JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 Vol.-16, No.-7, July (2021) pp 93-98 ISSN (Print) 0973-8975

ON SOME NEW HERMITE – HADAMARD DUAL INEQUALITIES

Muhammad Bilal[1], Asif R. Khan[2]

[1,2] Department of Mathematics, University of Karachi, University Road, Karachi-75270, Pakistan

Corresponding Author: **Muhammad Bilal**

E-mail: mbilalfawad@gmail.com

https://doi.org/10.26782/jmcms.2021.07.00008

(Received: May 7, 2021; Accepted: July 3, 2021)

Abstract

In this article, we would like to introduce some new types of p–convex function, which we named quasi–p–convex function and P–p–convex function. With the help of these new notions we would also state the well-known Hermite–Hadamard dual inequalities which we call Hermite–Hadamard dual inequality for quasi–p–convex function and P–p–convex function, respectively. In this way various new results related to Hermite–Hadamard inequalities would be obtained and some would be captured as special cases by varying different values of p.

Keywords: Hermite–Hadamard dual inequality, p–convex function, quasi-convex function, P–convex function.

I. Introduction:

The branch of Mathematical Inequality is attracting researchers day by day due to its many applications. During a detailed analysis, we found that the basic theory of convex function holds a very powerful solution to the problems faced by researchers. This field of mathematics is given considerable attention in the literature and provides an important contrivance in the growth of various branches of research. Convexity has its applications in various fields of professional and daily life like industrial, pharmaceutical research, arts, management sciences, architecture and many more. In literature, the Hermite–Hadamard dual inequalities are one of the most important applications of convexity which have several different applications, due to which it is a general need that one should study them, especially those involving p–convex functions. For further study related to the topic, we refer the reader following articles [I – III], [VII – IX] and [XI – XV].

Before we proceed further it is worth mentioning here, we would use I as a real interval in this article.

We shall start with some useful definitions and results:

Theorem 1.8. Let $f: I \longrightarrow \mathbb{R}$ be a convex function. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(y)dy \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This result is known as Hermite–Hadamard dual inequality for convex function. For concave function f, both inequalities would be in reverse order.

We recall here the definition of p −convex function from [II]:

Definition 1.2. A function $f: I \subset (0,\ \infty) \longrightarrow \mathbb{R}$ is said to be p −convex, if

$$f\left([tx^p + (1-t)y^p]^{1/p}\right) \leq tf(x) + (1-t)f(y),$$

$\forall\, x,\ y \in I$ and $t \in [0,\ 1]$.

Remark 1.3. If we choose $p = 1$ and $p = -1$ in **Definition 1.2.**, we get the definitions of ordinary convex function [IV] and harmonically convex function [VII], respectively.

Here, we are going to introduce some new types of p −convex function, which we call as quasi−p −convex function and $P - p$ −convex functions respectively.

Definition 1.4. Let $p \in \mathbb{R} \setminus \{0\}$. We say that $f: I \subset (0,\ \infty) \longrightarrow [0,\ \infty)$ is a quasi−p −convex function, if

$$f\left([tx^p + (1-t)y^p]^{1/p}\right) \leq \max\{f(x),\ f(y)\},$$

$\forall\, x,\ y \in I$ and $t \in [0,\ 1]$.

Remark 1.5. If we choose $p = 1$ and $p = -1$ in **Definition 1.4.**, we get the definitions of quasi-convex function [X] and harmonically quasi-convex function [XV], respectively.

Definition 1.6. Let $p \in \mathbb{R} \setminus \{0\}$. We say that $f: I \subset (0,\ \infty) \longrightarrow [0,\ \infty)$ is a $P - p$ −convex function, if

$$f\left([tx^p + (1-t)y^p]^{1/p}\right) \leq f(x) + f(y),$$

$\forall\, x,\ y \in I$ and $t \in [0,\ 1]$.

Remark 1.7. If we choose $p = 1$ and $p = -1$ in **Definition 1.6.**, we get the definitions of P −convex function and harmonically−P −convex function [V], respectively.

This article is organized into four sections, In the next section, we are going to state and prove the result related to Hermite−Hadamard dual inequality for quasi−p −convex function. The third section is devoted to state and prove the result related to Hermite−Hadamard dual inequality for $P - p$ −convex function. These results would contain some new inequalities and some of them would be stated in [V] and [IX] as special cases and the last section gives us a conclusion with some remarks and future ideas.

II. Hermite−Hadamard Dual Inequality for Quasi−p −Convex Function:

Now we are going to state and prove the result related to Hermite−Hadamard dual inequality for quasi−p −convex function.

Theorem 2.1. Let $f: I \longrightarrow \mathbb{R} \setminus \{0\}$ be quasi−p −convex function. Then following inequalities hold:

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \leq \frac{p}{b^p-a^p}\int_a^b f(y)dy \leq \max\{f(a),\ f(b)\}. \tag{2.1}$$

Proof. As f is quasi−p −convex function, we have,

$$f\left([ta^p + (1-t)b^p]^{1/p}\right) \leq \max\{f(a),\ f(b)\}, \qquad \forall\ t \in [0,\ 1].$$

Integrating this inequality over $t \in [0,\ 1]$, we get:

$$\int_0^1 f\left([ta^p + (1-t)b^p]^{1/p}\right) dt \leq \int_0^1 (\max\{f(a), f(b)\})\, dt$$

which gives us

$$\frac{p}{b^p-a^p}\int_a^b f(y)dy \leq \max\{f(a),\ f(b)\} \tag{2.2}$$

which proves the right side inequality. To prove the left side inequality we have,

$$f\left([tx^p + (1-t)y^p]^{1/p}\right) \leq \max\{f(x),\ f(y)\}, \qquad \forall\ t \in [0,\ 1].$$

Put $t = \frac{1}{2}$ we get,

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right) \leq \max\{f(x),\ f(y)\}.$$

By replacing $x = [ta^p + (1-t)b^p]^{1/p}$ and $y = [tb^p + (1-t)a^p]^{1/p}$ we get,

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \leq \max\left\{f\left([ta^p + (1-t)b^p]^{1/p}\right), f\left([tb^p + (1-t)a^p]^{1/p}\right)\right\}.$$

If both $f\left([ta^p + (1-t)b^p]^{1/p}\right)$ and $f\left([tb^p + (1-t)a^p]^{1/p}\right)$ have same integrals then by integrating this inequality w.r.t. t over the interval $[0,\ 1]$, we get,

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \leq \frac{p}{b^p-a^p}\int_a^b f(y)dy. \tag{2.3}$$

By combining (2.2) and (2.3), we get our required result.

Corollary 2.2. In **Theorem 2.1**, one can see the following interesting results:
(1) If one takes $p = 1$, then one has the following Hermite−Hadamard dual inequality for Quasi convex function:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(y)dy \leq \max\{f(a), f(b)\}.$$

(2) If one takes $p = -1$, then one has the following Hermite−Hadamard dual inequality for Harmonically quasi convex function:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b f(y)dy \leq \max\{f(a), f(b)\}.$$

III. Hermite−Hadamard Dual Inequality for $P-p$ −Convex Function:

Now we are going to state and prove the result related to Hermite−Hadamard dual inequality for $P-p$ −convex function.

Theorem 3.1. Let $f: I \longrightarrow \mathbb{R} \setminus \{0\}$ be a $P-p$ −convex function. Then following inequalities hold:

$$\frac{1}{2} f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \leq \frac{p}{b^p-a^p}\int_a^b \frac{f(y)}{y^{1-p}}dy \leq f(a)+f(b). \tag{3.1}$$

Proof. As f is $P-p$ −convex function, we have,

$$f\left([ta^p+(1-t)b^p]^{1/p}\right) \leq f(a)+f(b), \qquad \forall \ t \in [0,\ 1].$$

Integrating this inequality over $t \in [0,\ 1]$, we get:

$$\int_0^1 f\left([ta^p+(1-t)b^p]^{1/p}\right) dt \leq \int_0^1 (f(a)+f(b))\, dt$$

which gives us

$$\frac{p}{b^p-a^p}\int_a^b f(y)dy \leq f(a)+f(b) \tag{3.2}$$

which proves the right side inequality. To prove the left side inequality we have,

$$f\left([tx^p+(1-t)y^p]^{1/p}\right) \leq f(x)+f(y), \qquad \forall \ t \in [0,\ 1].$$

Put $t = \frac{1}{2}$ we get,

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right) \leq f(x)+f(y).$$

By replacing $x = [ta^p+(1-t)b^p]^{1/p}$ and $y = [tb^p+(1-t)a^p]^{1/p}$ we get,

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \leq f\left([ta^p+(1-t)b^p]^{1/p}\right) + \ f\left([tb^p+(1-t)a^p]^{1/p}\right)$$

by integrating this inequality w.r.t. t over the interval $[0,\ 1]$, we get,

$$\frac{1}{2} f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \leq \frac{p}{b^p-a^p}\int_a^b f(y)dy. \tag{3.3}$$

By combining (2.2) and (2.3), we get our required result.

Remark 3.2. Following well-known results will be obtained by putting different values of p.

(1) In **Theorem 3.1**, if we take $p = 1$, then we get Hermite−Hadamard dual inequality for P −convex function [V].

(2) In **Theorem 3.1**, if we take $p = -1$, then we get Hermite−Hadamard dual inequality for Harmonically−P −convex function [IX].

IV. Conclusion.

Hermite−Hadamard dual inequality is one of the most celebrated inequalities. We can find its various generalizations and variants in the literature. We have given its generalization by introducing a new generalized notion of quasi−p −convex function and $P - p$ −convex function. This new class of functions contains many important classes such as quasi-convex, harmonically quasi-convex, P −convex and harmonically −P −convex functions, respectively. In sections 2 and 3, we have stated and proved two different results related to Hermite−Hadamard dual inequality. These results capture various new results and already obtained results stated in articles [V] and [IX].

V. Remarks and Future Ideas.

(1) We can also state all the inequalities given in this article in the reverse direction for concave function by using simple relation f is concave iff $-f$ is convex.

(2) One may also work on Fejer inequality by introducing weights in Hermite−Hadamard dual inequality.

(3) One may do similar work by using various classes of functions.

(4) One may try to state all results stated in this article in the discrete case.

(5) By using different techniques one may also work to estimate the left and right bound of both results stated in this article.

Conflicts of Interest:

The authors declare that they have no conflicts of interest to report regarding the present study.

References:

I. Ambreen Arshad and Asif Raza Khan, Hermite−Hadamard−Fejer type inequalities for $s - p$ −convex functions of several senses, TJMM, **11** (2019), 25–40.

II. Asif Raza Khan, Inam Ullah Khan and Siraj Muhammad, Hermite−Hadamard type fractional inequalities for s −convex functions of mixed kind, TMCS, **1** (2021), 25–37.

III. Charles Hermite, Sur deux limites d'une intégrale définie, Mathesis, **3** (1883), 82.

IV. Edwin Ford Beckenback, Convex Functions, Bull. Amer. Math. Soc., **54** (1948), 439–460.

V. İmdat I$s$can, Hermite−Hadamard and Simpson like inequalities for differentiable harmonically convex functions, J. Math., 2014, (2014), Article 346305.

VI. İmdat I$s$can, Hermite−Hadamard type inequality for p −convex functions, Int. J. Anal. App., **11**(2), (2016), 137–145.

VII. İmdat I$s$can, Selim Nauman and Kerim Bekar, Hermaite−Hadamard and Simpson type inequalities for differentiable harmonically P −convex functions, British. J. Math. & Comp., **4** (14) (2014), 1908–1920.

VIII. Jaekeun Park, Hermite−Hadamard and Simpson−like type inequalities for differentiable Harmonically Quasi−convex functions, Int. J. Math. Anal., **8**(33), (2014), 1692–1645.

IX. Mehmet Kunt and İmdat I$s$can, Hermite−Hadamard−Fejer type inequalities for p −convex functions, Arab. J. Math., **23**(1), (2017), 215–230.

X. Muhammad Aslam Noor, Khalida Inayat Noor, Marcela Mihai and Muhammad Uzair Awan, Hermite−Hadamard inequalities for differentiable p −convex functions using hypergeometric functions, Publications de L'Institut Mathematique, **100**(114), (2015), 251–257.

XI. Muhammad Bilal and Asif Raza Khan, New Generalized Hermite−Hadamard Inequalities for p −convex functions in the mixed kind, EJPAM (Accepted), 2021.

XII. Muhammad Bilal, Muhammad Imtiaz Asif Raza Khan, Ihsan Ullah Khan and Muhammad Zafran, Generalized Hermite−Hadamard inequalities for s −convex functions in mixed kind, (Submitted), (2021).

XIII. Murat Emin Özdemír, Merve Avci and Havva Kavurmaci, Hermaite−Hadamard type inequalities via (α, m) convexity, Comp. Math. App., **61**, (2011), 2614–2620.

XIV. Mehmood Faraz, Asif R. Khan, & M. Azeem Ullah Siddique. : 'SOME RESULTS RELATED TO CONVEXIFIABLE FUNCTIONS'. J. Mech. Cont.& Math. Sci., Vol.-15, No.-12, December (2020) pp 36-45. DOI : 10.26782/jmcms.2020.12.00004

XV. Silvestru Sever Dragomir, Josip Pečarić and Lars-Erik Persson, Some inequalities of Hadamard type, Soochow J. Math., **21**(3) (1995), 335–341.