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# MAPS BETWEEN TANGENTIAL COMPLEXES FOR PROJECTIVE CONFIGURATIONS

**Sadaqat Hussain[1], Zahid Hussain[2], Shahid Hussain[3], Raziuddin Siddiqui[4]**

[1] Assistant Professor, Department of Mathematics, University of Baltistan Skardu, Pakistan

[2]Assistant Professor, Department of Mathematics, Karakoram International University Gilgit, Pakistan.

[3]Assistant Professor, Department of Mathematics, Karakoram International University Gilgit, Pakistan.

[4]Assistant Professor, Department of Mathematics, Institute of Business Administration, Karachi, Pakistan.

Email: sadaqat.hussain@uobs.edu.pk, zahid.hussain@kiu.edu.pk, shahid8310@yahoo.com, rsiddiqui@iba.edu.pk

Corresponding Author: **Shahid Hussain**

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## Abstract

*Grassmannian bi-complex contains two types of differential maps $d$ and $d'$. This complex is related to the Tangent complex by Siddiqui for the differential map. In this article, we try to find morphisms in tangential configuration space to relate Grassmannian complex and first-order tangent complex for differential map $d'$.*

**Keywords :** Grassmannian complex, Configuration, Vector Space, Cross-Ratio, Tangent Complex.

## I. Introduction

A.A.Suslin defined the Grassmannian complex over geometric configurations [XIV] whereas A.B. Goncharove related this complex to the famous Bloch-Suslin complex [V], [VI] as well as to his own motivic complexes. After that Cathelineau [II] worked on the "infinitesimal" and “Tangential” versions of Goncharov complexes by defining vector spaces $\boldsymbol{\beta}_n(F)$ and $T\mathcal{B}_n(F)$. Siddiqui brings the geometry of configuration spaces into infinitesimal and tangential settings. He computed the cross-ratios and Goncharov’s triple-ratios for the truncated polynomial rings $F[\varepsilon]_2$and $F[\varepsilon]_3$ [XI]. He connects the Grassmannian complex

$$d: C_m(A^2_{F[\varepsilon]_2}) \to C_{(m-1)}(A^2_{F[\varepsilon]_2}) \tag{1.1}$$

$$d: (x_0, \dots, x_m) \mapsto \sum_{i=0}^{m} (-1)^m (x_1, \dots, \hat{x_i}, \dots, x_m)$$

to the first order tangent complex of weight 2 .i.e.

$$\partial : T\mathcal{B}_2(F) \mapsto F \otimes F^\times + \wedge^2 F^\times$$
$$\partial (\langle a; b]_2) \mapsto \left(\frac{b}{a} \otimes (1-a) + \frac{b}{1-a} \otimes a\right) + (\frac{b}{1-a} \wedge \frac{b}{a})$$

by introducing maps $\tau^2_{0,\varepsilon}$ and $\tau^2_{1,\varepsilon}$. Also, he gives maps $\tau^3_{0,\varepsilon}$, $\tau^3_{1,\varepsilon}$ and $\tau^3_{2,\varepsilon}$ to connect the Grassmannian subcomplex (2) to the first order tangent complex of weight 3 .i.e.

$$T\mathcal{B}_3(F) \xrightarrow{\partial_\varepsilon} T\mathcal{B}_2(F) \otimes F^\times \oplus F \otimes \mathcal{B}_2(F) \xrightarrow{\partial_\varepsilon} F \otimes \wedge^2 F^\times + \wedge^3 F$$

Where

$$\partial_\varepsilon(\langle a; b]_3) \mapsto (\langle a; b]_2 \otimes a + \frac{b}{a} \otimes [a]_2)$$ , and

$$\partial_\varepsilon(\langle a; b]_2 \otimes c + x \otimes [y]_2) \mapsto \left(-\frac{b}{1-a} \otimes a \wedge c - \frac{b}{a} \otimes (1-a) \wedge c + x \otimes (1-y) \wedge y\right) + \left(\frac{b}{1-a} \wedge \frac{b}{a} \wedge x\right)$$

He also gives proof of the commutativity of resulting diagrams (see §4 $and$ §5 of [XI]. What we are going to do in this work is searching for suitable morphisms between the tangent complex and Grassmannian complex with projective differential map $d'$. For this purpose, we introduce maps $\pi^2_{0,\varepsilon}$ and $\pi^2_{1,\varepsilon}$ for weights 2 and$\pi^3_{0,\varepsilon}$, $\pi^3_{1,\varepsilon}$ and $\pi^3_{2,\varepsilon}$ for weight 3. After that, we give proof of the commutativity of the resulting diagrams. There should be no confusion about the maps $d$ and $d'$ because both these maps occur in the Grassmannian complex and are totally different from each other. That's why our work is different from that of the work in [XI] and [XII].

## ll. Preliminaries:

### Grassmannian Sub Complexes

For any set $X$, let $C_m(X)$ be a free abelian group generated by the elements $(x_0, \dots, x_m) \epsilon X^{m+1}$ . Then we have a simplicial complex $( C_m(X), d)$ , where

$$d: (x_0, \dots, x_m) \mapsto \sum_{i=0}^{m} (-1)^m (x_1, \dots, \hat{x}_i, \dots, x_m)$$

For any group $G$ acting on $X$, we call elements of the quotient set $G/X^m$ configurations of the elements of $X$. Let $C_m(n)$ be a free abelian group generated by the configurations of m-vectors
Of an $n$-dimensional vector space$V_n$. The configuration $(x_i | x_1, \dots, \hat{x_i}, \dots, x_m)$ is Called a projective configuration of $m - 1$ vector projected from $x_i$. Define a projective differential map $d'$ as

$$d: C_{(m+1)}(n+1) \to C_m(n)$$

$$d^{'}: (x_0, \dots, x_m) \mapsto \sum_{i=0}^{m} (-1)^m (x_i | x_1, \dots, \hat{x_i}, \dots, x_m)$$

Using the above two differential maps we can form the following Complexes known as Grassmannian complexes

$$\dots \xrightarrow{d} C_{n+3}(n) \xrightarrow{d} C_{n+2}(n) \xrightarrow{d} C_{n+1}(n)$$

$$\dots \xrightarrow{d'} C_{n+3}(n+2) \xrightarrow{d'} C_{n+2}(n+1) \xrightarrow{d'} C_{n+1}(n)$$

## Cathelineau's Complex

In [II] Cathelineau introduces another version of Goncharov and Bloch groups by using the infinitesimal procedure. He defines the infinitesimal vector spaces as

$$\beta_1(F) = F$$

Where $\boldsymbol{F}$ is any field of characteristic 0. Also

$$\beta_2(F) = \frac{F[F^{\bullet\bullet}]}{r_2(F)}$$

Where $F^{\bullet\bullet} = \mathrm{F} - \{0,1\}$ and $\boldsymbol{r_2(F)}$ is the subspace of $F[F]$ generated by the following

$$[a] - [b] + a\left[\frac{b}{a}\right] + (1-a)\left[\frac{1-b}{1-a}\right]$$

And there is a map $\delta_2: F[F^{\bullet\bullet}] \longrightarrow F^{\times} \otimes F^{\times}$defined as

$$[a] \mapsto a \otimes a + (1-a) \otimes (1-a)$$

Such that $r_2(F) \subseteq \ker\partial_2$. From the above setup, we have a complex

$$\delta: \beta_2(F) \longrightarrow F^{\times} \otimes F^{\times}$$

Called infinitesimal complex of weight 2, where $\partial$ is induced by $\delta_2$. Further, Infinitesimal complex for weight 3 can be defined as

$$\beta_3(F) = \frac{F[F^{\bullet\bullet}]}{r_3(F)}$$

where $r_3(F)$ is the kernel of the map

$$\delta_3: F[F^{\bullet\bullet}] \longrightarrow \beta_2(F) \otimes F^{\times} \oplus F \otimes \mathcal{B}_2(F)$$
$$[a] \mapsto \langle a \rangle_2 \otimes a + (1-a) \otimes [a]_2$$

where $\langle \boldsymbol{a} \rangle_{\boldsymbol{2}}$ is a class of $[\boldsymbol{a}]$ in $\beta_2(F)$, we can form a complex for the map $\delta$ as

$$\beta_3(F) \xrightarrow{\delta} \beta_2(F) \otimes F^{\times} \oplus (F \otimes \mathcal{B}_2(F)) \xrightarrow{\delta} F \otimes \wedge^3 F^{\times}$$

Where

$$\delta: \langle a \rangle_3 \mapsto \langle a \rangle_2 \otimes a + (1-a) \otimes [a]_2$$

and

$$\delta: \langle a \rangle_3 \otimes b + x \otimes [y]_2 \mapsto -(a \otimes a \wedge b + (1-a) \otimes (1-a) \wedge b) + x \otimes (1-y) \wedge y$$

where $\langle a \rangle_3$ is a class of $[a]$ in $\beta_3(F)$.

## Tangent complex:

The first order tangent group for weight 2 can be denoted by $T\mathcal{B}_2(F)$ (see [X] , [XI] ) and defined as a $\mathbb{Z}$ −module generated by the elements of the form

$$\langle a; b]_2 \in \mathbb{Z}[F[\varepsilon]_2] \text{ (see [11])}$$

and quotient by the expression

$$\langle a; a'] - \langle b; b'] + \left\langle \left(\frac{b}{a}\right); \left(\frac{b}{a}\right)' \right] - \left\langle \left(\frac{1-b}{1-a}\right); \left(\frac{1-b}{1-a}\right)' \right] + \left\langle \left(\frac{a(1-b)}{b(1-a)}\right); \left(\frac{a(1-b)}{b(1-a)}\right)' \right], \; a, b \neq 0$$

Where

$$\langle a; a'] = [a + a'\varepsilon] - [a] \;, \left(\frac{b}{a}\right)' = \frac{ab' - a'b}{a^2} \,, \left(\frac{1-b}{1-a}\right)' = \frac{a'(1-b) - b'(1-a)}{(1-a)^2}$$

and

$$\left(\frac{a(1-b)}{b(1-a)}\right)' = \frac{a'(1-b)b - b'(1-a)a}{(b(1-a))^2}$$

Using this group we can form the following complex

$$\partial : T\mathcal{B}_2(F) \mapsto F \otimes F^{\times} + \wedge^2 F^{\times}$$

$$\partial\, (\langle a; b]_2) \mapsto \left(\frac{b}{a} \otimes (1-a) + \frac{b}{1-a} \otimes a\right) + \left(\frac{b}{1-a} \wedge \frac{b}{a}\right) \qquad (1.2)$$

called Cathelineau's or tangential complex.

The group $T\mathcal{B}_3(F)$ is defined in [XI] and [XII] as a tangent group of weight 3. In the same paper, this group is used to obtain the cathelineau complex

$$T\mathcal{B}_3(F) \xrightarrow{\partial_\varepsilon} T\mathcal{B}_2(F) \otimes F^{\times} \oplus F \otimes \mathcal{B}_2(F) \xrightarrow{\partial_\varepsilon} F \otimes \wedge^2 F^{\times} + \wedge^3 F$$

where $\partial_\varepsilon(\langle a; b]_3) \mapsto (\langle a; b]_2 \otimes a + \frac{b}{a} \otimes [a]_2)$ (1.3)

and

$$\partial_\varepsilon(\langle a; b]_2 \otimes c + x \otimes [y]_2) \mapsto \left(-\frac{b}{1-a} \otimes a \wedge c - \frac{b}{a} \otimes (1-a) \wedge c + x \otimes (1-y) \wedge y\right) + \left(\frac{b}{1-a} \wedge \frac{b}{a} \wedge x\right) \qquad (1.4)$$

## Tangential configurations:

Let $F$ be any field, then we can define the $\nu^{th}$ truncated polynomial ring as

$$\mathrm{F}[\varepsilon]_\nu = \frac{F[\varepsilon]}{\varepsilon^\nu}$$

Also, define $A^n_{F[\varepsilon]_\nu}$ be an n-dimensional affine space over the truncated polynomial ring $\mathrm{F}[\varepsilon]_\nu$.

We take $= (a_1, a_2, a_3, \ldots, a_n) \in A^n_F \backslash\{(0,0,\ldots,0)\}$ , $l_\varepsilon = \left(a_{1,\varepsilon}, a_{2,\varepsilon}, a_{3,\varepsilon}, \ldots, a_{n,\varepsilon}\right) \in A^n_F$ and in a similar way we can take

$$l_{\varepsilon^{\nu-1}} = \left(a_{1,\varepsilon^{\nu-1}}, a_{2,\varepsilon^{\nu-1}}, a_{3,\varepsilon^{\nu-1}}, \ldots, a_{n,\varepsilon^{\nu-1}}\right) \in A^n_F \text{ ,}$$

then one can write $l^* = l + l_\varepsilon \varepsilon + l_{\varepsilon^2}\varepsilon^2 + \cdots + l_{,\varepsilon^{\nu-1}}\varepsilon^{\nu-1} \in A^n_{F[\varepsilon]_\nu}$ (see [IX]). Then we can define a differential map $d$ for the free abelian groups $C_m(\, A^n_{F[\varepsilon]_\nu})$ generated by the configurations of m-vectors in n-dimensional affine space over $\mathrm{F}[\varepsilon]_\nu$

$$d\colon C_{m+1}(A^n_{F[\varepsilon]_\nu}) \to C_{(m)}(A^n_{F[\varepsilon]_\nu}) \ \ ,(l^*_0, \ldots, l^*_m) \mapsto \sum_{i=0}^m (-1)^m (l^*_0, \ldots, \hat{l}^*_i, \ldots, l^*_m)$$

We also find another map $d'$ called projective differential map as

$$d'\colon C_{m+1}(A^n_{F[\varepsilon]_\nu}) \to C_{(m)}(A^{n-1}_{F[\varepsilon]_\nu})$$

$$(l^*_0, \ldots, l^*_m) \mapsto \sum_{i=0}^m (-1)^m (l^*_i | l^*_0, \ldots, \hat{l}^*_i, \ldots, l^*_m) \tag{1.5}$$

## Determinants over tangential configurations:

We denote the $2 \times 2$ determinant by $\Delta(l_i{}^*, l_j{}^*) \in \mathrm{F}[\varepsilon]_2$ with $l_i \in A^2_{F[\varepsilon]_2}$. Then

$$\Delta(l_i{}^*, l_j{}^*) = \Delta(l_i{}^*, l_j{}^*)_{\varepsilon^0} + \Delta(l_i{}^*, l_j{}^*)_{\varepsilon^1}$$

where

$$\Delta(l_i{}^*, l_j{}^*)_{\varepsilon^0} = \Delta\left(l_i, l_j\right) \ ; \Delta(l_i{}^*, l_j{}^*)_{\varepsilon^1} = \Delta\left(l_i, l_{j,\varepsilon}\right) + \Delta\left(l_{i,\varepsilon}, l_j\right)$$

In our main result we use a short notation for determinants as $\Delta\left(l_i{}^*, l_j{}^*\right) = \left(l_i{}^* l_j{}^*\right)$ that is we ignore delta and commas.

## Cross-ratio:

Let $(l^*_0, l^*_1, l^*_2, l^*_3) \in A^2_{F[\varepsilon]_2}$ be the configuration of four points, then the cross-ratio over the truncated polynomial ring $F[\varepsilon]_2$ can be defined as

$$r(l_o, l_1, l_2, l_3) = \frac{\Delta(l_o,l_3)\Delta(l_1,l_2)}{\Delta(l_o,l_2)\Delta(l_1,l_3)} \ \ r_\varepsilon(l^*_0, l^*_1, l^*_2, l^*_3) = \frac{\{\Delta(l_0{}^*,l_3{}^*)\Delta(l_1{}^*,l_2{}^*)\}_\varepsilon}{\Delta(l_o,l_2)\Delta(l_1,l_3)} - r(l_o, l_1, l_2, l_3)\frac{\{\Delta(l_0{}^*,l_2{}^*)\Delta(l_1{}^*,l_3{}^*)\}_\varepsilon}{\Delta(l_o,l_2)\Delta(l_1,l_3)}$$

See [11] and [9] for more details. It is clear that the cross-ratio is well defined as it depends neither upon the length of vectors nor upon the volume formed by these vectors.

## III. Results and Discussions

Our main result in this paper is to propose morphisms between the Grassmannian sub-complex of projective differential map $d'$ and the tangent complex. First of all, we discuss such types of morphisms for weight 2.

Since the tangent or Cathelineau's complex for weight 2 is

$$\partial : T\mathcal{B}_2(F) \mapsto F \otimes F^\times + \wedge^2 F^\times$$

Take the Grassmannian sub-complex of projective differential map $d'$ as

$$d' : C_4(A^2_{F[\varepsilon]_2}) \rightarrow C_3(A^1_{F[\varepsilon]_2})$$

Here we propose morphisms $\pi^2_{0,\varepsilon}$ and $\pi^2_{1,\varepsilon}$ to connect the above two complexes. Then we have the diagram (3.1a).

where

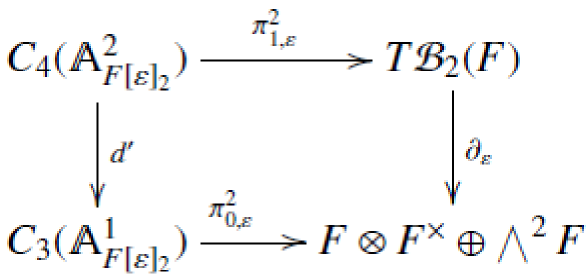
$$\partial_\varepsilon(\langle a; b]_2) \mapsto \left(\frac{b}{a} \otimes (1-a) + \frac{b}{1-a} \otimes a\right) + \left(\frac{b}{1-a} \wedge \frac{b}{a}\right) \tag{2.1}$$

$$\pi^2_{0,\varepsilon}(l_0^*, l_1^*, l_2^*) = \sum_{i=0}^{2}(-1)^i \left\{\frac{D(l_i^*)_\varepsilon}{l_i} \otimes \frac{l_{i+1}}{l_{i+2}} + \frac{(l_i{}^*)}{(l_{i+1}{}^*)} \wedge \frac{(l_{i+1}{}^*)}{(l_{i+2}{}^*)}\right\}, \ i \ \ mod \ \ 3 \tag{2.2}$$

$$\pi^2_{1,\varepsilon}(l_0^*, l_1^*, l_2^*, l_3^*) = \langle r(l_o, l_1, l_2, l_3), r_\varepsilon(l_0^*, l_1^*, l_2^*, l_3^*)] \tag{2.3}$$

It is very simple to verify that $\pi^2_{0,\varepsilon}$ and $\pi^2_{1,\varepsilon}$ are both well-defined because both these maps neither depend upon the length of the vectors nor upon the volume form of these vectors. So we directly come to verify the following results.

**Theorem 3.i.a.** The following diagram




is commutative. i.e.

$$\pi^2_{0,\varepsilon} \ o \ d' = \partial_\varepsilon o \ \pi^2_{1,\varepsilon}$$

**Proof:** Take the left-hand side and use the definition of $d'$, we get

$$\pi^2_{0,\varepsilon} \ o \ d'(l_0^*, l_1^*, l_2^*, l_3^*) = \tag{2.4}$$

$$\pi^2_{0,\varepsilon}(l_0^*|l_1^*, l_2^*, l_3^*) - \pi^2_{0,\varepsilon}(l_1^*|l_0^*, l_2^*, l_3^*) + \pi^2_{0,\varepsilon}(l_2^*|l_0^*, l_1^*, l_3^*) - \pi^2_{0,\varepsilon}(l_3^*|l_0^*, l_1^*, l_2^*)$$

Now using (2.2) on each term. The last term will become

$$\pi^2_{0,\varepsilon}(l_3^*|l_0^*,l_1^*,l_2^*)=\frac{D(l_0{}^*l_3{}^*)_\varepsilon}{(l_0,l_3)}\otimes\frac{(l_2,l_3)}{(l_1,l_3)}-\frac{D(l_1{}^*l_3{}^*)_\varepsilon}{(l_1,l_3)}\otimes\frac{(l_0,l_3)}{(l_2,l_3)}+\frac{D(l_2{}^*l_3{}^*)_\varepsilon}{(l_2,l_3)}\otimes\frac{(l_1,l_3)}{(l_0,l_3)}+\frac{(l_0{}^*l_3{}^*)}{(l_1{}^*l_3{}^*)}\wedge\frac{(l_1{}^*l_3{}^*)}{(l_2{}^*l_3{}^*)}-\frac{(l_1{}^*l_3{}^*)}{(l_2{}^*l_3{}^*)}\wedge\frac{(l_2{}^*l_3{}^*)}{(l_0{}^*l_3{}^*)}+\frac{(l_2{}^*l_3{}^*)}{(l_0{}^*l_3{}^*)}\wedge\frac{(l_0{}^*l_3{}^*)}{(l_1{}^*l_3{}^*)}$$

Similarly, we can easily calculate the value of the remaining three terms. Substitution of these values in (2.4) we get

$$\pi^2_{0,\varepsilon}\ o\ d'(l_0^*,l_1^*,l_2^*,l_3^*)=\frac{D(l_0{}^*l_1{}^*)_\varepsilon}{(l_0,l_1)}\otimes\frac{(l_0,l_3)}{(l_0,l_2)}-\frac{D(l_0{}^*l_2{}^*)_\varepsilon}{(l_0,l_2)}\otimes\frac{(l_0,l_1)}{(l_0,l_3)}+\frac{D(l_0{}^*l_3{}^*)_\varepsilon}{(l_0,l_3)}\otimes\frac{(l_0,l_2)}{(l_0,l_1)}-\frac{D(l_0{}^*l_1{}^*)_\varepsilon}{(l_0,l_1)}\otimes\frac{(l_1,l_3)}{(l_1,l_2)}+\frac{D(l_1{}^*l_2{}^*)_\varepsilon}{(l_1,l_2)}\otimes\frac{(l_0,l_1)}{(l_1,l_3)}-\frac{D(l_1{}^*l_3{}^*)_\varepsilon}{(l_1,l_3)}\otimes\frac{(l_1,l_2)}{(l_0,l_1)}+\frac{D(l_0{}^*l_2{}^*)_\varepsilon}{(l_0,l_2)}\otimes\frac{(l_2,l_3)}{(l_1,l_2)}-\frac{D(l_1{}^*l_2{}^*)_\varepsilon}{(l_1,l_2)}\otimes\frac{(l_0,l_2)}{(l_2,l_3)}$$

$$+\frac{D(l_2{}^*l_3{}^*)_\varepsilon}{(l_2,l_3)}\otimes\frac{(l_1,l_2)}{(l_0,l_2)}-\frac{D(l_0{}^*l_3{}^*)_\varepsilon}{(l_0,l_3)}\otimes\frac{(l_2,l_3)}{(l_1,l_3)}+\frac{D(l_1{}^*l_3{}^*)_\varepsilon}{(l_1,l_3)}\otimes\frac{(l_0,l_3)}{(l_2,l_3)}-\frac{D(l_2{}^*l_3{}^*)_\varepsilon}{(l_2,l_3)}\otimes\frac{(l_1,l_3)}{(l_0,l_3)}-\frac{(l_0{}^*l_3{}^*)}{(l_1{}^*l_3{}^*)}\wedge\frac{(l_1{}^*l_3{}^*)}{(l_2{}^*l_3{}^*)}+\frac{(l_1{}^*l_3{}^*)}{(l_2{}^*l_3{}^*)}\wedge\frac{(l_2{}^*l_3{}^*)}{(l_0{}^*l_3{}^*)}-\frac{(l_2{}^*l_3{}^*)}{(l_0{}^*l_3{}^*)}\wedge\frac{(l_0{}^*l_3{}^*)}{(l_1{}^*l_3{}^*)}+\frac{(l_0{}^*l_2{}^*)}{(l_1{}^*l_2{}^*)}\wedge\frac{(l_1{}^*l_2{}^*)}{(l_2{}^*l_3{}^*)}-\frac{(l_1{}^*l_2{}^*)}{(l_2{}^*l_3{}^*)}\wedge\frac{(l_2{}^*l_3{}^*)}{(l_0{}^*l_2{}^*)}+\frac{(l_2{}^*l_3{}^*)}{(l_0{}^*l_2{}^*)}\wedge\frac{(l_0{}^*l_2{}^*)}{(l_1{}^*l_2{}^*)}+\frac{(l_0{}^*l_1{}^*)}{(l_0{}^*l_2{}^*)}\wedge\frac{(l_0{}^*l_2{}^*)}{(l_0{}^*l_3{}^*)}-\frac{(l_0{}^*l_2{}^*)}{(l_0{}^*l_3{}^*)}\wedge\frac{(l_0{}^*l_3{}^*)}{(l_0{}^*l_1{}^*)}+\frac{(l_0{}^*l_3{}^*)}{(l_0{}^*l_1{}^*)}\wedge\frac{(l_0{}^*l_1{}^*)}{(l_0{}^*l_2{}^*)}-\frac{(l_0{}^*l_1{}^*)}{(l_1{}^*l_2{}^*)}\wedge\frac{(l_1{}^*l_2{}^*)}{(l_1{}^*l_3{}^*)}+\frac{(l_1{}^*l_2{}^*)}{(l_1{}^*l_3{}^*)}\wedge\frac{(l_1{}^*l_3{}^*)}{(l_0{}^*l_1{}^*)}-\frac{(l_1{}^*l_3{}^*)}{(l_0{}^*l_1{}^*)}\wedge\frac{(l_0{}^*l_1{}^*)}{(l_1{}^*l_2{}^*)}$$

Now we come to calculate $\partial_\varepsilon o\ \pi^2_{1,\varepsilon}$ . Here we use (2.1) and (2.3)

$$\partial_\varepsilon o\ \pi^2_{1,\varepsilon}(l_0^*,l_1^*,l_2^*,l_3^*)=\partial_\varepsilon\{\langle r(l_o,l_1,l_2,l_3),r_\varepsilon(l_0^*,l_1^*,l_2^*,l_3^*)]\}$$

$$=\left(\frac{r_\varepsilon(l_0^*,l_1^*,l_2^*,l_3^*)}{r(l_o,l_1,l_2,l_3)}\otimes\big(1-r(l_o,l_1,l_2,l_3)\big)+\frac{r_\varepsilon(l_0^*,l_1^*,l_2^*,l_3^*)}{1-r(l_o,l_1,l_2,l_3)}\otimes r(l_o,l_1,l_2,l_3)\right)+\left(\frac{r_\varepsilon(l_0^*,l_1^*,l_2^*,l_3^*)}{1-r(l_o,l_1,l_2,l_3)}\wedge\frac{r_\varepsilon(l_0^*,l_1^*,l_2^*,l_3^*)}{r(l_o,l_1,l_2,l_3)}\right)$$

The values of $\frac{r_\varepsilon(l_0^*,l_1^*,l_2^*,l_3^*)}{r(l_o,l_1,l_2,l_3)}$ and $\frac{r_\varepsilon(l_0^*,l_1^*,l_2^*,l_3^*)}{1-r(l_o,l_1,l_2,l_3)}$ are given in [XI] and [XII], therefore we can write above as

$$\left\{\frac{D(l_0{}^*l_3{}^*)_\varepsilon}{(l_0,l_3)}+\frac{D(l_1{}^*l_2{}^*)_\varepsilon}{(l_1,l_2)}-\frac{D(l_0{}^*l_2{}^*)_\varepsilon}{(l_0,l_2)}-\frac{D(l_1{}^*l_3{}^*)_\varepsilon}{(l_1,l_3)}\right\}\otimes\frac{\Delta(l_o,l_1)\Delta(l_2,l_3)}{\Delta(l_o,l_2)\Delta(l_1,l_3)}$$
$$+\left\{\frac{D(l_0{}^*l_2{}^*)_\varepsilon}{(l_0,l_2)}+\frac{D(l_1{}^*l_3{}^*)_\varepsilon}{(l_1,l_3)}-\frac{D(l_0{}^*l_1{}^*)_\varepsilon}{(l_0,l_1)}-\frac{D(l_2{}^*l_3{}^*)_\varepsilon}{(l_2,l_3)}\right\}$$
$$\otimes\frac{\Delta(l_o,l_3)\Delta(l_1,l_2)}{\Delta(l_o,l_2)\Delta(l_1,l_3)}$$
$$+\left\{\frac{D(l_0{}^*l_2{}^*)_\varepsilon}{(l_0,l_2)}+\frac{D(l_1{}^*l_3{}^*)_\varepsilon}{(l_1,l_3)}-\frac{D(l_0{}^*l_1{}^*)_\varepsilon}{(l_0,l_1)}-\frac{D(l_2{}^*l_3{}^*)_\varepsilon}{(l_2,l_3)}\right\}$$
$$\wedge\left\{\frac{D(l_0{}^*l_3{}^*)_\varepsilon}{(l_0,l_3)}+\frac{D(l_1{}^*l_2{}^*)_\varepsilon}{(l_1,l_2)}-\frac{D(l_0{}^*l_2{}^*)_\varepsilon}{(l_0,l_2)}-\frac{D(l_1{}^*l_3{}^*)_\varepsilon}{(l_1,l_3)}\right\}$$

Expansion of this expression will give a value identical to that of $\pi^2_{0,\varepsilon}$ $o\ d'(l_0^*, l_1^*, l_2^*, l_3^*)$.

We have discussed about tangent complex or weight 3 in section (2.3). Here we will give suitable morphisms to connect this complex to the Grassmannian subcomplex with projective differential map $d'$. Consider the diagram (3.1b) ,where

$$\pi^3_{0,\varepsilon}(l_0^*,\dots,l_3^*)=\sum_{i=0}^{3}(-1)^i\left\{\frac{(l_i^*)_\varepsilon}{(l_i)}\otimes\frac{(l_{i+1})}{(l_{i+2})}\wedge\frac{(l_{i+3})}{(l_{i+2})}+\wedge_{\substack{j=0\\ j\neq i}}^{3}\frac{(l_j{}^*)}{(l_j)}\right\},\ i\ \ mod\ \ 4 \qquad (2.5)$$

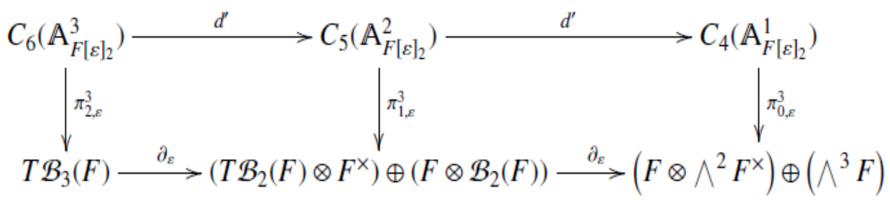
$$\pi^3_{1,\varepsilon}(l_0^*,\dots,l_4^*)=-\frac{1}{3}\sum_{i=0}^{4}(-1)^i\{\langle r(l_o,\dots\hat{l}_i^*\dots,l_4),r_\varepsilon(l_o,\dots\hat{l}_i^*\dots,l_4)]\otimes \prod_{\substack{i\neq j\\ j=0}}^{4}(i,j)+$$
$$\prod_{\substack{i\neq j\\ j=0}}^{4}(i,j)\otimes[r(l_o,\dots\hat{l}_i^*\dots,l_4)]\} \qquad (2.6)$$

$$\pi^3_{2,\varepsilon}(l_0^*,\dots,l_5^*)=\frac{2}{45}Alt_6\langle r(l_o,\dots\hat{l}_i^*\dots,l_4),r_\varepsilon(l_o,\dots\hat{l}_i^*\dots,l_4)]_3 \qquad (2.7)$$

Where $(i,j)=\Delta(l_i,l_j)$ and $\Delta(l_i{}^*,l_j{}^*)=(i^*,j^*)$

The map $\partial_\varepsilon$ is already defined in (1.3) and (1.4) and $d'$ is defined in (1.5).

**Theorem 3.1.2.** The right square of the below diagram




is commutative. i.e.

$$\pi^3_{0,\varepsilon}\ o\ d'(l_0^*,\dots,l_4^*)=\ \partial_\varepsilon o\ \pi^3_{1,\varepsilon}\ (l_0^*,\dots,l_4^*)$$

**Proof:** First of all we split the map $\pi^3_{0,\varepsilon}$ into two parts as $\pi^3_{0,\varepsilon} = \pi^1+\pi^2$ .The left-hand side can be written as

$$\pi^3_{0,\varepsilon} o\, d'(l_0^*, \dots, l_4^*) = \pi^1 o\, d'(l_0^*, \dots, l_4^*) + \pi^2 o\, d'(l_0^*, \dots, l_4^*)$$

The first part can be written as

$$\pi^1 o\, d'(l_0^*, \dots, l_4^*) \;=\; \pi^1(l_0^*|l_1^*, l_2^*, l_3^*, l_4^*) - \pi^1(l_1^*|l_0^*, l_2^*, l_3^*, l_4^*) + \pi^1(l_2^*|l_0^*, l_1^*, l_3^*, l_4^*) - \pi^1(l_3^*|l_0^*, l_1^*, l_2^*, l_4^*) + \pi^1(, l_4^*|l_0^*, l_1^*, l_2^*, l_3^*) \quad (2.8)$$

The above expression has five summands and the expansion of each summand through the map $\pi^3_{0,\varepsilon}$ gives us four terms of the form $\frac{(i^*,j^*)_\varepsilon}{(i,j)} \otimes \frac{x}{y} \wedge \frac{u}{v}$. Using the property $\wedge \frac{u}{v} = x \wedge u - x \wedge v$ , $x \wedge uv = x \wedge u + x \wedge v$ and $x \wedge x = 0$ we get 12 terms of the form$\frac{(i^*,j^*)_\varepsilon}{(i,j)} \otimes u \wedge v$. Doing the same for all summands we obtain a total of 60 terms. We combine the terms with a common factor$\frac{(i^*,j^*)_\varepsilon}{(i,j)} \otimes$. For example the term with a common factor $\frac{(0,4)_\varepsilon}{(0,4)} \otimes$ will be

$$\frac{(0^*,4^*)_\varepsilon}{(0,4)} \otimes \{(14) \wedge (34) + (01) \wedge (03) - (14) \wedge (24) + (24) \wedge (34) - (01) \wedge (02) - (02) \wedge (03)\}$$

This complete the calculation of $\pi^1 o\, d'(l_0^*, \dots, l_4^*)$. We define the second part as

$$\pi^2 o\, d'(l_0^*, \dots, l_4^*) = \widetilde{Alt}_5 \left\{ \sum_{i=0}^{3} (-1)^i \bigwedge_{\substack{j=0 \\ i\neq j}}^{3} \frac{(l_j^*)}{(l_j)} \right\}$$

Where $\widetilde{Alt}_5$ is the alternation sum. First, we expand the inner sum

Now we compute the right-hand side $\partial_\varepsilon o\, \pi^3_{1,\varepsilon}\, (l_0^*, \dots, l_4^*)$. Using (2.6) we can write

$$\partial_\varepsilon o\, \pi^3_{1,\varepsilon}\, (l_0^*, \dots, l_4^*) = \partial_\varepsilon \left\{ -\frac{1}{3} \sum_{i=0}^{4} (-1)^i \left\{ \langle r(l_o, \dots \hat{l}_i^* \dots, l_4), r_\varepsilon(l_o, \dots \hat{l}_i^* \dots, l_4) ] \otimes \prod_{\substack{i\neq j \\ j=0}}^{4} (i,j) + \prod_{\substack{i\neq j \\ j=0}}^{4} (i,j) \otimes [r(l_o, \dots \hat{l}_i^* \dots, l_4)] \right\} \right\} \quad (2.9)$$

We split the map $\partial_\varepsilon$ into two parts as $\partial_\varepsilon = \partial^1 + \partial^2$. The first part $\partial^1 o\, \pi^3_{1,\varepsilon}\, (l_0^*, \dots, l_4^*)$ will be

$$= -\frac{1}{3} \sum_{i=0}^{4} (-1)^i \left( -\frac{r_\varepsilon(l_o, \dots \hat{l}_i^* \dots, l_4)}{1 - r(l_o, \dots \hat{l}_i \dots, l_4)} \otimes r(l_o, \dots \hat{l}_i \dots, l_4) \wedge \prod_{\substack{i\neq j \\ j=0}}^{4} (i,j) - \frac{r_\varepsilon(l_o, \dots \hat{l}_i^* \dots, l_4)}{r(l_o, \dots \hat{l}_i \dots, l_4)} \otimes (1 - r(l_o, \dots \hat{l}_i \dots, l_4)) \wedge \prod_{\substack{i\neq j \\ j=0}}^{4} (i,j) + \prod_{\substack{i\neq j \\ j=0}}^{4} (i,j) \otimes (1 - r(l_o, \dots \hat{l}_i \dots, l_4)) \wedge r(l_o, \dots \hat{l}_i \dots, l_4) \right) \quad (2.10)$$

And the second part $\partial^1 o\, \pi^3_{1,\varepsilon}\, (l_0^*, \dots, l_4^*)$ is

$$= -\frac{1}{3}\sum_{i=0}^{4}(-1)^i \left\{ \frac{r_\varepsilon(l_o,\ldots \hat{l}_i^* \ldots, l_4)}{1-r(l_o,\ldots \hat{l}_i \ldots, l_4)} \wedge \frac{r_\varepsilon(l_o,\ldots \hat{l}_i^* \ldots, l_4)}{r(l_o,\ldots \hat{l}_i \ldots, l_4)} \wedge \prod_{\substack{i \neq j \\ j=0}}^{4} (i,j) \right\} \qquad (2.11)$$

The values of $\frac{r_\varepsilon(l_o,\ldots \hat{l}_i^* \ldots, l_4)}{1-r(l_o,\ldots \hat{l}_i \ldots, l_4)}$ and $\frac{r_\varepsilon(l_o,\ldots \hat{l}_i^* \ldots, l_4)}{r(l_o,\ldots \hat{l}_i \ldots, l_4)}$ are given in [11]. Using these values we expand the sums (1.10) and (1.11) for $i = 0, \ldots, 4$. After the expansion of (1.10) we combine the terms with a common factor $\frac{(i^*,j^*)_\varepsilon}{(i,j)} \otimes \ldots$. For example the terms with a common factor $\frac{(0^*,4^*)_\varepsilon}{(0,4)} \otimes$ will be

$$\frac{(0^*,4^*)_\varepsilon}{(0,4)} \otimes \{(14) \wedge (34) + (01) \wedge (03) - (14) \wedge (24) + (24) \wedge (34) - (01) \wedge (02) - (02) \wedge (03)\}$$

And so on. Similarly, the expansion of (2.11) for $i = 0, \ldots, 4$ gives us the terms of the form $\frac{(i^*,j^*)_\varepsilon}{(i,j)} \wedge x \wedge y$. After combining the expanded forms of (3.10) and (3.11) we get the final value of $\partial_\varepsilon o\, \pi_{1,\varepsilon}^3\, (l_0^*, \ldots, l_4^*)$ which is identical with that of the value of $\pi_{0,\varepsilon}^3 o\, d'(l_0^*, \ldots, l_4^*)$.

**Theorem 3.i.b.** The left square of the below diagram

$$\begin{array}{ccccc}
C_6(\mathbb{A}^3_{F[\varepsilon]_2}) & \xrightarrow{d'} & C_5(\mathbb{A}^2_{F[\varepsilon]_2}) & \xrightarrow{d'} & C_4(\mathbb{A}^1_{F[\varepsilon]_2}) \\
\downarrow \pi_{2,\varepsilon}^3 & & \downarrow \pi_{1,\varepsilon}^3 & & \downarrow \pi_{0,\varepsilon}^3 \\
T\mathcal{B}_3(F) & \xrightarrow{\partial_\varepsilon} & (T\mathcal{B}_2(F) \otimes F^\times) \oplus (F \otimes \mathcal{B}_2(F)) & \xrightarrow{\partial_\varepsilon} & \left(F \otimes \bigwedge^2 F^\times\right) \oplus \left(\bigwedge^3 F\right)
\end{array}$$

is commutative. i.e.

$$\pi_{1,\varepsilon}^3\ o\ d'(l_0^*, \ldots, l_4^*) = \partial_\varepsilon o\, \pi_{2,\varepsilon}^3\ (l_0^*, \ldots, l_4^*)$$

**Proof:** Since there are no significant changes in the map of $\pi_{2,\varepsilon}^3$ and this map is defined in [XV] so we are referring to Theorem 3.11 of [XV] for the proof.

## IV. Conclusion

This work shows that we can relate Grassmannian Complex to the first order tangent complex for the projective map " $d'$ ".

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**Conflicts of Interest:**

The authors declare that they have no conflicts of interest to report regarding the present study.

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