JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 Vol.-16, No.-9, September (2021) pp 46-58 ISSN (Print) 0973-8975

A NEW CONSTRUCTION OF OS OF SUBALGEBRAS AND INVARIANT SOLUTION OF THE BLACK-SCHOLES EQUATION

Zahid Hussain[1], Sadaqat Hussain[2], Suhail Abbas[3], Shams-ur-Rehman[4] Shahid Hussain[5]

[1,3,4,5] Assistant Professor, Department of Mathematical Science, Karakoram International University, Gilgit-Baltistan, Pakistan.

[2]Assistant Pofessor, University of Baltistan , Skardu Main Campus , Gilgit-Baltistan, Pakistan

[3]Assistant Professor, Ghizer Campus, Karakoram International University, Gilgit-Baltistan, Pakistan.

Email: [1]zahid.hussain@kiu.edu.pk, [2]sadaqatku@gmail.com, [3]suhail.abbas@kiu.edu.pk, [4]shams.rahman@kiu.edu.pk, [5]shahid8310@yahoo.com

Corresponding Author: **Shahid Hussain**

https://doi.org/10.26782/jmcms.2021.09.00005

(Received: July 9, 2021; Accepted: September 2, 2021)

Abstract

In this manuscript, the Lie group technique is applied to construct a new OS and invariant solutions of a one-dimensional LA, which describes the symmetries properties of a nonlinear Black-Scholes model. The structure of LA depends on one parameter. We have shown a novel way to construct the so-called OS of subalgebras of the Black-Scholes equation by utilizing the given symmetries. We transform the symmetries of the Black-Scholes equation into a simple ordinary differential equation called the Lie equation, which provides us a way through which to construct a new optimal scheme of subalgebras of the Black-Scholes through applying the concept of LE. The OS which consists of minimal representatives is utilized to develop the invariant solution for the Black-Scholes equation. The fundamental use of the Lie group analysis to the differential equation is the categorization of group invariant solutions of differential equations via OS. Finally, we have utilized the OS to construct the invariant solution of the Black-Scholes equation.

Keywords: Black-Scholes Equation, Generators, LE, OS, Invariant solution

I. Introduction

In the after mid-period of the 19th c., the renowned Norwegian Scientist Sophus Lie developed "the Lie theory of symmetry group for differential equations" which is the utmost contribution in applied group analysis. This theory is the most effective and swift solution method to tackle the nonlinear problem in the arena of applied mathematics. The fundamental concept of Lie's theory is founded on the invariance of the equation under transformation groups of the independent and dependent variables, called Lie groups. One of the chief achievements of Lie was to recognize that the properties of the global transformation of the group are wholly and distinctively determined using the infinitesimal transformation around the identity transformation. Stampfli and Victor [X] and Wilmot et al. [VI] gives a detailed analytical approach to the Black-Scholes model. In the current era, finance has to turn out to be one of the fastest emerging areas in the contemporary banking and cooperate world. Given this, together with modern sophisticated financial products (this development along with modern refined financial products), provides a swiftly growing impetus for novel mathematical models and methods. The eminent Black-Scholes model produced by Fischer Black and Myron Scholes in 1973 [XI] is used in nearly all financial markets to determine the prices of options. The Lie group analysis of the Black-Scholes equation is provided in Bordag [IV]. In applied group analysis, the Lie theory of symmetry group instead of differential equations represented by Sophus Lie is the utmost significant method to solve the nonlinear problems in the arena of applied mathematics. The rudiments of Lie's theory are founded on the invariance of the equation under transformation groups of independent and dependent variables so-called Lie groups.

In the preceding century, the application of the Lie group approach has been developed by many mathematicians; Ovsyannikov [IV], Olver [II], Ibragimov [III], Baumann [VIII], and Bluman and Anco [IX] are some of them who gave enormous amount of contribution to this domain of mathematics. The presence of symmetries of differential equations under the Lie group of transformations frequently permits those equations to get reduced to simple equations. One of the key accomplishments of Lie was to find that the features of the global transformations of the group are entirely and distinctively determined by the infinitesimal transformations around the identity transformation. This permits the nonlinear relations to deal with global transformation equations for the recognition of invariance groups. We use differential operators which can also be termed as the group generators whose exponentiation produces the action of the group. The set of these differential operators establishes the base for the LA. The Lie groups and the associated LAs have a one-to-one correspondence between them. A rudimentary problem regarding the group invariant solution is its division given by Yadav and Ali [I]. A Lie group (or LA) typically contains infinitely numerous subgroups (or subalgebras) of a similar dimensional that's why it is essential to classify them up to the attainment of some equivalent relation. The categorization of group invariant solutions of differential equations utilizing so-called OS is primarily coined by Ovsiannikov [V]. OS comprises representative elements of each equality class. Discussion on it can be found in the works of Ibragimov [XI] and Ovsiannikov [XII]. Some illustrations of OS can also be observed in Ibragimov's [XIII]. Moreover, OS of subalgebras in Ovsyannikov's [XIV], OS of sub=algebras of one solvable algebra L_7 in Chupakin's [XV], Group Invariant solutions and OSs for multidimensional hydrodynamics given by Coggeshall et al. [XVI] and Invariant Solutions of the Black-Scholes Equation is suggested by Pooe et al [XVII] are also worth mentioning.

In the current paper, we construct a new OS and invariant solution of the Black-Scholes equation by utilizing the given symmetries. The organization of the rest of this work is as follows. Section 2, is devoted to constructing generators and LE of the

Black -Scholes equation by using given symmetries. Section 3 contains the constructions of OS of subalgebras of the Black-Scholes equation. Subsequently, the invariant solution of the Black-Scholes equation is constructed in section 4 of the study. At last, the Conclusion is made in section 5.

II. Construction of Generators and Lie-equations

The Black-Scholes equation Black and Scholes [III]

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0 \qquad (1)$$

Where A, B, and C are constants and the equation admits/declares/states the LA L_6 spanned

by the succeeding six operators Gazizov and Ibragimov1 [XVIII] and Ibragimov [XIX]

$$S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\frac{\partial}{\partial x}, \quad S_3 = 2t\frac{\partial}{\partial t} + (lnx + Pt)x\frac{\partial}{\partial x} + 2Ctu\frac{\partial}{\partial u},$$
$$S_4 = A^2tx\frac{\partial}{\partial x} + (lnx - Pt)u\frac{\partial}{\partial u}, \quad S_5 = 2A^2t^2\frac{\partial}{\partial t} + 2A^2txlnx\frac{\partial}{\partial x} + \left[(lnx - Pt)^2 + 2A^2Ct^2\right]u\frac{\partial}{\partial u}, \quad (2)$$
$$S_6 = u\frac{\partial}{\partial u}.$$

Where

$$P = B - \frac{A^2}{2}.$$

We use the formula Ibragimov [XX] and Ibragimov [XXI] to calculate the commutators.

$$\left[S_\alpha, S_\beta\right] = S_\alpha S_\beta - S_\beta S_\alpha = \sum_{i=1}^{n}\left(S_\alpha\left(\xi_\beta^i\right) - S_\beta\left(\xi_\alpha^i\right)\right)\frac{\partial}{\partial x^i} \qquad (3)$$

where $i = 1, 2, 3$ and $\alpha, \beta = 1, 2, ..., 6.$

The commutators of the symmetries (2) of the Black-Scholes equation (1) has the following form:

Table 1: The commutators table for symmetries (2)

	S_1	S_2	S_3	S_4	S_5	S_6
S_1	0	0	$2S_1+PS_2+2C S_6$	$A^2 S_2-P S_6$	$2A^2 S_3-2P S_4 -A^2 S_6$	0
S_2	0	0	S_2	S_6	$2S_2$	0
S_3	$-2S_1-PS_2-2C S_6$	$- S_2$	0	S_4	$2S_5$	0
S_4	$-A^2 S_2+P S_6$	$- S_6$	$-S_4$	0	0	0
S_5	$-2A^2 S_3+2P S_4 +A^2 S_6$	$-2 S_4$	$- 2S_5$	0	0	0
S_6	0	0	0	0	0	0

$$S = l^1 S_1 + \ldots + l^6 S_6 \qquad (4)$$

which rests on six arbitrary constants $l^1 + \ldots + l^6$.
To make this problem wieldy Ovsyannikov [V] and [XII] presented the idea of OSs of subalgebras by mentioning that if two algebras are similar i.e. linked to each other by the transformation of the symmetry group, then their corresponding invariant solutions are linked with each other via the same transformation. Thus, it is enough to deal with an OS of invariant solutions which are acquired in our case as follows. We placed all similar operators' $S \in L_6$ into one class and choose a representative of each class. The collection of the representatives of all these classes is an OS of one-dimensional subalgebras.

Let us make an OS of one-dimensional subalgebras of LA L_6 pursuing the same simple method which is used by Ovsyannikov [XII] and Ibragimov [XX]. The transformations contained in groups of symmetry with the LA $S \in L_6$ gives the group having 6-parameters of linear transformations of the operators L_6 or equivalently, linear transformations of the vector

$$l = \left(l^1 + \ldots + l^6 \right) \qquad (5)$$

where $l^1 + \ldots + l^6$ are obtained from the arbitrary operator provided by equation (3.1). To find these linear transformations we make use of their generating elements as by Ibragimov in [VII] and [I, XIX],

$$G_\mu = c_{\mu\nu}^{\lambda} l^\nu \frac{\partial}{\partial l^\lambda}, \qquad \mu = 1, \ldots, 6 \qquad (6)$$

where the $c_{\mu\nu}^{\lambda}$ of the LA L_6 are structure constants which can be defined as

$$\left[S_\mu, S_\nu \right] = c_{\mu\nu}^{\lambda} S_\lambda \qquad (7)$$

Now, let us find, the operator G_1. According to the equation (6), this is written as

$$G_1 = c_{1\nu}^{\lambda} l^\nu \frac{\partial}{\partial l^\lambda} \qquad (8)$$

Where $c_{1\nu}^{\lambda}$ are defined by the commutators

$$\left[S_1, S_\nu \right] = c_{1\nu}^{\lambda} S_\lambda \qquad (9)$$

i.e. by the first row from table 1, viz., the non-vanishing $c_{\mu\nu}^{\lambda} S_\lambda$ is appeared by setting $\nu = 3$ and $\lambda = 1, 2, 6$

$$\left[S_1, S_3 \right] = c_{13}^{1} S_1 + c_{13}^{2} S_2 + c_{13}^{6} S_6 \qquad (10)$$

where $c_{13}^{1} = 2$, $c_{13}^{2} = P$ and $c_{13}^{6} = 2C$ are non-vanishing constant. Similarly, we can easily find the rest of the LAs defined in (7) as follows

$\left[S_1, S_5 \right] = c_{15}^{3} S_3 + c_{15}^{4} S_4 + c_{15}^{6} S_6$ and $\left[S_1, S_3 \right] = c_{13}^{1} S_1 + c_{13}^{2} S_2 + c_{13}^{6} S_6$

From (8), we have the operator G_1,

$$G_1 = c_{13}^{\lambda} l^3 \frac{\partial}{\partial l^{\lambda}} + c_{14}^{\lambda} l^4 \frac{\partial}{\partial l^{\lambda}} + c_{15}^{\lambda} l^5 \frac{\partial}{\partial l^{\lambda}}$$

$$= c_{13}^{1} l^3 \frac{\partial}{\partial l^1} + c_{13}^{2} l^3 \frac{\partial}{\partial l^2} + c_{13}^{6} l^3 \frac{\partial}{\partial l^6} + c_{14}^{2} l^4 \frac{\partial}{\partial l^2} + c_{14}^{6} l^4 \frac{\partial}{\partial l^6} + c_{15}^{3} l^5 \frac{\partial}{\partial l^3} + c_{15}^{4} l^5 \frac{\partial}{\partial l^4} + c_{15}^{6} l^5 \frac{\partial}{\partial l^6} \tag{11}$$

If we Place the values of non-vanishing constants from commutator table 1 in (11), then we have

$$G_1 = 2l^3 \frac{\partial}{\partial l^1} + (Pl^3 + A^2 l^4) \frac{\partial}{\partial l^2} + 2A^2 l^5 \frac{\partial}{\partial l^3} - 2Pl^5 \frac{\partial}{\partial l^4} + \left(2Cl^3 - Pl^4 - A^2 l^5\right) \frac{\partial}{\partial l^6} \tag{12}$$

Similarly, we can find generators $G_2 + G_6$ by utilizing table 1 and (8) as follows

$$G_2 = l^3 \frac{\partial}{\partial l^2} + 2l^5 \frac{\partial}{\partial l^4} + l^4 \frac{\partial}{\partial l^6} \tag{13}$$

$$G_3 = -2l^1 \frac{\partial}{\partial l^1} - (Pl^1 + l^2) \frac{\partial}{\partial l^2} + l^4 \frac{\partial}{\partial l^4} + 2l^5 \frac{\partial}{\partial l^5} - 2Cl^1 \frac{\partial}{\partial l^6} \tag{14}$$

$$G_4 = -A^2 l^1 \frac{\partial}{\partial l^2} - l^3 \frac{\partial}{\partial l^4} + \left(Pl^1 - l^2\right) \frac{\partial}{\partial l^6} \tag{15}$$

$$G_5 = -2A^2 l^1 \frac{\partial}{\partial l^3} + 2(Pl^1 - l^2) \frac{\partial}{\partial l^4} - 2l^3 \frac{\partial}{\partial l^5} + A^2 l^1 \frac{\partial}{\partial l^6} \tag{16}$$

where the generator G_6 is trivial.

Now, we will construct LE by utilizing the transformations obtained from the generating elements (12) - (16). The transformation in the form of LE with the parameter a_1 for the generator G_1 in (12) can be denoted as

$$\left.\begin{array}{lll} \frac{d\tilde{l}^1}{da_1} = 2\tilde{l}^3, & \frac{d\tilde{l}^2}{da_1} = \left(P\tilde{l}^3 + A^2\tilde{l}^4\right), & \frac{d\tilde{l}^3}{da_1} = 2A^2\tilde{l}^5, \\ \frac{d\tilde{l}^4}{da_1} = 2P\tilde{l}^5, & \frac{d\tilde{l}^5}{da_1} = 0, & \frac{d\tilde{l}^6}{da_1} = \left(2C\tilde{l}^3 - P\tilde{l}^4 + A^2\tilde{l}^5\right) \end{array}\right\} \tag{17}$$

On integrating the (17) and utilizing the initial condition $\tilde{l}\,|_{a_1=0} = l$. We get

$$\left.\begin{array}{l} G_1 : \tilde{l}^1 = 2A^2 l^5 a_1^2 + 2a_1 l^3 + l^1, \quad \tilde{l}^2 = \left(Pl^3 + A^2 l^4\right) a_1 + l^2, \\ \quad \tilde{l}^3 = 2A^2 l^5 a_1 + l^3, \quad \tilde{l}^4 = -2Pl^5 a_1 + l^4, \quad \tilde{l}^5 = l^5, \\ \quad \tilde{l}^6 = l^5\left(2A^2 C + P^2\right) a_1^2 + \left(2Cl^3 - Pl^4 - A^2 l^5\right) a_1 + l^6 \end{array}\right\} \tag{18}$$

Considering the operators (13) – (16), the transformations in the form of Lie equation with the parameter a_1 for the generators G_2 □ G_6 can be described as

$$\left.\begin{array}{l} G_2 : \tilde{l}^1 = l^1, \quad \tilde{l}^2 = l^3 a_2 + l^2, \quad \tilde{l}^3 = l^3, \quad \tilde{l}^4 = 2l^5 a_2 + l^4, \\ \quad \tilde{l}^5 = l^5, \quad \tilde{l}^6 = l^5 a_2^2 + l^4 a_2 + l^6 \end{array}\right\} \tag{19}$$

$$\left.\begin{aligned} G_3: \tilde{l}^1 = l^1 a_3^{-2}, \quad \tilde{l}^2 = P l^1 a_3^{-2} + \left(l^2 - P l^1\right) a_3^{-1}, \quad \tilde{l}^3 = l^3, \\ \tilde{l}^4 = l^4 a_3, \quad \tilde{l}^5 = l^5 a_3^2, \quad \tilde{l}^6 = C\left(a_3^{-2} - 1\right) l^1 + l^6 \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} G_4: \tilde{l}^1 = l^1, \quad \tilde{l}^2 = -A^2 l^1 a_4 + l^2, \quad \tilde{l}^3 = l^3, \quad \tilde{l}^4 = -l^3 a_4 + l^4, \\ \tilde{l}^5 = l^5, \quad \tilde{l}^6 = \frac{1}{2} A^2 l^1 a_4^2 + \left(P l^1 - l^2\right) a_4 + l^6 \end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned} G_5: \tilde{l}^1 = l^1, \quad \tilde{l}^2 = l^2, \quad \tilde{l}^3 = -2A^2 l^1 a_5 + l^3, \quad \tilde{l}^4 = 2\left(P l^1 - l^2\right) a_5 + l^4, \\ \tilde{l}^5 = 2A^2 l^1 a_5^2 - 2 l^3 a_5 + l^5, \quad \tilde{l}^6 = A^2 l^1 a_5 + l^6 \end{aligned}\right\} \tag{22}$$

$$G_6: \tilde{l}^1 = l^1, \quad \tilde{l}^2 = l^3 l^2, \quad \tilde{l}^3 = l^3, \quad \tilde{l}^4 = l^4, \quad \tilde{l}^5 = l^5, \quad \tilde{l}^6 = l^6 \Big\} \tag{23}$$

It is pertinent to mention that the transformations (18) - (23) maps the vector $S \in L_6$ given by (4) to the vector $\tilde{S} \in L_6$ is denoted by the following relation

$$\tilde{S} = \tilde{l}^1 S_1 + \ldots + \tilde{l}^6 S_6 \tag{24}$$

Upon integrating the equations

$G_\mu(J) = 0$, where $\mu = 1, \ldots, 6$. This provides us enough information about the invariant that is given by the following

$$J = \left(l^3\right)^2 - 2A^2 l^1 l^5 \tag{25}$$

Knowledge of invariant (25) significantly simplifies further calculations.

The last equation (23) shows that if $l^1 \neq 0$, we obtain $l^5 = 0$ by solving the quadratic equation

$2A^2 l^1 a_5^2 - 2 l^3 a_5 + l^5 = 0$. Solving for a_5 we get

$$a_5 = \frac{l^3 \pm \sqrt{J}}{2A^2 l^1} \tag{26}$$

where J is invariant (24). It is worthy to mention here that we can only use (26) if $J \geq 0$.

III. Construction of an OS of subalgebras

Now, we start to construct the OS. The technique we use demands a simplification of the general vector (5) through the transformations (18) - (23). Consequently, we will obtain the representatives of each class of similar vectors in the simplest form (5). Substituting these representatives in (4), we will get the OS of one-dimensional subalgebras L_6. We will split the construction of OS into the following several cases.

Now, we form the OS of the Black-Scholes equation by using given symmetries as follows

(1) The case $l^1 = 0$.

Now, we bifurcate the case (1) into the succeeding subclasses namely, (a) $l^5 \neq 0$ and (b) $l^5 = 0$. Firstly, case (1) $l^1 = 0$ is being discussed with subcase (a) $l^5 \neq 0$. Considering the vector (5) and let $l^1 = 0$ in (5), we get the following form

$$\left(0, l^2, l^3, l^4, l^5, l^6\right) \text{ provided that } l^5 \neq 0$$

On using $l^5 \neq 0$ and the transformation (18), we obtain the following reduced vector of the form

$$\left(0, l^2, 0, l^4, l^5, l^6\right)$$

To further reduce the above vector, we use the transformation (20) and get the following reduced form

$$\left(0, l^2, 0, 0, l^5, l^6\right)$$

(a) (i)If $l^2 \neq 0$, then considering (21) and (20), we get the following representatives for the OS

$$S_2 + S_5 \text{ and } S_2 - S_5 \tag{27}$$

(a) (ii) If we assume that $l^2 = 0$, then the vector reduced to the following form

$$\left(0, 0, 0, 0, l^5, l^6\right)$$

If we further consider $l^6 \neq 0$ and divide it by l^5, we find the reduced vector

$$\left(0, 0, 0, 0, 1, k\right)$$

and the representative contributed towards an OS

$$S_5 + kS_6 \tag{28}$$

If $l^6 = 0$, then we get the following reduced vector

$$\left(0, 0, 0, 0, 1, 0\right)$$

which provides us the following representative to the OS

$$S_5 \tag{29}$$

Secondly, we consider the case (A) $l^1 = 0$ with subcase (b) $l^5 = 0$. Considering the vector (5) and let $l^1 = 0$ and $l^5 = 0$ in (5), we get the following form

$$\left(0, l^2, l^3, l^4, 0, l^6\right)$$

(b) (i) Suppose that $l^3 \neq 0$ and utilizing the transformations (19), (21), and (18), we have

$$\left(0, 0, l^3, 0, 0, 0\right)$$

and which results in the acquirement of another representative for the OS

$$S_3 \tag{30}$$

(b) (ii) Now, we consider the case when $l^3 = 0$. If $l^3 = 0$, we get the vector of the form

$$\left(0, l^2, 0, l^4, 0, l^6\right)$$

Now, we assume that if $l^4 \neq 0$, then by using transformations (19) and (18), a new reduce vector is obtained as given below;

$$\left(0, 0, 0, l^4, 0, 0\right)$$

The above vector contributed the following representative for the OS

$$S_4 \tag{31}$$

Now, let us take into consideration the case when $l^4 = 0$, we get the reduced vector

$$\left(0, l^2, 0, 0, 0, l^6\right)$$

(b) (ii)º If $l^2 \neq 0$ then from the equation (21), we obtain some reduced vectors which are;

$$\left(0, l^2, 0, 0, 0, 0\right) \text{ and } \left(0, 0, 0, 0, 0, l^6\right)$$

which provides us with the representatives given below for constructing an OS

$$S_2 \text{ and } S_6 \tag{32}$$

Now, we utilize the invariant J to find the more suitable representatives for OS.

(2) The case $l^2 \neq 0$ with $J < 0$. It is evident from the (2) that

$$J = \left(l^3\right)^2 - 2A^2 l^1 l^5 \neq 0$$

provided that $l^5 \neq 0$. Therefore, we can successfully use the transformations (22), (19), (18), and (21) with $a_5 = -\frac{l^6}{A^2 l^1}$, $a_2 = -\frac{l^4}{2l^5}$, $a_1 = -\frac{l^3}{2A^2 l^5}$ and $a_4 = \frac{l^2}{A^2 l^1}$ respectively. Then, we obtain $l^2 = l^3 = l^4 = l^6 = 0$ which yields the following reduced vector

$$\left(l^1, 0, 0, 0, l^5, 0\right)$$

The components l^1 and l^5 of the directly above vectors share a common emblem since the condition $J < 0$ produces $l^1 l^5 > 0$. Therefore, utilizing the transformation (20) with an apt value of the parameter a_3 and invoking that we can multiply the vector l by any constant, we find $l^1 = l^5 = 1$ and get the following representative for the OS

$$S_1 + S_5 \tag{33}$$

(3) The case $l^2 \neq 0$ with $J = 0$. We use (26) $a_5 = \frac{l^3}{2A^2 l^1}$. If $l^3 \neq 0$ then we can utilize (22) with the transformation $a_5 = \frac{l^3}{2A^2 l^1}$ and let $\tilde{l}^3 = 0$. Due to the

invariance, J we deduce that the equation $J=0$ yields $\left(\tilde{l}^3\right)^2 - 2A^2\tilde{l}^1\tilde{l}^5 = 0$. Since $\tilde{l}^3 = 0$ follows that $\tilde{l}^5 = 0$. In consequence, we can deal with the vector of the form

$$\left(l^1, l^2, 0, l^4, 0, l^6\right)$$

Furthermore, if $l^3 = 0$, we have $J = 2A^2l^1l^5$ and the equation $J=0$ yields $l^5 = 0$. Given that $l^1 \neq 0$, we derive the same vector as (28). By utilizing (21), we have $a_4 = \frac{l^2}{A^2l^1}$ and this produces $l^2 = 0$. Again from (21) we get $2Pl^1a_5 + l^4 = 0$ where $a_5 = -\frac{l^4}{2Pl^1}$ and therefore $l^4 = 0$. Likewise, from (22) we make use of $A^2l^1a_5 + l^6 = 0$ which provides us $a_5 = -\frac{l^6}{A^2l^1}$ with yields $l^6 = 0$. Choosing $l^1 = 1$, we obtain the new reduced vector as follows

$$\left(l^1, 0, 0, 0, 0, 0\right)$$

Which contribution for the OS is

$$S_1 \tag{34}$$

The case $l^2 \neq 0$ with $J > 0$. Since J is invariant under the transformation (18)-(23) The condition $J > 0$ results in the vector mentioned above that is either-or $l^5 \neq 0$. If $l^5 \neq 0$ then l^1 and l^5 should have opposite signs $l^1l^5 < 0$. Since l^1 and l^5 having opposite signs for the condition $J > 0$ to hold. Assuming this possibility holds and that is $l^1, l^5 \neq 0$, from (18), (19), (21), and (22), we deduce that $a_1 = -\frac{l^3}{2A^2l^5}$, $a_2 = -\frac{l^4}{2l^5}$, $a_4 = \frac{l^2}{A^2l^1}$ and $a_5 = -\frac{l^6}{A^2l^1}$ respectively. The reckoning shows that l^2, l^3, l^4 and l^6 are equal to zero. In consequence, we have the following reduced vector of the form

$$\left(l^1, 0, 0, 0, l^5, 0\right)$$

Therefore, it contributes the following representative for the construction of the OS

$$\left(l^1, 0, 0, 0, 0, 0\right)$$

Which contribution for the OS is

$$S_1 - S_5 \tag{35}$$

Finally, collecting all the operators (27) - (35), we arrive at the OS

S_1, S_2, S_3, S_4, S_5, S_6,

$S_1 + S_5$, $S_1 - S_5$, $S_2 + S_5$, $S_2 - S_5$, $S_5 + kS_6$

where k is an arbitrary parameter.

The constructed OS of subalgebras can be utilized to develop an invariant solution of the Black-Scholes equation.

IV. Novel invariant solution of Black-Scholes equation

In this section, we utilize each representative of the constructed OS to develop a new invariant solution of the Black-Scholes and also categorize the group invariant solutions of differential equations by utilizing OS as under;

$S_1: \quad u = C_1 x^{\lambda_1} + C_2 x^{\lambda_2},$

where

$$\lambda_1 = -\frac{P + \sqrt{P^2 + 2A^2C}}{A^2}, \quad \lambda_1 = -\frac{P - \sqrt{P^2 + 2A^2C}}{A^2}.$$

$S_2: \quad u = Ke^{Ct}$

$$S_3: \quad u = Ke^{Ct} \int exp(\frac{J_2^2}{2A^2}) dJ_2 + K_1 e^{Ct};$$

OR

$$S_3: \quad u = AKe^{Ct} \sqrt{\frac{\pi}{2}} erfi\left(\frac{J_2}{A\sqrt{2}} \right) + K_1 e^{Ct}, \ J_2 = P\sqrt{t} - \frac{lnx}{\sqrt{t}}.$$

$$S_4: \quad u = \frac{K}{\sqrt{t}} exp\left(\frac{(lnx)^2}{2A^2 t} - \frac{Plnx}{A^2} + t\left(\frac{P^2}{2A^2} + C \right) \right).$$

$$S_5: \quad u = \frac{1}{\sqrt{t}} \left(\frac{K_1}{t} lnx + K_2 \right) exp\left(\frac{P^2}{2A^2 t} (lnx - Pt)^2 + Ct \right).$$

$$S_1 + S_5: \quad u = exp\left(-\frac{Plnx}{A^2} + \frac{t(lnx)^2}{2A^2t^2+1} + \frac{P^2 t}{2A^2} - \frac{P^2}{2^{\frac{3}{2}} A^3} tan^{-1}\sqrt{2}At + Ct - \frac{C}{\sqrt{2}A} tan^{-1}\sqrt{2}A^2 t \right.$$
$$\left. -\frac{1}{4} ln(2A^2t^2+1) \varphi(J_1) \right)$$

$$J_1 = (2A^2t^2+1)^{\frac{-1}{2}} (lnx), \quad \varphi'' + \frac{2}{A^2}(J_1 - K)\varphi = 0,$$

where

$$K = \frac{P^2}{2A^2} + C.$$

$$S_1 - S_5: \quad u = exp\left(-\frac{Plnx}{A^2} + \frac{t(lnx)^2}{2A^2t^2+1} + \frac{P^2 t}{2A^2} - \frac{P^2}{2^{\frac{3}{2}} A^3} tanh^{-1}\sqrt{2}At + Ct - \frac{C}{\sqrt{2}A} tanh^{-1}\sqrt{2}At \right.$$
$$\left. -\frac{1}{4} ln(2A^2t^2-1) \varphi(J_1) \right)$$

$$J_1 = (2A^2t^2-1)^{\frac{-1}{2}} (lnx), \quad \varphi'' - \frac{2}{A^2}\left(J_1^2 - \frac{P^2}{2A^2} - C \right)\varphi = 0. \text{ where } K = \frac{P^2}{2A^2} + C.$$

$$S_2+S_5: \quad u=\frac{1}{\sqrt{t}}\left(C_1A_i(z)+C_2B_i(z)\right)exp\left(\frac{1}{2A^2}\left(\frac{1}{6A^4t^3}+\frac{lnx}{A^2t^2}-\frac{P}{A^2t}+\frac{(lnx)^2}{t}+P^2t-2Plnx\right)+Ct\right)$$

$$A_i(z)=\frac{1}{\pi}\int_0^{\infty}cos\left(z\tau+\frac{1}{3}\tau^3\right)d\tau$$

$$B_i(z)=\frac{1}{\pi}\int_0^{\infty}exp\left(z\tau-\frac{1}{3}\tau^3\right)+sin\left(z\tau+\frac{1}{3}\tau^3\right)d\tau.$$

$$S_2-S_5: \quad u=\frac{1}{\sqrt{t}}\left(C_1A_i(z)+C_2B_i(z)\right)exp\left(\frac{1}{2A^2}\left(\frac{1}{6A^4t^3}-\frac{lnx}{A^2t^2}+\frac{P}{2A^2t}+\frac{(lnx)^2}{t}+P^2t-2Plnx\right)+Ct\right)$$

$$A_i(z)=\frac{1}{\pi}\int_0^{\infty}cos\left(z\tau+\frac{1}{3}\tau^3\right)d\tau$$

$$B_i(z)=\frac{1}{\pi}\int_0^{\infty}exp\left(z\tau-\frac{1}{3}\tau^3\right)+sin\left(z\tau+\frac{1}{3}\tau^3\right)d\tau.$$

$$S_5+kS_6: \quad u=\frac{1}{\sqrt{t}}\left(k_1+k_2x^{\frac{-\alpha}{t}}\right)exp\left(\frac{t}{2A^2}\left(\frac{1}{t}lnx-P\right)^2+Ct-\frac{k}{2A^2t}\right).$$

V. Conclusion

In this article, we have constructed a novel OS of the Black-Scholes equation. The given symmetries of the Black-Scholes equation are being utilized to construct the commutators table which gave us a way to construct generators and Lie-equations. We developed OS where the method necessitates a simplification of the general vector (5) through the transformations (18) - (23). As a consequence, we have found the simplest representatives of each class of similar vectors (5). Moreover, by placing these representatives in (4), we have obtained the OS of one-dimensional subalgebras of L_6 which is utilized to develop the invariant solution of the Black-Scholes equation. The chief application of the Lie group analysis to the differential equation via OS is the division of group invariant solutions of differential equations.

VI. Acknowledgement

This manuscript is an extract of our master's thesis and we are very grateful to Professor Nail H. Ibragimov (Late) for his productive guidance and deliberations.

Note: *Throughout the text, we use the abbreviations BSE, OS, LE and LA for the terms Black-Scholes equation, Optimal system, Lie equation and Lie algebra respectively.*

Conflicts of Interest:

There is no conflict of interest regarding this article

Reference

I. A.K. Yadav,. And, A.T Ali,. "An OS and Invariant solution of Dark Energy Models in cylindrically symmetric space-time", *The European Physical Journal plus*, *Eur. Phys. J. Plus*, pp. 129-179 (2014).

II. A.P. Coggeshall and Meyer-ter-Vehn, "Group Invariant solutions and OSs for multidimensional hydrodynamics", J. Math. Phys 33,(1994).

III. A.P. hupakin. "OS of sub algebras of one solvable algebra L7", Lie Groups and Their Applications 1 (1994).

IV. C.A. Pooe. Mahomed F.M. and Wafo Soh C. Invariant Solutions of the Black- Scholes Equation, Math. & Compt. Apps 8, p p. 63- 70 (2003).

V. F. Black. and M. Scholes, "The pricing of options and corporate liabilities", *J. Pol. Econ*. **81,** pp. 637 - 659 (1973).

VI. G. Baumann, "Symmetry Analysis of Differential Equations with Mathematica. Telos, *Springer Verlag", New York*, (2000).

VII. G.W. Bluman,.and S.C. Anco,. "Symmetry and Integration Methods for Differential Equations", *Springer-Verlag, New York* (2002).

VIII. J. Stampfli, and Victor, "The Mathematics of finance, in: Brooks/ Cole series in Advance Mathematics, Brooks/ cole, Pacific Grove", CA, *Modeling and Hedging*, (2001).

IX. L. A. Bordag. "Option-valuation in illiquid markets: invariant solutions to a nonlinear model in mathematical Control Theory and Finance", *eds.A. Sarychev, M. Guerra and M. R. Grossinho*, **81,** pp.72-94, "Springer",2008.

X. L.V Ovsyannikov,. "Group analysis of differential equations", *Nuaka, Moscow, 1978. English transl., ed W.F. Ames, Academic Press, New York* (1982).

XI. L.V. Ovsyannikov ,"Group properties of differential equations", *Siberian Branch, USSR Academy of Sciences, Novosibirsk* (1962).

XII. Mohammad Asif Arefin, Biswajit Gain, Rezaul Karim, Saddam Hossain. : 'A COMPARATIVE EXPLORATION ON DIFFERENT NUMERICAL METHODS FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS'. J. Mech. Cont.& Math. Sci., Vol.-15, No.-12, December (2020) pp 1-11. DOI : 10.26782/jmcms.2020.12.00001.

XIII. N. H. Ibragimov. "CRC Handbook of Lie group analysis of differential equations. Volume1,2,3 CRC Press", *Boca Raton, Ann Arbor, London, Tokyo,* (1994,1995,1996).

XIV. N.H. bragimov. "Exercises for courses based on Lie group analysis", ALGA *Publication, Karlskrona* (2008).

XV. N.H. Ibragimov, "Analytical Methods in Mathematical Modelling", *ALGA Publication Karlskrona* (2008).

XVI. N.H. Ibragimov, "Elementary Lie group analysis and ordinary differential equations", John Wiley and Sons, Chichester (1999).

XVII. N.H. Ibragimov, "Transformation Groups Applied to Mathematical Physics CRC Press*", Boca Raton, F.L*, (1985).

XVIII. N.H. bragimov, "A practical course in differential equations and mathematical modelling", Third Edition, ALGA Publication, Karlskrona (2006)

XIX. Ovsyannikov,. OS of subalgebras, *Lie Groups and Their Applications 1* (1994).

XX. P. J. Olver,. "Applications of Lie Groups to Differential Equations", *GTM 107, Second edn., Springer-Verlag* (1993).

XXI. P. Wilmot, S. Howison, and Dewynne, "The Mathematics of financial Derivatives", *Cambridge University Press, Cambridge,* (1997).

XXII. R. K, Gazizov. and , N.H Ibragimov1, "Lie symmetry analysis of differential equations in finance", Nonlinear Dynamics, 17, pp. 387 – 407(1998).

XXIII. Sivaram. B. : 'COMPARATIVE STUDY OF SOLUTION METHODS OF NON-HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS'. *J. Mech. Cont.& Math. Sci., Vol.-16, No.-1, January (2021) pp 1-18.* DOI : 10.26782/jmcms.2021.01.00001.