



ON EXACT ANALYTICAL SOLUTIONS OF THE TIMOSHENKO BEAM MODEL UNDER UNIFORM AND VARIABLE LOADS

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Abstract

In this research work, we consider the mathematical model of the Timoshenko beam (TB) problem in the form of a boundary-value problem of a system of ordinary differential equations. Instead of numerical solutions using finite difference and finite volume methods, an attempt is made to derive the exact analytical solutions of the model with boundary feedback for a better and more explicit description of the rotation and displacement parameters of the TB structure model. The explicit analytical solutions have been successfully found for uniform and real-time variable load cases. The rotation and displacement profiles obtained through the analytical solutions accurately picture the structure of the beam under uniform and variable loads.

Keywords : Timoshenko beam, Analytical solution, Rotation, Displacement, Uniform load, Variable load.

I. Introduction

Beam theory has a long history and various engineers, scientists, etc have developed numerical schemes and tested different approaches to knowing important characteristics about the structure and performance of the beams subjected to loads [II]. At the beginning of the 20th century, a new beam theory was developed by a Ukrainian-born scientist named Stephen Timoshenko [X]. Due to his name, this theory was named as Timoshenko beam (TB) theory. In the TB model, both shear deformation and rotational inertia effects were taken into account. Thus the TB describes the behavior of various beams such as short beams, composite sandwiched

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beams or the beams which can be excited by high frequency so that the wavelength of excitation becomes shorter and thus approaches the thickness of the beam.

To solve the Reissner-Mindlin problem which is more complex, researchers frequently use a numerical approximation of the TB model as a starting point so that a better understanding can be developed. Some bad behavior called locking phenomenon [III] arises when these problems are solved with finite difference method or standard Galerkin finite methods. This locking phenomenon is due to small parameters, and researchers have proposed different schemes which should be uniform concerning the small parameters. For example, [VIII], proposed a formulation that is linked to Petrov-Galerkin, and [III], [IV], [V], [IX] proposed a mixed formulation which results in approximations due to the use of the reduced integration. In [I], the authors proposed finite difference schemes for the TB model. The use of the least-squares finite element method was discussed by [VI]. The p and h - p versions of the finite element method have also been tested for the TB model by [VII].

The problem with the discussed approaches is that these are numerical, and do not provide an exact profile of the problem in terms of the analytical solutions. The numerical schemes need more precise values perturbation parameter and strictly lesser step sizes and a considerable processing time to reach a bit of accuracy in the results. The stability regimes of the numerical schemes are not always unconditional, and the numerical results should be validated through analytical results, if available. In this research study, we focus the computation of exact analytical solutions of the TB model [X] in explicit form for better understanding the structure of the model and analyzing exactly the rotation and displacement profiles when the beam is subjected to different loads. We consider uniform as well as variable loads with boundary conditions to derive analytical solutions and observe the effect of such loadings on the structure of the TB model causing displacements and rotations. The obtained results can help in accurately understanding the TB behavior while subjected to different loads.

II. Mathematical Model of the Timoshenko Beam Problem

Consider a Timoshenko beam clamped at both ends. When a load $p(\bar{x})$ is applied, which is distributed along the beam at different points. Due to this distribution of load, the moment $m(\bar{x})$ is also distributed along the beam i.e. at each point of the TB where the load is applied, there will be different moments. So, the beam will bend within the plane as a result of this distribution. We take the length of the beam as 'L' and its area of cross-section 'A'. We use 'E' for Young's modulus, 'G' for modulus of rigidity. The terms $M(\bar{x})$, $Q(\bar{x})$ and $\theta(\bar{x})$ are also used for bending moment, shear force and rotation of cross-section, respectively. $W(\bar{x})$ stands for displacement as transversed by the beam and κ is the correction factor for shear. For the TB problem, the mathematical model in the form of a system of ordinary differential equations, for $\bar{x} \in (0, L)$, can be described as in equations (1)-(3).

$$-\frac{dQ}{d\bar{x}} = p \quad (1)$$

$$-EI \frac{d^2 \theta}{d\bar{x}^2} - Q = 0 \quad (2)$$

$$-\frac{Q}{kGA} + \frac{dW}{d\bar{x}} - \theta = 0 \quad (3)$$

Due to no motion at the two ends, the conditions at the boundary can be specified as:

$$W(0) = W(L) = 0; \quad \theta(0) = \theta(L) = 0$$

The non-dimensionalized problem for (1)-(3) can be obtained by introducing the following change of variables:

$$x = \frac{\bar{x}}{L}, \quad w = \frac{W}{L}, \quad \sigma = \frac{QL^2}{EI}, \quad f = \frac{pL^3}{EI}, \quad xL = \bar{x}, \quad wL = W, \quad Q = \frac{\sigma EI}{L^2}$$

Thus, from (1), we have:

$$-\frac{dQ}{d\bar{x}} = p \quad \Rightarrow -\frac{d\left(\frac{\sigma EI}{L^2}\right)}{d(xL)} = p \quad \Rightarrow -\frac{d\sigma\left(\frac{EI}{L^2}\right)}{dx(L)} = p$$

OR

$$-\frac{d\sigma EI}{dx L^3} = p \Rightarrow -\frac{d\sigma}{dx} = \frac{pL^3}{EI}$$

Finally, $-\sigma' = f$

From (2), we have:

$$\begin{aligned} -EI \frac{d^2 \theta}{d(xL)^2} &= Q \\ \Rightarrow -\frac{d^2 \theta}{dx^2} \frac{1}{L^2} &= \frac{Q}{EI} \\ \Rightarrow -\frac{d^2 \theta}{dx^2} &= \frac{QL^2}{EI} \\ \Rightarrow -\theta'' &= \sigma \\ \Rightarrow -\theta'' - \sigma &= 0 \end{aligned}$$

From (3)

$$\begin{aligned} -\frac{Q}{kGA} + \frac{dW}{d\bar{x}} - \theta &= 0 \\ \Rightarrow -\frac{\frac{\sigma EI}{L^2}}{kGA} + \frac{d(wL)}{d(xL)} - \theta &= 0 \\ \Rightarrow -\sigma \varepsilon^2 + w' - \theta &= 0 \end{aligned}$$

where

$$\varepsilon^2 = \frac{EI}{kGAL^2}$$

The original problem can now be transformed to the following non-dimensionanlized model problem in (4)-(6).

$$-\sigma' = f \quad (4)$$

$$-\theta'' - \sigma = 0 \quad (5)$$

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$$-\varepsilon^2 \sigma + w' - \theta = 0 \quad (6)$$

The problem (4)-(6) demands to find w , θ and σ such that in x lies in $(0,1)$.

The system of three differential equations can be further reduced into a system of two differential equations. For this, we use (6), as:

$$\begin{aligned} -\varepsilon^2 \sigma &= \theta - w' \\ \Rightarrow \sigma &= -\varepsilon^{-2}(\theta - w') \end{aligned}$$

Putting the above in (4) and (5) gives the final form of non-dimensionanlized TB model is:

$$-\theta'' + \varepsilon^{-2}(\theta - w') = 0 \quad (7a)$$

$$\varepsilon^{-2}(\theta' - w'') = f \quad (7b)$$

together with the boundary conditions

$$w(0) = w(1) = 0; \quad \theta(0) = \theta(1) = 0 \quad (8)$$

The parameter $\varepsilon^2 = EI/kGAL^2$ is a constant proportional to the ratio of the thickness to the length of the beam. In most realistic applications $\varepsilon \ll 1$.

III. Exact Analytical Solutions of TB Model

First, for the uniform load case $f = 1$; $0 < x < 1$, we have derived exact solution analytically with this load for Timoshenko beam model. Considering the model of the TB problem (9)-(11) as:

$$-\sigma' = f \quad (9)$$

$$-\theta'' - \sigma = 0 \quad (10)$$

$$-\varepsilon^2 \sigma + w' - \theta = 0 \quad (11)$$

Subject to the conditions:

$$w(0) = w(1) = 0; \quad \theta(0) = \theta(1) = 0$$

From (9)

$$\begin{aligned} \sigma' &= -f \\ \Rightarrow \sigma' &= -1 \quad \because f = 1 \end{aligned}$$

Integrating

$$\sigma = -x + c_1 \quad (12)$$

From (10)

$$\begin{aligned} \theta'' &= -\sigma \\ \Rightarrow \theta'' &= x - c_1 \end{aligned}$$

Integrating,

$$\theta' = \frac{x^2}{2} - c_1 x + c_2$$

Integrating again,

$$\theta = \frac{x^3}{6} - \frac{c_1 x^2}{2} + c_2 x + c_3 \quad (13)$$

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$$\theta(0) = 0 \Rightarrow x = 0, \quad \theta = 0, \quad \text{Equation (13) becomes}$$

$$0 = c_3 \quad \{\Rightarrow c_3 = 0\}$$

$$\text{Also, } \theta(1) = 0 \Rightarrow x = 1, \quad \theta = 0, \quad \text{Equation (13) becomes}$$

$$0 = \frac{1}{6} - \frac{c_1}{2} + c_2 + 0 \Rightarrow 1 - 3c_1 + 6c_2 = 0 \quad \{\Rightarrow 3c_1 - 6c_2 = 1\} \quad (14)$$

$$\text{From (11)} \quad w' = \theta + \varepsilon^2 \sigma$$

$$w' = \frac{x^3}{6} - \frac{c_1 x^2}{2} + c_2 x + c_3 + \varepsilon^2 (-x) + \varepsilon^2 c_1$$

$$w' = \frac{x^3}{6} - \frac{1}{2} x^2 c_1 + \varepsilon^2 c_1 + c_2 x - \varepsilon^2 x \quad \because c_3 = 0$$

Integrating

$$w = \frac{x^4}{24} - \frac{x^3}{6} c_1 + \varepsilon^2 c_1 x + \frac{c_2 x^2}{2} - \frac{\varepsilon^2 x^2}{2} + c_4 \quad (15)$$

$$w(0) = 0 \Rightarrow x = 0, \quad w = 0, \quad \text{Equation (15) becomes}$$

$$0 = 0 - 0 + 0 + 0 - 0 + c_4 \quad \{\Rightarrow c_4 = 0\}$$

$$w(1) = 0 \Rightarrow x = 1, \quad w = 0, \quad \text{Equation (15) becomes}$$

$$0 = \frac{1}{24} - \frac{1}{6} c_1 + \varepsilon^2 c_1 + \frac{1}{2} c_2 - \frac{1}{2} \varepsilon^2$$

$$\Rightarrow 1 - 4c_1 + 24 \varepsilon^2 c_1 + 12c_2 - 12 \varepsilon^2 = 0$$

$$\Rightarrow -4(1 - 6 \varepsilon^2) c_1 + 12c_2 = 12 \varepsilon^2 - 1 \quad (16)$$

Multiplying (14) by 2 and adding (16) in it, we have

$$(2 + 24 \varepsilon^2) c_1 = 1 + 12 \varepsilon^2$$

$$\Rightarrow 2(1 + 12 \varepsilon^2) c_1 = 1 + 12 \varepsilon^2 \Rightarrow c_1 = \frac{1}{2}, \text{ putting in (14)}$$

$$\Rightarrow c_2 = \frac{1}{12}$$

Hence,

$$\theta = \frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12} + 0$$

$$\Rightarrow \theta = \frac{x}{12} (2x^2 - 3x + 1)$$

$$\Rightarrow \theta = \frac{x}{12} (2x - 1)(x - 1)$$

and

$$w = \frac{x^4}{24} - \frac{x^3}{12} + \frac{1}{2} \varepsilon^2 x + \frac{x^2}{24} - \frac{1}{2} \varepsilon^2 x^2 + 0$$

$$= \frac{x^4}{24} - \frac{x^3}{12} + \frac{x^2}{24} + \frac{1}{2} \varepsilon^2 x - \frac{1}{2} \varepsilon^2 x^2$$

$$= \frac{x^2}{24} (x^2 - 2x + 1) + \frac{1}{2} \varepsilon^2 x (1 - x)$$

$$= \frac{x^2}{24} (x - 1)^2 + \frac{1}{2} \varepsilon^2 x (1 - x)$$

$$w = \frac{1}{24} x^2 (1 - x)^2 + \frac{1}{2} \varepsilon^2 x (1 - x)$$

Finally, for the uniform load case, The exact solutions of the TB model are

$$\theta(x) = \frac{1}{12}x(1-x)(1-2x) \quad (17)$$

$$w(x) = \frac{1}{24}x^2(1-x)^2 + \frac{1}{2}\varepsilon^2x(1-x) \quad (18)$$

Now, for the variable load case, say, $f(x) = 100(e^x + x)$; $0 < x < 1$, we have derived the exact solution analytically with this load for the Timoshenko beam model. Consider:

$$-\sigma' = 100(e^x + x) \quad (19)$$

$$-\theta'' - \sigma = 0 \quad (20)$$

$$-\varepsilon^2\sigma + w' - \theta = 0 \quad (21)$$

Subject to:

$$w(0) = w(1) = 0; \quad \theta(0) = \theta(1) = 0$$

From (19)

$$\begin{aligned} \sigma' &= -100(e^x + x) \\ \Rightarrow \frac{d\sigma}{dx} &= -100(e^x + x) \end{aligned}$$

Integrating,

$$\begin{aligned} \int d\sigma &= -100 \int (e^x + x) dx \\ \Rightarrow \sigma &= -100\left(e^x + \frac{x^2}{2}\right) + c_1 \quad \left\{ \Rightarrow \sigma = -100\left(e^x + \frac{x^2}{2}\right) + c_1 \right\} \end{aligned}$$

From (20)

$$\begin{aligned} \theta'' &= -\sigma \\ \Rightarrow \theta'' &= 100\left(e^x + \frac{x^2}{2}\right) - c_1 \end{aligned}$$

Integrating

$$\theta' = 100\left(e^x + \frac{x^3}{6}\right) - c_1x + c_2$$

Integrating again

$$\begin{aligned} \theta(x) &= 100\left(e^x + \frac{x^4}{24}\right) - c_1\frac{x^2}{2} + c_2x + c_3 \\ \left\{ \Rightarrow \theta(x) &= 100\left(e^x + \frac{x^4}{24}\right) - c_1\frac{x^2}{2} + c_2x + c_3 \right\} \end{aligned}$$

Now,

$$\theta(0) = 0; \quad \theta(1) = 0$$

$$\theta(0) = 0$$

$$0 = 100(1) + c_3 \quad \left\{ \Rightarrow c_3 = -100 \right\}$$

$$\theta(1) = 0$$

$$0 = 100\left(e + \frac{1}{24}\right) - \frac{1}{2}c_1 + c_2 - 100$$

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$$\begin{aligned}
 &\Rightarrow \frac{-c_1+2c_2}{2} = 100 - 100\left(e + \frac{1}{24}\right) = 100\left(1 - e - \frac{1}{24}\right) = 100\left(\frac{23}{24} - e\right) \\
 &\Rightarrow -c_1 + 2c_2 = 200\left(\frac{23}{24} - e\right) \\
 &\Rightarrow c_1 - 2c_2 = 200\left(e - \frac{23}{24}\right) = 200e - 200\frac{23}{24} = 200e - \frac{575}{3} \\
 &\Rightarrow c_1 - 2c_2 = 200e - \frac{575}{3} \quad (22)
 \end{aligned}$$

From (21)

$$\begin{aligned}
 w' &= \theta + \varepsilon^2 \sigma \\
 \Rightarrow w' &= 100\left(e^x + \frac{x^4}{24}\right) - c_1 \frac{x^2}{2} + c_2 x - 100 - 100\varepsilon^2\left(e^x + \frac{x^2}{2}\right) + \varepsilon^2 c_1
 \end{aligned}$$

Integrating

$$\{w(x) = 100\left(e^x + \frac{x^5}{120}\right) - c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} - 100x - 100\varepsilon^2\left(e^x + \frac{x^3}{6}\right) + \varepsilon^2 c_1 x + c_4\}$$

Now,

$$\begin{aligned}
 w(0) &= 0; \quad w(1) = 0 \\
 w(0) &= 0 \\
 0 &= 100(1) - 100\varepsilon^2(1) + c_4 \Rightarrow -100 + 100\varepsilon^2 = c_4 \quad \{\Rightarrow c_4 = 100(\varepsilon^2 - 1)\} \quad w(1) = 0 \\
 0 &= 100\left(e + \frac{1}{120}\right) - \frac{1}{6}c_1 + \frac{1}{2}c_2 - 100 - 100\varepsilon^2\left(e + \frac{1}{6}\right) + \varepsilon^2 c_1 + c_4 \\
 &\Rightarrow -\frac{1}{6}c_1 + \varepsilon^2 c_1 + \frac{1}{2}c_2 = 100\varepsilon^2\left(e + \frac{1}{6}\right) - 100\left(e + \frac{1}{120}\right) + 100 - 100(\varepsilon^2 - 1) \\
 &\Rightarrow \frac{-c_1 + 6\varepsilon^2 c_1 + 3c_2}{6} = 100\varepsilon^2 e - 100e + \frac{100}{6}\varepsilon^2 - 100\varepsilon^2 - \frac{100}{120} + 100 + 100 \\
 &\Rightarrow \frac{(-1 + 6\varepsilon^2)c_1 + 3c_2}{6} = 100(\varepsilon^2 - 1)e - \frac{500}{6}\varepsilon^2 + \frac{2390}{12} \\
 &\Rightarrow (-1 + 6\varepsilon^2)c_1 + 3c_2 = 600(\varepsilon^2 - 1)e - 500\varepsilon^2 + 1195 \quad (23)
 \end{aligned}$$

Multiplying (22) by 3 and (23) by 2 and adding gives:

$$\begin{aligned}
 (1 + 12\varepsilon^2)c_1 &= 600(2\varepsilon^2 - 1)e - 1000\varepsilon^2 + 1815 \\
 \Rightarrow c_1 &= \frac{1}{1+12\varepsilon^2} \{600(2\varepsilon^2 - 1)e - 1000\varepsilon^2 + 1815\} \quad \text{put in equation (38)} \\
 \Rightarrow c_2 &= \frac{1}{2}c_1 - 100e + \frac{575}{6} \\
 &= \frac{600(2\varepsilon^2 - 1)e}{2(1+12\varepsilon^2)} - 100e + \frac{1}{2(1+12\varepsilon^2)} \{-1000\varepsilon^2 + 1815\} + \frac{575}{6} \\
 &= \frac{1200\varepsilon^2 e - 600e - 200e - 2400\varepsilon^2 e}{2(1+12\varepsilon^2)} + \frac{1}{2(1+12\varepsilon^2)} \{-1000\varepsilon^2 + 1815\} + \frac{575}{6} \\
 \Rightarrow c_2 &= \frac{1}{2(1+12\varepsilon^2)} \{-400(2 + 3\varepsilon^2)e - 1000\varepsilon^2 + 1815\} + \frac{575}{6} \\
 \left\{ \Rightarrow c_2 \right. &= \frac{1}{2(1 + 12\varepsilon^2)} \{-400(2 + 3\varepsilon^2)e - 1000\varepsilon^2 + 1815\} + \frac{575}{6} \left. \right\}
 \end{aligned}$$

Therefore, the exact solutions for the variable load considered are:

$$\theta(x) = 100 \left(e^x + \frac{x^4}{24} \right) - c_1 \frac{x^2}{2} + c_2 x + c_3 \quad (24)$$

$$w(x) = 100 \left(e^x + \frac{x^5}{120} \right) - c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} - 100x - 100 \varepsilon^2 \left(e^x + \frac{x^3}{6} \right) + \varepsilon^2 c_1 x + c_4 \quad (25)$$

Where,

$$c_1 = \frac{1}{1+12\varepsilon^2} \{600(2\varepsilon^2-1)e - 1000 \varepsilon^2 + 1815\} \quad (26)$$

$$c_2 = \frac{1}{2(1+12\varepsilon^2)} \{-400(2+3\varepsilon^2)e - 1000 \varepsilon^2 + 1815\} + \frac{575}{6} \quad (27)$$

$$c_3 = -100, \quad (28)$$

$$c_4 = 100(\varepsilon^2 - 1) \quad (29)$$

IV. Results and Discussion

For the case of uniform load, the exact analytical solutions of the rotation and displacement variables for $x = 0.1, 0.5$ and 0.8 have been computed for reference through (17)-(18) and displayed in Table-1 for $\varepsilon = 0.5, 0.1$ and 0.01 .

Regarding rotation of the TB at $\varepsilon = 0.5$, the value of $\theta(x)$ up to 4 decimal places at the point $x = 0.1$ is 0.0060. At $x = 0.5$, which is the mid-point of the beam, the value of $\theta(x)$ decrease to 0 since the load is uniform, so at equal distances to the right and left of $x = 0.5$, the rotations will be equal but in opposite direction, thus the net effect is 0. This shows that at this point $\theta(x)$ is zero or we can say that there will be no rotation at $x = 0.5$. At $x = 0.8$, the value of $\theta(x)$ is negative. The negative sign shows that the beam will rotate in opposite direction at this point to that of at $x = 0.1$. Thus rotation to the right and left of $x = 0.5$ are opposite in direction.

Regarding displacement of the TB at $\varepsilon = 0.5$, we see that at $x = 0.1$, $w(x) = 0.0116$. At $x = 0.5$, $w(x) = 0.0339$ and at $x = 0.8$, $w(x) = 0.0211$. So we see that as we go from $x = 0.1$ to $x = 0.5$ the value of $w(x)$ increases and then from $x = 0.5$ to $x = 0.8$ again value of $w(x)$ decreases. At $\varepsilon = 0.1$, we observe the same pattern for rotation and displacement. Finally, for $\varepsilon = 0.01$, the values of rotation depict the same picture as for the other two cases. Generally, as the value of ε decreases from 0.5 to 0.01, the value of rotation remains the same at each point from top to bottom. On the other hand, the values of displacement decrease from top to bottom at each point, as shown in Table-1.

For the case of variable load, the exact analytical solutions of the rotation and displacement variables for $x = 0.1, 0.5$ and 0.8 have been computed for reference through (24)-(29) and displayed in Table-2 for $\varepsilon = 0.1, 0.01, 0.001$ and 0.0001 . For this load, the values of rotation and displacement are given in Table-2 for different values of ε at three different points of beam.

At $\varepsilon = 0.1$, from Table-2 we notice that the value of $\theta(x)$ decreases from $x = 0.1$ to $x = 0.5$ and then increases from $x = 0.5$ to $x = 0.8$. Further the values of $\theta(x)$ will be negative to the right of $x = 0.5$ due to opposite rotation. On the other hand, the value of displacement increases from point $x = 0.1$ to point $x = 0.5$ and then decreases from $x = 0.5$ to $x = 0.8$. The maximum displacement is at $x = 0.5$. For $\varepsilon = 0.01, 0.001$ and 0.0001 the same pattern is seen in the Table-2 for both rotation and displacement. We

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also notice that the load will be minimum at left hand at $x = 0$ and increases from left to right. It will be maximum at $x = 1$. But since both ends are fixed so there will be no rotation and displacement at $x = 0$ and $x = 1$.

Table 1: Exact solutions for the case: $f(x) = 1, 0 < x < 1$

Exact Solutions: ($\theta(x)/w(x)$)	$x = 0.1$	$x = 0.5$	$x = 0.8$
$\varepsilon = 0.5$	(0.0060/0.0008)	(0.0000/0.0339)	(- 0.008/0.0019)
$\varepsilon = 0.1$	(0.0060/0.0003)	(0.0000/0.0026)	(- 0.00800/0.0011)
$\varepsilon = 0.01$	(0.0060/0.0003)	(0.0000/0.0026)	(- 0.008/0.0011)

Table 2: Exact solutions for the case: $f(x) = 100(e^x + x); 0 < x < 1$

Exact Solutions: ($\theta(x)/w(x)$)	$x = 0.1$	$x = 0.5$	$x = 0.8$
$\varepsilon = 0.1$	(1.2209/0.1468)	(0.1988/0.8397)	(- 1.7744/0.4507)
$\varepsilon = 0.01$	(1.1997/0.0673)	(0.1396/0.5695)	(- 1.8122/0.2518)
$\varepsilon = 0.001$	(1.1994/0.0665)	(0.1390/0.5668)	(- 1.8126/0.2498)
$\varepsilon = 0.0001$	(1.1994/0.0665)	(0.1390/0.5668)	(- 1.8127/0.2498)

V. Conclusion

In this research work, we have derived exact analytical solutions of the TB model for two types of load. One load is fixed load, $f(x)=1$ and other load is a variable load, $f(x) = 100(e^x + x)$. With these loads, we have derived and computed exact solutions for rotation and displacement. From the analysis of these results, we conclude that there were no locking phenomena. These results were obtained for different values of ε , the only dependent parameter of the problem. The results obtained show that the formulation can be applied to more complicated problems of beam deformation.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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