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NUMERICAL HYBRID ITERATIVE TECHNIQUE FOR SOLVING NONLINEAR EQUATIONS IN ONE VARIABLE

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Abstract

In recent years, some improvements have been suggested in the literature that has been a better performance or nearly equal to existing numerical iterative techniques (NIT). The efforts of this study are to constitute a Numerical Hybrid Iterative Technique (NHIT) for estimating the real root of nonlinear equations in one variable (NLEOV) that accelerates convergence. The goal of the development of the NHIT for the solution of an NLEOV assumed various efforts to combine the different methods. The proposed NHIT is developed by combining the Taylor Series method (TSM) and Newton Raphson's iterative method (NRIM). MATLAB and Excel software has been used for computational purpose. The developed algorithm has been tested on variant NLEOV problems and found that convergence is better than the bracketing iterative method (BIM), which does not observe any pitfall and is almost equivalent to NRIM.

Keywords: Numerical hybrid iterative technique; Nonlinear equations in one variable; Bracketing iterative method; Newton Raphson's iterative method; Taylor series method

I. Introduction

In the 21st century, the sciences and even the arts are accepted as the basics essential to scientific computations. The era of computational mathematics is extended to advanced studies in mathematics and computer science and such studies have produced various novel algorithms for solving problems numerically. Such applied problems have logically determined real-world algebra, geometry and calculus in engineering, biological, natural sciences, social sciences, and physical sciences. The awareness of the convergence study of the numerical applications is essential for the research process, which is concerned with the derivation, analysis, and implementation of methods for estimating the reliable degree of numerical accuracy, which may be close to the exact analytical solution of mathematical models. Many modifications are proposed in the literature, and these are providing better performance or are almost equal to existing BIM and open iterative methods (OIM).

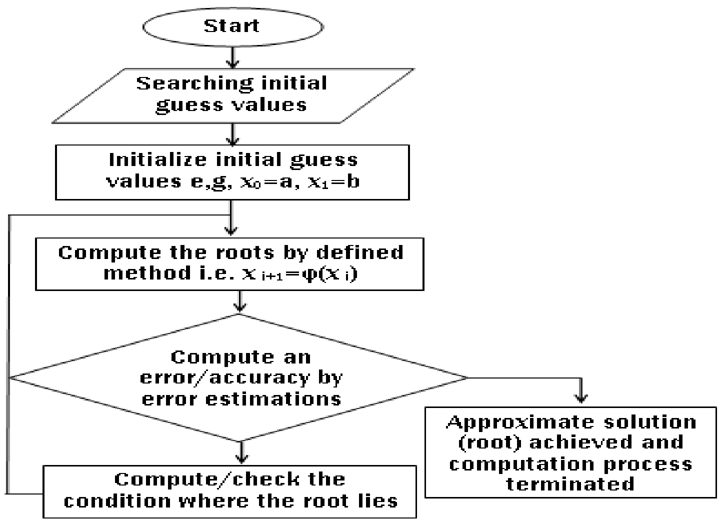
The iterative numerical analysis is a mathematical technique that produces an arrangement of successful approximate solutions for NLEOV problems. A specific execution procedure of NIT that including the termination norms of the NIT algorithms with an initial guess (s) are used, [XIII]. The numerical analysis technique says convergent whenever the corresponding classification converges at the given initial(s). The NIT may converge or diverges, and if the convergence occurs, the method should be cyclical until the appropriate accuracy is achieved because there may have a solution, and if the divergence occurs, the method should be terminated because there may be no solution. The following three points are required for iterative methods to determine the approximate real root of the nonlinear equations $f(x)=0$;

i. Initial guesses values of the method, where the real root lies in between the guess values.

ii. An algorithm (systematic procedure) of the method should be applied for approximation.

iii. For terminating the computations, either accuracy criteria have been achieved or the number of iterations.

In case of convergence, the Flowchart-1 (criteria) should be applied for computations terminating. Continue the computations procedure until the required or pre-assigned accuracy (accuracy measure by error estimations); this iterative computational procedure is defined in flowchart-1.



Flowchart-1: Iterative computational process

In applied mathematics, several complex problems contain the algebraic and transcendental types in the NLEOV, often to used NIT to estimate the approximate solution of such problems because its exact solution is not always possible to determine by analytical methods or usual algebraic procedure [VII]. A literature survey is a research-stayed effort for researchers in any era because the development of algorithms has been a historically significant initiative. In the past time, great researchers and mathematicians have persisted in deep touch with scientific applications, and in many cases, this has succeeded in the discovery of NIT still in use today. Several NIT approaches have been suggested and analyzed in the literature for the approximate real root or solution for solving NLEOV $f(x)=0$ [I].

To estimates the real root of nonlinear equations, the NHIM has been derived and compared with the existing NIM, which may be an efficient method. The iterative structure NHIT has been developed by using the thoughts of TSM and NRIM. In addition to being compatible with the NHIT, it provides, in fact, a fabulous stability characteristic. To estimate the approximate real root with the essential role of computer programming languages, MATLAB, MATHEMATICA, MAPEL, FORTRAN / Force, and C / C++ can also be used EXCEL. Therefore, in this study, for estimations have been used MATLAB and excel applications. The solution of the several problems of NLEOV i.e. $f(x)=0$ by newly developed NIT that gave the most powerful systematically desired response as compared to existing NIT.

II. Methodology

This section highlights the methodology of the study with research-supported efforts of the various researchers for the approximate solution (real root) of nonlinear equations $f(x)=0$.

II.i. Taylor's Series Method (TSM)

The methodology of the series was developed by the Scottish mathematician James Gregory and officially presented in 1715 by mathematician Brook Taylor. Taylor series is a way for estimating the significance of a function by compelling the sum of its derivatives at a specified point. It's an extension of a sequence around a level when the Taylor series is defined at zero, which is the unique and different case of the Taylor series. That case of the series is called a Maclaurin series, after the 18th-century Scottish mathematician Colin Maclaurin, who made significant and general use of this unique and different case of the Taylor series.

A function can be estimated by the Taylor series method using its number of finite terms. Taylor's series method provides measurable approximations on the error presented by the use of such an estimate. The Taylor polynomial is made by appearing some initial terms in the series. The Taylor series of a function has the boundary of the Taylor polynomials that function as the degree increases till it occurs. An open interval function equal to its Taylor series is regarded in that interval as an analytical function. The power series is the Taylor series of a real or complex function markedly differentiable at point "a".

$$\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n} \tag{1}$$

Many researchers have developed the NIT schemes for the solution of NLEOV $f(x)=0$ by using the well-known Taylor series (TS) or interpolating polynomials. A new iterative technique by Taylor series expansion to order three to solve NLEOV problems further explained that iterative technique is earlier convergence than NR method, and new NIT proposed iteration needs the minimum number of calculations in the given functions. Many NIT has been found using quite different techniques such as Taylor's series, Quadrature formulas, Homotopy, interpolation, Decomposition, and its various amendments discussed [XII, XIV].

II.ii. Newton Raphson's Iterative Method (NRIM)

More than four centuries have passed since a technique for the solution of the algebraic nonlinear equation was suggested by (Newton in 1669) and after that by (Raphson in 1690), according to [XI], who was "Raphson"?, the procedure of the method is now known as Newton's method or Newton Raphson's (NR) method and quiet a dominant method for the solution of nonlinear equations.

Newton's method is known as Newton Raphson's method, also said to be the N-R method by which is well-known a root-determining technique that evaluates approximating roots (or zeroes) of a real-value function established by named after Isaac Newton and Joseph Raphson in numerical analysis. The iterative scheme defines a function for a single-variable x i.e. $f(x)$, differentiation of the function i.e. $f'(x)$ for a real root of a function at an initial guess x_0. The NRM scheme is a type of open iterative algorithm, whereas the algorithm requirement is one initial guess value for estimating the approximate real root[IX]. This technique is quite often used to improve the results obtained from other iterative approaches. The iterative technique (2) formed by the Taylor series (TS) also can be derived by using the slope (tangent) definition.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

In recent years, many modifications have been proposed in the literature, which has almost equal or better performance than NRM. Among other existing methods, NRM is well known the rate of convergence is quadratic, faster and efforts for computer programming code is easy for solving the nonlinear equation $f(x)=0$. However, having pitfall of this method in some cases at any point of computation that NRM does not work properly or diverges if slope at an initial guess is very small or lies away from the real root or tangent is parallel to the x-axis i.e. $f'(x)=0$. It is well known that derivative is used in NRIM, but some researchers have suggested without derivative NIT to govern the real root for solving NLEOV [XII, XX]. Several NIT was developed to the improvement of quadratically convergent NRIM to obtain the best convergence order it. Previously, several procedures have been done to estimate the approximate real root of nonlinear algebraic and transcendental equations $f(x)=0$, debated [IV, XV]. For the similar estimation of the equations, the alternatives of Newton's formulas have explained [III], however [V] proposed the multi-step iterative technique for it. The NRIT is the best prevalent and the most well-known method for solving nonlinear equations, few historical points about this method were discussed [XV, XVII]. Modification for the iterative method to determine multiple roots or simple zeros of $f(x)=0$, where f is a real-valued continuous function on $[a,b]\subseteq R$, several examples solve by numerically and compared with modified Newton's method and Ujević methods [XII] in the identical accuracy with the number of iterations and time of Computer machine.

Various NIT for approximations NLEOV discussed by researchers in recent years. These methods have been developed using accuracy examination methodologies, quadrature rules, or other techniques for the solution of NLEOV [XVIII, VI]. The efficiency of the developed algorithms was analyzed by error estimations, quadrature procedures, or other numerical procedures explored [V, VIII, IX, XIX]

II.iii. Working systematic theme of proposed NHIT for estimation of NLEOV

As proposed NHIT belongs to BIM and its solution procedure same as the Bisection method (BM) and False Position method (FPM). The following steps are described as the working theme of the proposed NHIT to determines the approximate real root of the NLEOV.

a) Determine two points or intervals or two initial guesses x_0 and x_1 by hit and trial method satisfying the following convergence criteria or inequality i.e.

$$f(x_0)\,f(x_1)<0 \tag{3}$$

Let $[x_0, x_1]=[a, b]$ are the two initial guesses for the approximate real root, such that

$$f(x_0)\,f(x_1)=f(a)\,f(b)<0 \tag{4}$$

or in other words, $f(x)$ (the ordinate) changes sign between $x_0=a$ and $x_1=b$ or can say the approximate real root lies in between $x_0=a$ and $x_1=b$.

b) Estimate the approximate real root x_n of NLEOV $f(x)=0$, by using proposed NHIT i.e.

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_{n-1})+(x_n-x_{n-1})\,f''(x_{n-1})} \tag{5}$$

Evaluate $f(x_n)$ and check convergence criteria

i. If $f(x_n)=0$, then real root x_n, is an exact root of NLEOV $f(x)=0$ and stop the procedure.

ii. If $f(x_n)\neq 0$, then further steps will proceed and two other procedural conditions are possible:

- If $f(a)f(x_n)<0$, then estimate x_{n+1}, the new iterative approximate real root lies in between a and x_n; then $b=x_n$ and a remain same, using (5).
- If $f(x_n)f(b)<0$, then estimate x_{n+1}, the new iterative approximate real root lies in between x_n and b; then $a=x_n$ and b remain same, using (5).

c) Again evaluate $f(x_{n+1})$ and check convergence criteria as above.

d) Estimate required accuracy by using as per the given type of approximate error either use absolute error i.e.

$$|\in| = \left| x_n^{new} - x_n^{old} \right| \tag{6}$$

or use percentage error i.e.

$$|\in| = \left| \frac{x_n^{new} - x_n^{old}}{x_n^{new}} \right| \times 100 \tag{7}$$

Where $x_n^{new} =$ estimated an approximate real root from the present iteration and $x_n^{old} =$ estimated an approximate real root from the previous iteration.

e) Compare the approximate error $|\in|$ with the pre-defined tolerance accuracy or a required level of tolerance accuracy $\in_s$. If $|\in| > \in_s$;

- Then the process is repeated and goes to step c) either until the obtained accuracy or achieved a better result or repeated according to the number of iterations.
- Else stop the procedure.

III. Results and Discussions

In this section, the various results based on efforts are taken for new algorithms are explored to determine a real root of nonlinear equations (NLE) $f(x)=0$. The numerical comparison shows that the proposed hybrid iterative algorithm for solving nonlinear (algebraic/transcendental) equations in one variable (HIA) is comparable with the existing numerical iterative algorithms and is fruitful.

The study merging the various methods of bracketing iterative methods (BIM) and open iterative methods (OIM) was discussed and analyzed, rendering the detailed literature review and introduction. This research is focused on various parameters and it has been observed that proposed NHIT for estimation of the algebraic and transcendental type of functions of NLEOV have been an appropriate technique. It also has one more advantage; it works properly and converges for the functions where NRM diverges. It is better than BIM. Further, it is noticed that its convergence is about equivalent to NRM. With the combination of the NRM and two terms of the TSM, the proposed technique NHIT for solving NLEOV has been constructed the powerful iterative technique to determine the real root. Furthermore, this new proposed Hybrid Iterative Algorithm (HIA) is close converging to NRM. It is a good achievement of the research study to estimate the approximate real root for the solution of NLE $f\left(x\right)=0$.

In Tab-01, illustrative, the different types of the algebraic and transcendental NLEOV problems are tested and demonstrate the obtained results by conventional and proposed NHIT based on accuracy and efficiency. In Tab-01, the comparison of the results is highlighted by the number of iterations of existing numerical approaches with NHIT to determine the approximate real root of various NLOVE with zero tolerance accuracy. The numerical comparison shows that NHIT is the powerful and more reliable NIT algorithm for the complex mathematical problems for the solution of NLOVE $f\left(x\right)=0$.

Table 1: Results and Comparisons

Sr. No	Polynomials	Initial guess	f(x)	Approximate Real Root	No. of Iterations till to Zero tolerance			
					BIM	FP IM	NR IM	NHIT
1	x^3 - x + 9	-2	3	-2.240040987	40	23	6	**6**
		-3	-15					
2	$0.2x^4 + x^2 – 1$	0	-0.1	0.313171217	>70	70	Fail	**7**
		1	1.1					
3	$x^3 + 2x^2 – 1$	0	-1	0.618033989	>32	32	Fail	**7**
		1	2					
4	x^3 - 9x + 1	2	-9	2.942820058	52	11	5	**7**
		3	1					
5	$xe^x – 2$	0	-2	0.852605502	>23	23	5	**7**
		1	0.718281828					
6	e^x - 4Sin(x)	0	1	0.370558096	25	21	Fail	**20**
		1	-0.647602111					
7	e^x - Sin(x) - 2Cos(x) + ½	0	-0.5	0.547127317	47	33	Fail	**12**
		1	1.296206232					
8	Cos(x) - Sinh(x)	0	1	0.703290659	53	17	6	**6**
		1	-0.634898888					
9	2Cosh(x)Sin (x) – 1	0.4	-0.158021178	0.466833756	57	18	5	**5**
		0.5	0.081225371					
10	e^{-x} - Sin(x)	0.5	-0.209272207	0.588532744	>11	11	5	**8**
		0.75	0.127105121					

Tab-02 compares various conventional iterative techniques and proposed NHIT for estimation of NLEOV and shows that the proposed technique is working tremendously and its convergence is fast and assured, whereas NRIM diverges.

Table 2: Conclusions

Numerical Methods	Convergence	Stability	Accuracy	Comments
Numerical Hybrid Iterative Technique (NHIT)	**Fast**	**Always**	**Good**	• **The hybrid model was developed as BIM.** • **This method is the combination of the NRM and Taylor series.** • **Where NRM fails/diverge, but this developed Hybrid method work positively.** • **Convergence assured.**
BM	Slow	Always	Good	• BIM • Linearly convergence
FPM	Medium	Always	Good	• BIM • Linearly convergence
NRM	Fast	Not always/ May diverge	Good	• OIM • NRM usually converges quadratically. • Requires two functions for each iteration, i.e. $f(x_n)$ and $f'(x_n)$. • Complexity of calculating $f'(x_n)$. • May converge slowly at first, however, as iterates come nearer to the real root, the speed of convergence increase.
FPIM	Slow	Not always/ May diverge	Good	• OIM • Time-consuming in choosing $g(x_n)$. • The possible behaviour of this method iterates x_n for various sizes of $g(x_n)$.
SM	Medium	Always	Good	• BIM • Similar to FP • Super Linearly convergence

IV. Conclusions

The various algebraic and transcendental problems tested on proposed NHIT and on existing iterative techniques are explored to determine the real root of NLEOV and observed $f(x)=0$. The proposed NHIT work properly and the real root with zero tolerance accuracy also observed in NHIT improved the iterations compared to existing methods presented in Tab-01. In the case where the existing algorithms of bracketing Iterative methods (BIM) are slow converge but proposed NHIT has fast convergence, it is further analyzed that proposed NHIT is successful converge where NIRM has numerical difficulties or fail to converge. Furthermore, this new proposed NHIT is close converging to NRIM; this is also a good achievement of the present research study to estimate the approximate real root for the solution of NLEOV $f(x)=0$. Hence, the NHIT is an improved technique to determine the approximate real root for algebraic, trigonometric and transcendental types of polynomials.

As iterative methods for solving nonlinear equations, $f(x)=0$ one variable plays a dynamic role in solving the problems/application problems rising from several science fields. Therefore, this proposed Hybrid Iterative Algorithm is a more reliable tool for complex problems for the solution of NLEOV $f(x)=0$.

V. Authors' Contributions/Acknowledgement

All the authors mutually contributed to the research thoughts for this framework and brought it to an end. The first, fourth, and fifth authors discussed and developed the nonlinear one variable equations algorithm, i.e. $f(x)=0$. The first author has computationally coded the algorithm and obtained results and then wrote this manuscript. Thereafter, the second and fourth have verified the results. Finally, the third and fifth authors have critically reviewed and acknowledged the findings of this work and encouraged the methodology of the manuscript.

Conflicts of Interest:

The authors declare that they have no conflicts of interest to report regarding the present study.

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