



## AN EFFICIENT FINITE DIFFERENCE SCHEME FOR THE NUMERICAL SOLUTION OF TIMOSHENKO BEAM MODEL

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### Abstract

*We propose and implement a finite difference scheme for the numerical solution of the Timoshenko beam model without locking phenomenon. The averaging concept is used in approximating the function, and thus developing the scheme for elements. Finally, the system is discretized into the algebraic system using the proposed scheme and the numerical solution is attained. The numerical solutions are attained for a constant load and a variable load comprising linear and exponential functions. The mathematical model of the Timoshenko beam (TB) problem in the form of a boundary-value problem has been solved successfully for the rotation and displacement parameters. The results agree with other schemes in the literature for various values of the parameter and step size.*

**Keywords:** Timoshenko beam, Finite-difference solution, Rotation, Displacement, Constant load, Variable load, Interpolation.

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### I. Introduction

There are various engineers, scientists etc who had a long history in the development of numerical schemes, they performed different approaches subjected to loads to examine the performance and important characteristics of the beam [IV], [VI]. At the beginning of the 20<sup>th</sup> century, a new beam theory was developed by a Ukrainian-born scientist named Stephen Timoshenko [XII]. The theory was named as Timoshenko beam (TB) theory due to his name. Here, both shear deformation and rotational inertia effects were taken into account, in the TB model. The other beams' behavior is also described by the TB model, which includes short beams, composite sandwiched beams or the beams which can be excited by high frequency so that the wavelength of excitation becomes shorter.

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For the solution of more complex Reissner-Mindlin problem, as a starting point, TB model was used for the numerical approximation for better understanding. Sometimes locking phenomenon [III], [XIII] arises when the problems were solved using the finite difference method or standard Galerkin methods. To overcome the problem of the locking phenomenon, researchers proposed different schemes which are uniform concerning the small parameters. A formulation linked to Petrov-Galerkin was proposed in [IX], due to the use of the reduced integrations mixed formulations were proposed in [I], [II], [III], [X]. Finite difference schemes were proposed by authors [III], [XIII] which were for the TB model. In [V], the authors discussed the use of the least-squares finite element method. In [VII] authors tested the  $p$  and  $h$ - $p$  versions of the finite element method for the TB model. In [XI], a second-order accurate difference scheme was derived by the method of the reduction of order on non-uniform meshes and proves the stability and second-order convergence with the theoretical results. A linearized three-level difference scheme of the TB equations on uniform meshes was derived in [XV] by the method of reduction of order and obtained convergence order in maximum norm is of order two in both space and time. In [XVI] generalized eigenvector system of a Timoshenko beam with some linear boundary feedback concluded that the closed-loop system exhibits exponential stability. A finite-difference for a Timoshenko beam was derived in [XVII] by the method of reduction of order on uniform meshes. [XVII] established the efficiency and accuracy of the finite element method for calculating the eigenvalues and eigenmodes and created interesting phenomena concerning the damped vibration spectrum and the associated eigenmodes. [VIII] derived linearized three-level difference scheme of the Timoshenko beam equations on uniform meshes by the method of reduction of order on uniform meshes and proved unique solvability, unconditional stability and convergence of the difference scheme by the discrete energy method.

In terms of analytical solutions, the TB problem with the discussed approaches does not provide an exact profile as these are numerical. To have reduced processing time to achieve a bit of accuracy in the results the numerical schemes should be concise with parameters and small step sizes. The numerical results should be validated through analytical results as the stability of the numerical schemes are not always unconditional. In this research work, the focus is on the computations from a proposed numerical scheme, referred to as scheme-III ahead, and two previously existing schemes referred to as Schemes I and II. Also, the exact results are obtained from the analytical solution of TB model [XIV], [XV] when the beam is subjected to different loads which are uniform as well as variable loads, analyzing exactly the rotation and displacement profiles on the structure of the TB model. The approximate results obtained from the existing schemes as well as proposed scheme III were compared with the exact results, to examine the performance of the proposed scheme.

## **II. Mathematics of the TB Model**

In a TB model with clamping on both ends, consider a uniformly distributed load  $p(\bar{x})$ . Consequently, at each point of the beam the moment  $m(\bar{x})$  is apparent due to load  $p(\bar{x})$ . Hence, the beam bends within the plane as a result of load and moment. Further, we assume 'L' and 'A' are the length and area of the cross-section of the

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beam, 'E' is Young's modulus and 'G' is the modulus of rigidity. Let  $M(\bar{x})$ ,  $Q(\bar{x})$  and  $\theta(\bar{x})$  denote the bending moment, shear force and rotation of cross-section, respectively. Let  $W(\bar{x})$  stands for displacement as transversed by the beam and  $\kappa$  is the correction factor for shear. For the TB problem, the mathematical model in the form of a system of ordinary differential equations, for  $\bar{x} \in (0, L)$ , can be described as in equations (1)-(3).

$$-\frac{dQ}{d\bar{x}} = p \quad (1)$$

$$-EI\frac{d^2\theta}{d\bar{x}^2} - Q = 0 \quad (2)$$

$$-\frac{Q}{kGA} + \frac{dW}{d\bar{x}} - \theta = 0 \quad (3)$$

Due to static condition of both end points, the conditions at the boundary are:

$$W(0) = W(L) = 0; \quad \theta(0) = \theta(L) = 0$$

The non-dimensionalized problem for (1)-(3) can be described as:

$$-\sigma' = f \quad (4)$$

$$-\theta'' - \sigma = 0 \quad (5)$$

$$-\varepsilon^2\sigma + w' - \theta = 0 \quad (6)$$

The problem (4)-(6) demands to find  $w$ ,  $\theta$  and  $\sigma$  such that in  $x$  lies in  $(0,1)$ . The model (4)-(6) can further be simplified to get (7)-(8):

$$-\theta'' + \varepsilon^{-2}(\theta - w') = 0 \quad (7)$$

$$\varepsilon^{-2}(\theta' - w'') = f \quad (8)$$

subject to:  $w(0) = w(1) = 0; \quad \theta(0) = \theta(1) = 0$ .

The parameter  $\varepsilon^2 = EI/kGAL^2$  is a constant proportional to the ratio of the thickness to the length of the beam. In most realistic applications  $\varepsilon \ll 1$ .

### III. Existing and Proposed Finite Difference Schemes for TB model

There exist two finite difference schemes in literature [I] described as follows:

#### Existing Scheme I

In element  $\Delta_i = (x_{i-1}, x_i)$ , we let  $f \approx f_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}})$  be a constant. Then, solving (4)-(6) and (7)-(8) in  $\Delta_i$ , one has:

$$-\sigma' = f_{i-\frac{1}{2}} \quad (9)$$

$$-\theta'' - \sigma = 0 \quad (10)$$

$$-\varepsilon^2\sigma + w' - \theta = 0 \quad (11)$$

From (9), (10) and (11), we have the linear system, for  $i = 1, 2, \dots, N-1$ ,

$$\begin{aligned} \frac{12}{12\varepsilon^2+h^2} \left\{ \frac{\theta_{i-1}+2\theta_i+\theta_{i+1}}{4} - \frac{w_{i+1}-w_{i-1}}{2h} \right\} - \frac{\theta_{i-1}-2\theta_i+\theta_{i+1}}{h^2} &= \frac{1}{12}h \left( f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right) \\ \frac{12}{12\varepsilon^2+h^2} \left\{ \frac{\theta_{i+1}-\theta_{i-1}}{2h} - \frac{w_{i-1}-2w_i+w_{i+1}}{h^2} \right\} &= \frac{1}{2} \left( f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}} \right) \end{aligned} \quad (12)$$

and with the boundary conditions

$$\theta_0 = \theta_N = 0, \quad w_0 = w_N = 0 \quad (13)$$

### **Existing Scheme II**

Let  $\Delta^i = (x_{i-1}, x_{i+1})$ , we let  $f \approx \bar{f}_i = \frac{1}{2} \left( f(x_{i-\frac{1}{2}}) + f(x_{i+\frac{1}{2}}) \right)$  be a constant. Then, using these in (4)-(8), the other scheme was attained in the element  $\Delta^i$  as in (17) using (14)-(16):

$$-\sigma' = \bar{f}_i \quad (14)$$

$$-\theta'' - \sigma = 0 \quad (15)$$

$$-\varepsilon^2 \sigma + w' - \theta = 0 \quad (16)$$

$$\begin{aligned} \frac{3}{3\varepsilon^2+h^2} \left\{ \frac{\theta_{i-1}+\theta_{i+1}}{2} - \frac{w_{i+1}-w_{i-1}}{2h} \right\} - \frac{\theta_{i-1}-2\theta_i+\theta_{i+1}}{h^2} &= 0 \\ \frac{12}{12\varepsilon^2+h^2} \left\{ \frac{\theta_{i+1}-\theta_{i-1}}{2h} - \frac{w_{i-1}-2w_i+w_{i+1}}{h^2} \right\} &= \bar{f}_i \end{aligned} \quad (17)$$

We have the linear system, for  $i = 1, 2, \dots, N-1$  in (17) for the final numerical solution.

The boundary conditions are:

$$\theta_0 = \theta_N = 0, \quad w_0 = w_N = 0 \quad (18)$$

### **Proposed Scheme III**

In this research work, we are analyzing a new scheme whose pattern is parallel to [X]. Here both rotation and transverse displacement are converted into linear finite elements. For both, these variables' error is of optimal order. By comparing this error with a thickness of the beam, we obtain a uniformity.

Let  $\Delta^{ii} = (x_{i-2}, x_{i+2})$ , we let  $f \approx \bar{f}_i = \frac{1}{2} (f(x_{i-1}) + f(x_{i+1}))$  be a constant.

Then we solve (4)-(8) in  $\Delta^{ii}$ .

$$-\sigma' = \bar{f}_i \quad (19)$$

$$-\theta'' - \sigma = 0 \quad (20)$$

$$-\varepsilon^2 \sigma + w' - \theta = 0 \quad (21)$$

From (19), (20) and (21) the equations for numerical computations are

$$\frac{6}{3\varepsilon^2+4h^2} \left\{ (\theta_{i-2} + \theta_{i+2}) - \frac{(w_{i+2}-w_{i-2})}{2h} \right\} - \frac{(\theta_{i-2}-2\theta_i+\theta_{i+2})}{h^2} = 0$$

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$$\frac{3}{3\varepsilon^2+h^2}\left\{\frac{\theta_{i+2}-\theta_{i-2}}{4h}-\frac{w_{i-2}-2w_i+w_{i+2}}{4h^2}\right\}=\bar{f}_i \quad (22)$$

with the boundary conditions

$$\theta_0 = \theta_N = 0, \quad w_0 = w_N = 0 \quad (23)$$

It is important to verify the sensitivity of the numerical solutions by existing schemes and the proposed ones. For this, we theoretically establish the norm expression (24) using numerical approximations.

If the pair  $(\theta, w)$  represent solution of (4)- (8), then there exists a constant  $C$  independent of  $\varepsilon$  such that

$$\|\theta\|^\infty + \|w\|^\infty \leq C\|f\|^\infty \quad (24)$$

where  $\|\cdot\|^\infty$  is the uniform norm on the interval  $(0,1)$ .

From (4), we have

$$-\sigma(x) = -F(x) + C_1, \quad F(x) = \int_0^x f(t)dt$$

From (5) and the boundary conditions, we get

$$\theta(x) = -\frac{1}{2}C_1x^2 + C_2x + G(x)$$

with

$$G(x) = \int_0^x \left( \int_0^s F(t) dt \right) ds - x \int_0^1 \left( \int_0^s F(t) dt \right) ds$$

From (6), we obtain

$$w(x) = -\frac{1}{6}C_1x^3 + \frac{1}{4}C_2x^2 + \varepsilon^2 C_3x - \varepsilon^2 \int_0^x F(t) dt + \int_0^x G(t) dt$$

Applying the conditions  $w(0) = w(1) = 0$ , we can derive

$$C_1 = \frac{\varepsilon^2 \int_0^1 F(t) dt - \int_0^1 G(t) dt}{\varepsilon^2 + \frac{1}{12}}$$

Thus for  $0 < \varepsilon < 1$ , we can verify

$$\|F\|^\infty \leq C\|f\|^\infty, \quad \|G\|^\infty \leq C\|f\|^\infty, \quad |C_1| \leq C\|f\|^\infty$$

Thus (24) is established.

Finally, denoting the piecewise constant function  $f_h(x)$  from  $f(x)$  by  $f_h(x) = \left(x_{i-\frac{1}{2}}\right)$ ,  $x_{i-1} < x < x_i$ , we consider the equations:

$$-\tilde{\theta}'' + \varepsilon^{-2}(\tilde{\theta} - \tilde{w}') = 0 \quad (25)$$

$$\varepsilon^{-2}(\tilde{\theta} - \tilde{w}') = f_h \quad (26)$$

together with the boundary conditions

$$\tilde{w}(0) = \tilde{w}(1) = 0, \quad \tilde{\theta}(0) = \tilde{\theta}(1) = 0$$

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we can see that the

$$\tilde{\theta}(x_i) = \tilde{\theta}_i, \quad \tilde{w}(x_i) = \tilde{w}_i, \quad i = 0, 1, 2, \dots, N$$

for the solutions of (12) and (13). Therefore, as the  $\theta(x) - \tilde{\theta}(x)$  and  $w(x) - \tilde{w}(x)$  are the solution of (7)-(8) with the replacement  $f(x)$  by  $f(x) - f_h(x)$ , then we apply (24) theoretically to get:

$$\max_i |\theta(x_i) - \tilde{\theta}_i| + \max_i |w(x_i) - \tilde{w}_i| \leq C \|f - f_h\|^\infty$$

It is the uniform bounds concerning the small parameter  $\varepsilon$ .

#### IV. Results and Discussion

Here, we consider two types of loads, the first one is a uniform load for  $0 < x < 1$ , and the second one is a combination of linear and exponential loads. To solve the TB model, we have performed numerical experiments by using two existing schemes: schemes I and II, and the proposed one scheme III. The results depend upon the variation in step size and dependent parameter  $\varepsilon$ , also the comparison of proposed scheme III obtained results with existing schemes I and II are discussed.

For the uniform load case,  $f = 1$ ;  $0 < x < 1$  the exact solutions of the TB model are

$$\theta(x) = \frac{1}{12} x(1-x)(1-2x) \quad (27)$$

$$w(x) = \frac{1}{24} x^2(1-x)^2 + \frac{1}{2} \varepsilon^2 x(1-x) \quad (28)$$

For the variable load case,  $f(x) = 100(e^x + x)$ ;  $0 < x < 1$

The exact solutions for the variable load considered are:

$$\theta(x) = 100 \left( e^x + \frac{x^4}{24} \right) - c_1 \frac{x^2}{2} + c_2 x + c_3 \quad (29)$$

$$w(x) = 100 \left( e^x + \frac{x^5}{120} \right) - c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} - 100x - 100 \varepsilon^2 \left( e^x + \frac{x^3}{6} \right) + \varepsilon^2 c_1 x + c_4 \quad (30)$$

Where,

$$c_1 = \frac{1}{1+12\varepsilon^2} \{600(2\varepsilon^2-1)e - 1000 \varepsilon^2 + 1815\} \quad (31)$$

$$c_2 = \frac{1}{2(1+12\varepsilon^2)} \{-400(2+3\varepsilon^2)e - 1000 \varepsilon^2 + 1815\} + \frac{575}{6} \quad (32)$$

$$c_3 = -100, \quad (33)$$

$$c_4 = 100(\varepsilon^2 - 1) \quad (34)$$

The exact solutions have been quoted from the literature for reference, as these are discussed ahead to explain the approximate values obtained by the proposed as well as existing schemes. We may notice that in the exact solution of uniform load, the rotation profile is free of the parameter  $\varepsilon$ , which is shown in (27), so due to this reason, the approximate values of rotation for  $\varepsilon = 0.5, 0.1, 0.01$  obtained by all schemes are same, and the graphs shown in Figs 1, 3 and 5 are same. Another observation while computing the approximate values by all schemes is that when the

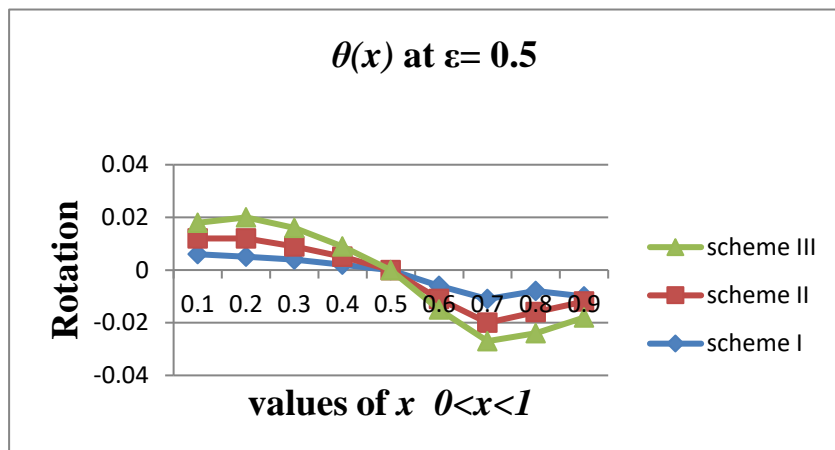
only dependent parameter  $\varepsilon$  is reduced from 0.001 to 0.0001, while computing variable load by the schemes, as a result for both these values of  $\varepsilon$ , the rotation and displacement values remains same. This means, when  $\varepsilon$  becomes smaller then the values obtained remain the same.

Figs 1,3 and 5 shows the graphs of rotation of uniform load at  $\varepsilon = 0.5, 0.1, 0.01$ , it can be easily noticed that all these three graphs are same, this happens because, in the exact solution of rotation of uniform load shown in (27),  $\varepsilon$  is not present, so the proposed schemes, as well as existing schemes, also give same values at  $\varepsilon = 0.5, 0.1, 0.01$ . But the values obtained from proposed scheme III show the exact rotation curve of TB graphically as compared to the existing schemes I and II curves.

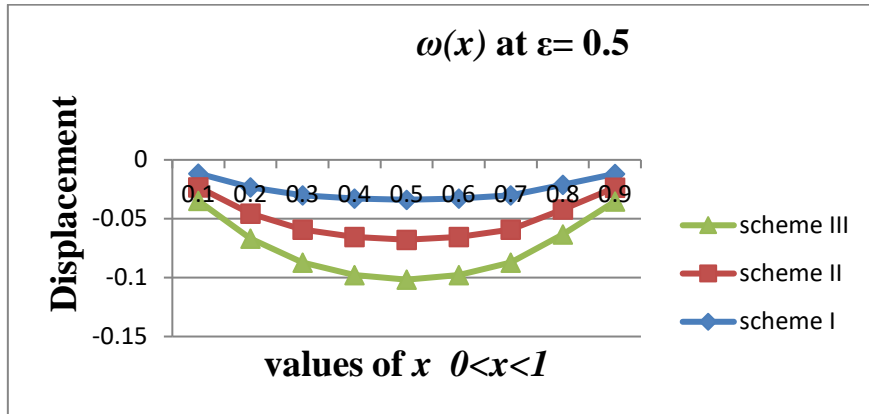
Figs 2,4 and 6 show graphs of displacement of uniform load at  $\varepsilon = 0.5, 0.1, 0.01$ , it can be easily noticed, that the curve obtained from the approximate values of proposed scheme III in  $0 < x < 1$  shows the exact displacement curve of TB, as when the load is placed over TB, then due to the load as well as the force of gravity of earth, the beam will be bent around, and as it is an especial type of beam, which has more elasticity as compared to an ordinary beam, so it can be bent to a large extent without becoming rigid, which is proved by the curve of computed values from proposed scheme III.

A similar efficient performance can be seen by the proposed scheme III in variable load also, in Figs 7,9,11, and 13, the graphs of rotation are shown at  $\varepsilon = 0.1, 0.01, 0.001, 0.0001$  and the curves plotted by the obtained approximate values from scheme III shows the perfect rotation curve of TB graphically as compared to the curves plotted from the obtained approximate values from existing schemes I and II.

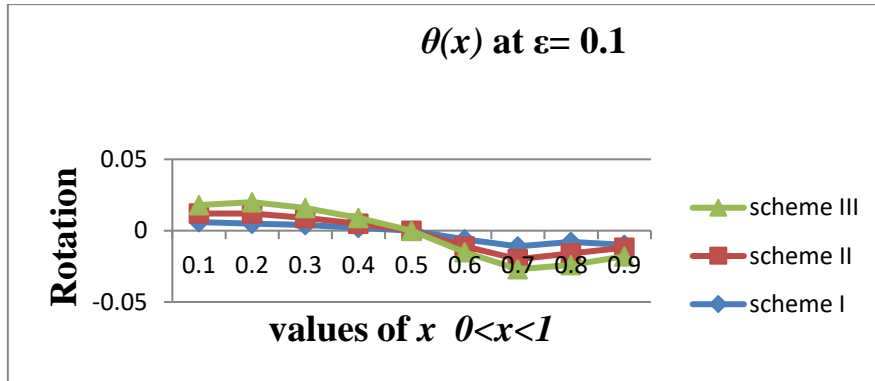
The same is the case with the graphs of displacement of variable load shown in Figs 8, 10, 12 and 14, the exact displacement curve can be seen from the approximate obtained values from proposed scheme III as compared to the existing schemes I and II.



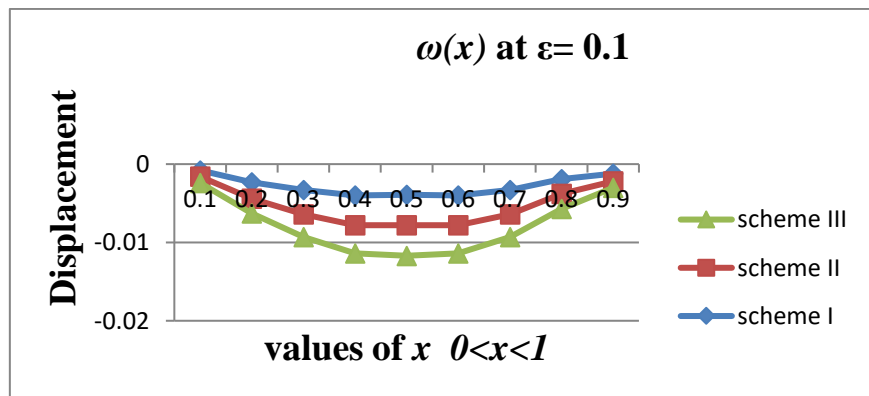
**Figure 1.** Numerical solutions of rotation for the uniform load  $f(x) = 1$ ,  $\varepsilon = 0.5$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.



**Figure 2.** Numerical solutions of displacement for the load  $f(x) = 1$ ,  $\varepsilon = 0.5$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.

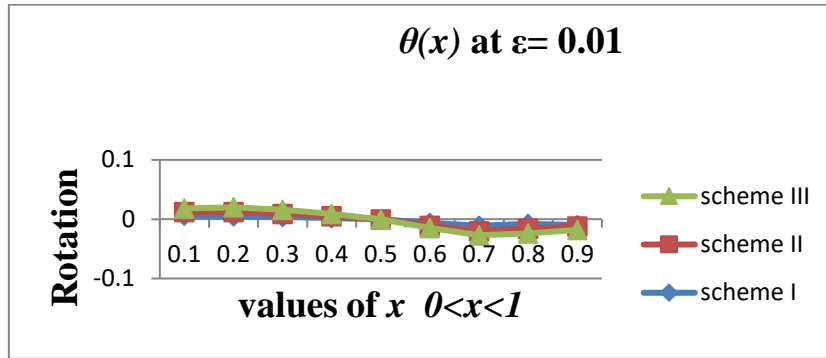


**Figure 3.** Numerical solutions of rotation for the load  $f(x) = 1$ ,  $\varepsilon = 0.1$ ,  $0 < x < 1$ ,  $h = 1/10$  by the schemes I-III.

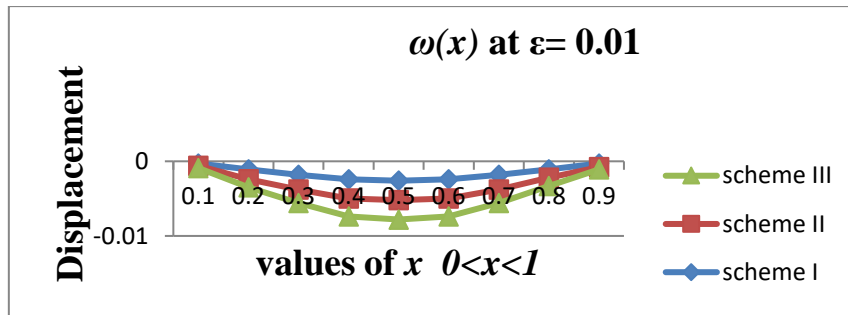


**Figure 4.** Numerical solutions of displacement for the load  $f(x) = 1$ ,  $\varepsilon = 0.1$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.

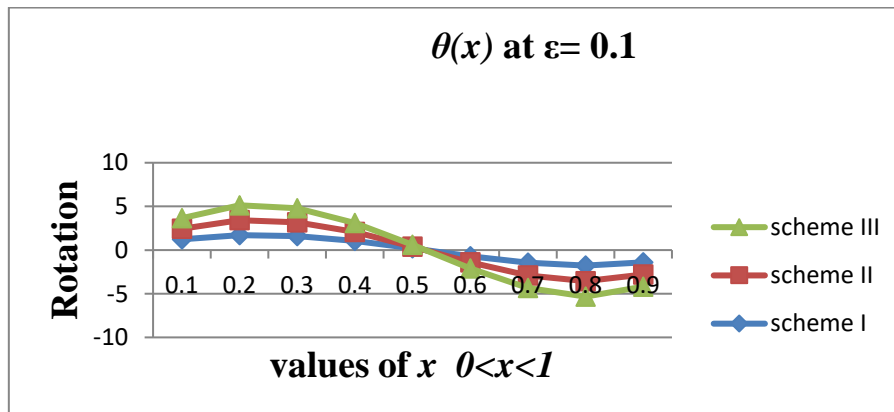




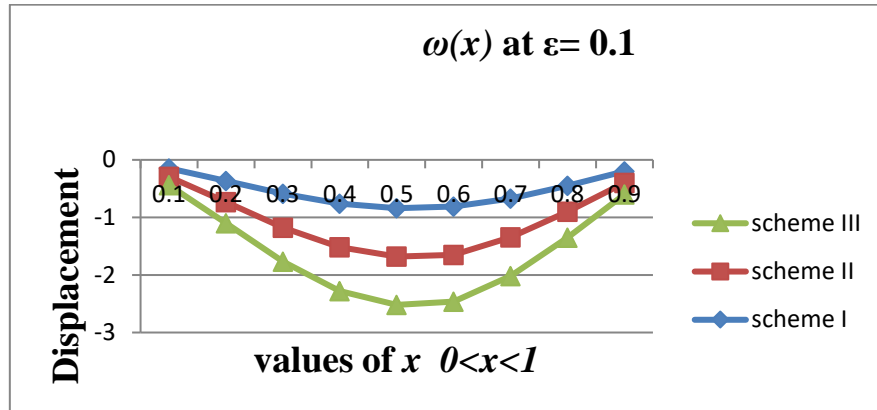
**Figure 5.** Numerical solutions of rotation for the load  $f(x) = 1$ ,  $\varepsilon = 0.01$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.



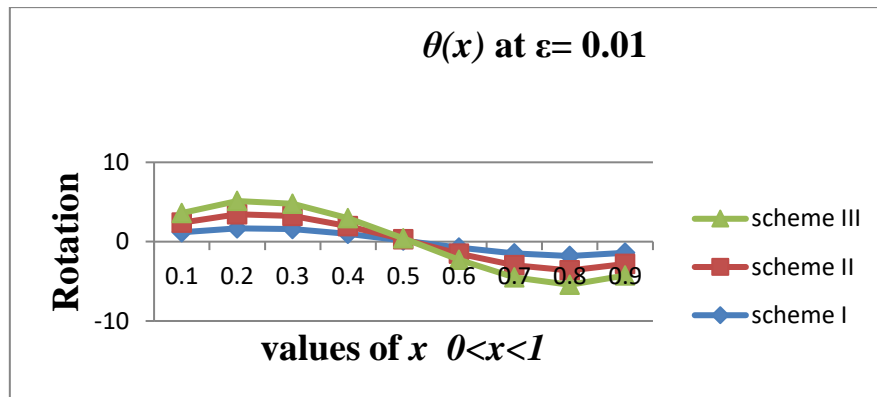
**Figure 6.** Numerical solutions of displacement for the load  $f(x) = 1$ ,  $\varepsilon = 0.01$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.



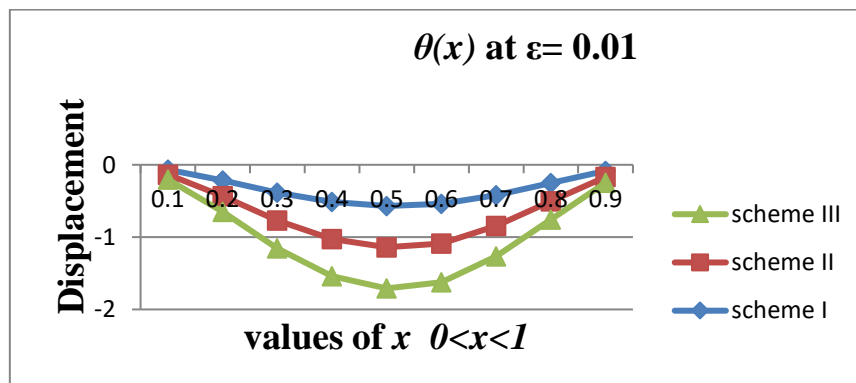
**Figure 7.** Numerical solutions of rotation for the load  $f(x) = 100(e^x + x)$ ,  $\varepsilon = 0.1$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.



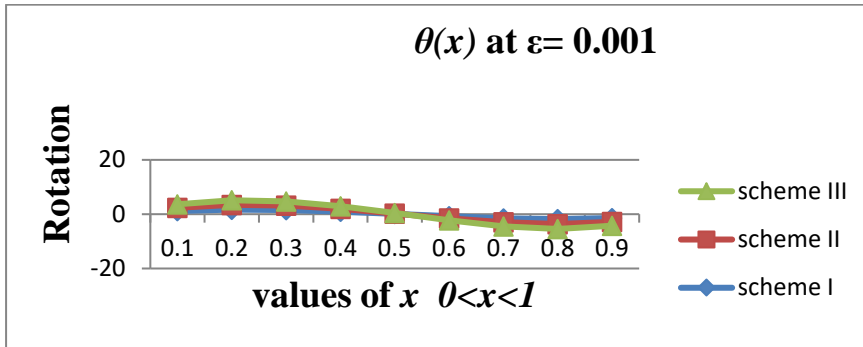
**Figure 8.** Numerical solutions of displacement for the load  $f(x) = 100(e^x + x)$ ,  $\varepsilon = 0.1$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.



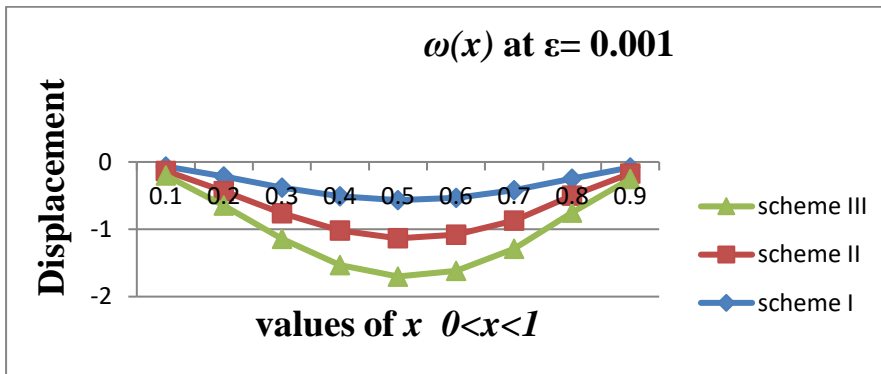
**Figure 9.** Numerical solutions of rotation for the load  $f(x) = 100(e^x + x)$ ,  $\varepsilon = 0.01$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.



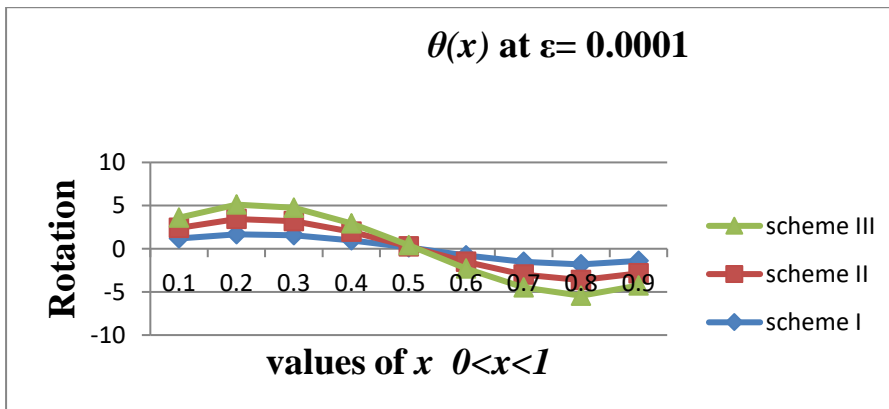
**Figure 10.** Numerical solutions of displacement for load  $f(x) = 100(e^x + x)$ ,  $\varepsilon = 0.01$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.



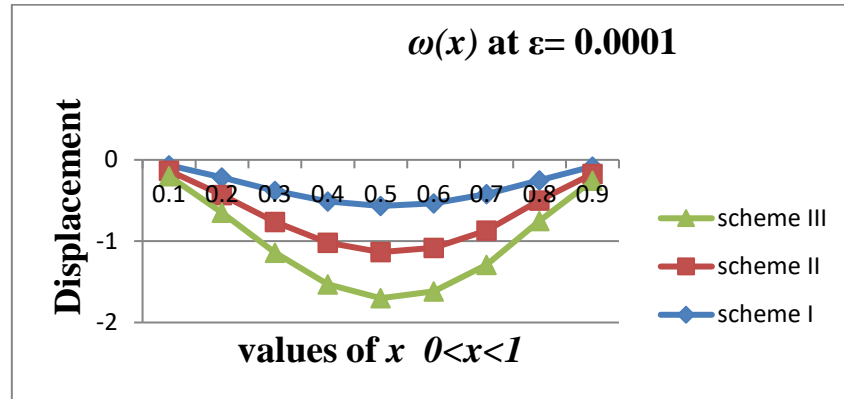
**Figure 11.** Numerical solutions of rotation for the load  $f(x) = 100(e^x + x)$ ,  $\varepsilon = 0.001$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.



**Figure 12.** Numerical solutions of displacement for the load  $f(x) = 100(e^x + x)$ ,  $\varepsilon = 0.001$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.



**Figure 13.** Numerical solutions of rotation for the load  $f(x) = 100(e^x + x)$ ,  $\varepsilon = 0.0001$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.



**Figure 14.** Numerical solutions of displacement for the load  $f(x) = 100(e^x + x)$ ,  $\varepsilon = 0.0001$ ,  $0 < x < 1$ ,  $h = 1/10$  by schemes I-III.

## V. Conclusion

In this research work, non-standard finite difference schemes have been developed for the solution of TB. Numerical experiments were performed on two types of loads. One load is constant and the other is variable load. Numerical solutions for rotation and displacement are computed using proposed scheme III as well as existing schemes I and II. The rotation and displacement curves plotted by the obtained approximate values from the proposed scheme III are graphically more closer to exact values as compared to the curves plotted by the approximate values from the existing schemes I and II, which proves the efficient performance of the proposed scheme III.

In the future, for the solution of any beam model, the proposed scheme can be implemented to obtain better accuracy.

## Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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