsilon)}{\partial x^2} + \frac{1}{2\varepsilon}V(x,t)W(x,t;\varepsilon)$$

Step.6: Have a system of algebraic equations $C(\varepsilon)a(\varepsilon) = b^*(\varepsilon)$, where:

$$a(\varepsilon) = [a_1(t; \varepsilon), a_2(t; \varepsilon), \ldots, a_n(t; \varepsilon)]^T$$

and $C(\varepsilon)$ is $n\times n$ constant matrix defined as:

$$C(\varepsilon)=\begin{bmatrix} \int_a^b\int_0^T f_{11}(x,t;\varepsilon)dtdx & \int_a^b\int_0^T f_{12}(x,t;\varepsilon)dtdx & \cdots & \int_a^b\int_0^T f_{1n}(x,t;\varepsilon)dtdx \\ \int_a^b\int_0^T f_{21}(x,t;\varepsilon)dtdx & \int_a^b\int_0^T f_{22}(x,t;\varepsilon)dtdx & \cdots & \int_a^b\int_0^T f_{2n}(x,t;\varepsilon)dtdx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b\int_0^T f_{n1}(x,t;\varepsilon)dtdx & \int_a^b\int_0^T f_{n2}(x,t;\varepsilon)dtdx & \cdots & \int_a^b\int_0^T f_{nn}(x,t;\varepsilon)dtdx \end{bmatrix}$$

and

$$b^*(\varepsilon) = \begin{bmatrix} \int_a^b \int_0^T d_1(x,t;\varepsilon)dtdx \\ \int_a^b \int_0^T d_2(x,t;\varepsilon)dtdx \\ \vdots \\ \int_a^b \int_0^T d_n(x,t;\varepsilon)dtdx \end{bmatrix}$$

By using the Gauss-elimination algorithm for the system of algebraic equations, one can solve the problem of n-unknown variables as the solution of the system of algebraic equations, and since the set of functions are a complete set of functions and which are linearly independent functions, the system $C(\varepsilon)a(\varepsilon) = b^*(\varepsilon)$ has a unique solution.

Step.7: On using the solution (29), then:

$$u(x,t) = -2\varepsilon \frac{\frac{\partial}{\partial x}\psi(x,t)}{\psi(x,t)}$$

to find an approximate solution of the original problem (1)-(3).

IV. Illustrations

Problem (1):

Consider a non-homogeneous one-dimensional Burger's equation:

$$\frac{\partial}{\partial t}u(x,t) - \varepsilon\frac{\partial^2}{\partial x^2}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) = 4xt^4,\ 0 < x < 1,\ t > 0 \tag{33}$$

with the initial condition

$$u(x,0) = x,\ 0 < x < 1 \tag{34}$$

and the boundary conditions:

$$\left.\begin{aligned} u(0,t) &= 4t \\ u(1,t) &= \cos t \end{aligned}\right\},\ t > 0 \tag{35}$$

Case One: If $\varepsilon = 1$. Then the heat problem (17)-(19) becomes:

$$\frac{\partial\psi(x,t)}{\partial t} = \frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{1}{2}\psi(x,t)V(x,t),\ 0 < x < 1,\ t > 0 \tag{36}$$

where $V(x, t) = 2x^2t^4$, with initial condition:

$$\psi(x, 0; \varepsilon) = e^{-\frac{1}{4}x^2}, \quad 0 < x < 1 \qquad (37)$$

and boundary conditions:

$$\left.\begin{aligned} \psi(0, t; \varepsilon) &= e^{-t^2} \\ \text{and} \qquad\qquad & \qquad , t > 0 \\ \psi(1, t; \varepsilon) &= e^{\left(-\frac{1}{4}-\frac{1}{2}\sin t\right)} \end{aligned}\right\} \qquad (38)$$

Table 1: The approximate solution and the error of problem (36)-(38).

x	*t*	*ψ(x, t)*	*Error = E*
0.1	0.4	0.8562	0.0114
0.1	0.9	0.4776	0.0055
0.2	0.4	0.8539	0.0211
0.2	0.8	0.5720	0.0062
0.3	0.8	0.5846	0.0248
0.4	0.8	0.5913	0.0073
0.5	0.5	0.7585	0.0063
0.5	0.8	0.5924	0.0012
0.6	0.7	0.6377	0.0064
0.6	0.5	0.7373	0.0063
0.6	0.8	0.5888	0.00035
0.6	0.9	0.5416	0.0077
0.7	0.4	0.7544	0.0066
0.7	0.6	0.6675	0.0087
0.7	0.7	0.6238	0.0091
0.7	0.8	0.5812	0.0015
0.7	0.9	0.5400	0.0074
0.8	0.1	0.8240	0.0289

Table (2) show the approximate solution of the non-linear Burger's equation (33)-(35), when $\varepsilon = 1$.

Table 2: The approximate solution of problem (33)-(35), when ε = 1.

x	*t*	*Approximate solution u(x, t)*	*x*	*t*	*Approximate solution u(x, t)*
0.1	0.4	−0.0211	0.6	0.8	0.1945
0.1	0.9	−1.2158	0.6	0.9	0.0015
0.2	0.4	0.1269	0.7	0.4	0.8638
0.2	0.8	−0.5538	0.7	0.6	0.6639
0.3	0.8	−0.3276	0.7	0.7	0.5054
0.4	0.8	−0.1279	0.7	0.8	0.3171
0.5	0.5	0.4914	0.7	0.9	0.1070
0.5	0.8	0.0461	0.8	0.1	0.9377
0.6	0.5	0.6399	0.8	0.9	0.1700
0.6	0.7	0.3727	0.9	0.7	0.7223

Case Two: If ε = 0.1. Then the heat problem (17)-(19) becomes:

$$\frac{\partial\psi(x,t)}{\partial t} = 0.1\frac{\partial^2\psi(x,t)}{\partial x^2} - 5\psi(x,t)V(x,t),\ 0 < x < 1,\ t > 0 \qquad (39)$$

where $V(x, t) = 2x^2t^4$, with initial condition:

$$\psi(x, 0; \varepsilon) = e^{-2.5x^2},\ 0 < x < 1 \qquad (40)$$

and boundary conditions:

$$\left.\begin{array}{l}\psi(0, t; \varepsilon) = e^{-10t} \\ \text{and} \qquad\qquad ,\ t > 0\end{array}\right\} \qquad (41)$$

Table 3: The approximate solution and the error of problem (39)-(41).

x	*t*	*ψ(x, t)*	*Error = E*
0.1	0.7	1.0321	0.0097
0.2	0.8	0.9147	0.0072
0.3	0.7	0.8046	0.0970
0.4	0.8	0.6324	0.0083
0.4	0.9	0.5898	0.0096
0.4	0.6	0.6824	0.0137
0.5	0.1	0.5264	0.0546
0.5	0.3	0.5470	0.0373
0.5	0.4	0.5542	0.0097
0.6	0.2	0.3998	0.0571
0.6	0.3	0.4122	0.0515
0.6	0.8	0.3662	0.0689
0.7	0.3	0.2949	0.0858
0.7	0.8	0.2664	0.0075
0.7	0.9	0.2264	0.0822
0.8	0.3	0.1997	0.0750
0.8	0.7	0.2082	0.0040
0.9	0.2	0.1132	0.0992
0.9	0.6	0.1279	0.0087
0.9	0.5	0.1461	0.0279

Table (4) show the approximate solution of the non-linear Burger's equation (33)-(35), when $\varepsilon = 0.1$.

Table 4: The approximate solution of problem (33)-(35), when ε = 0.1.

x	*t*	*Approximate solution u(x, t)*	*x*	*t*	*Approximate solution u(x, t)*
0.1	0.7	0.1540	0.6	0.3	0.6172
0.2	0.8	0.2720	0.6	0.8	0.6154
0.3	0.7	0.3437	0.7	0.3	0.7230
0.4	0.8	0.4641	0.7	0.8	0.6518
0.4	0.9	0.5174	0.7	0.9	0.6989
0.4	0.6	0.4196	0.8	0.3	0.8405
0.5	0.1	0.5253	0.8	0.7	0.6853
0.5	0.3	0.5147	0.9	0.2	1.1061
0.5	0.4	0.5055	0.9	0.6	0.8191
0.6	0.2	0.6374	0.9	0.5	0.8557

V. Conclusions

The Cole-Hopf transformation is employed as a major rule to transform a nonlinear partial differential equation to a nonsymmetric (concerning inner product) linear partial differential equation. Due to the difficulties arising in solving many nonsymmetric linear partial differential equations with initial and boundary conditions, we are suggesting the method of non-classical approach to overcome the difficulties. The numerical results can be improved and obtain a better degree of accuracy when a large number of basis functions are selected or even when a type of a basis function is of a suitable linearly independent class of functions. It should be noted that the proposed procedure for solving non-homogeneous Burger's problem is easily and effectively to be implemented. As one can see from the numerical results, the number ε (kinematics viscosity) plays an important role in determining the approximate solution of Burger's problem, where the solution is completely dependent on ε.

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Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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