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A NEW HARMONIC MEAN DERIVATIVE-BASED SIMPSON'S 1/3-TYPE SCHEME FOR RIEMANN-STIELTJES INTEGRAL

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Abstract

In this research paper, a new harmonic mean derivative-based Simpson's 1/3 scheme has been presented for the Riemann-Stieltjes integral (RS-integral). The basic and composite forms of the proposed scheme with local and global error terms have been derived for the RS-integral. The proposed scheme has been reduced using g(t) = t for Riemann integral. Experimental work has been discussed to verify the theoretical results of the new proposed scheme against existing schemes using MATLAB. The order of accuracy, computational cost and average CPU time (in seconds) of the new proposed scheme have been computed. Finally, it is observed from computational results that the proposed scheme is better than existing schemes.

Keywords : Quadrature rule, Riemann-Stieltjes, Harmonic Mean, Simpson's 1/3 rule, Composite form, Local error, Global error, Cost-effectiveness, Time-efficiency

I. Introduction

Numerical integration is the study of algorithms that finds an approximate solution for problems. Numerical integration is used to find the area under a curve, the surface area of a solid, the volume of the solid figure, the arc length of a graph, central points, moment of inertia and many useful things. Unfortunately for a definite integral $I(f) = \int_a^b f(x)\,dx$, some functions $f(x) = e^{x^2}$ or $\sin x^2$ have no simple antiderivatives in such cases if the value of a definite integral is needed it will have to be approximated. Such definite integrals can be evaluated by numerical methods. The process of finding the approximate value of a definite integral is known as quadrature. The Riemann-Stieltjes integral is a modification of the Riemann integral. Suppose $f(x)$ and $\alpha(x)$ be two bounded functions on $[a, b]$ in which $\alpha(x)$ is monotonically increasing in $[a, b]$, the RS-integral can be defined in [III] as:

$$RS(f(x):\alpha;a,b)=\int_a^b f(x)d\alpha(x) \tag{1}$$

where $f(x)$ is integrand and $\alpha(x)$ is an integrator.

RS-integrals can be applied in Statistics and theory of Probability, Complex analysis, Functional analysis, the theory of the Operator, and other fields. Numerical quadrature is widely used for the Riemann integral in literature as in [XVII] introduced a new family of closed Newton-Cotes quadrature schemes with Midpoint derivative for the Riemann integral. Authors in [XIII], [XIV], [XV], [XVI] derived derivative-based closed Newton-Cotes quadrature schemes using different means such as geometric, harmonic, heronian and centroidal at the evaluation of function derivative.

Quadrature schemes have been extended for cubature. Authors in [IV], proposed some new and efficient derivative-based schemes for numerical cubature. Also in [V], derived error of closed Newton-Cotes cubature schemes for double integrals.

In the past, however, only a few works have been focused on the numerical approximation for the RS-integrals. At first in [X], Hadamard inequality can be described in the approximation of Trapezoid-type for the RS-integral. Also, in [XI], some important inequalities were proposed for the approximation of the RS-integral using the midpoint and Simpson rules, based on the relative convexity concept.

The midpoint derivative-based quadrature scheme of trapezoid-type for the RS-integral was first proposed in [XVIII]. After that, without numerical verifications, the authors proposed the composite form of trapezoidal rule for the RS-integral in [XIX]. New efficient derivative-based and derivative-free quadrature schemes for the RS-integral have recently been proposed in [VI], [VII], [VIII], [IX] with numerical verifications.

In this study, a new harmonic mean derivative-based Simpson's 1/3-type scheme is developed for the RS-integral. The derivations of the proposed scheme are derived in basic and composite forms with error terms. The performance of the proposed scheme is checked by numerical experiments in terms of cost efficiency, time efficiency and rapid convergence.

II. Existing Quadrature Schemes for the RS-Integral

Some existing quadrature schemes: T [X], ZT [XVIII], MZT [VI], HeMS13 [VIII], CMS13 [IX], can be described for the RS-integral in basic form in (2)-(6) as:

$$T\approx\left(\frac{1}{b-a}\int_a^b g(t)dt-g(a)\right)f(a)+\left(g(b)-\frac{1}{b-a}\int_a^b g(t)dt\right)f(b) \tag{2}$$

$$\begin{aligned}ZT\approx&\left(\frac{1}{b-a}\int_a^b g(t)dt-g(a)\right)f(a)+\left(g(b)-\frac{1}{b-a}\int_a^b g(t)dt\right)f(b)\\&+\left(\int_a^b\int_a^t g(x)dxdt-\frac{b-a}{2}\int_a^b g(t)dt\right)f''(c_{ZT})\end{aligned} \tag{3}$$

where

$$c_{ZT} = \frac{(-2b^2+a^2-ab)\int_a^b g(t)dt + 6b\int_a^b\int_a^t g(x)dxdt - 6\int_a^b\int_a^t\int_a^y g(x)dxdydt}{6\int_a^b\int_a^t g(x)dxdt - 3(b-a)\int_a^b g(t)dt}.$$

$$MZT \approx \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right)f(b)$$
$$+\left(\int_a^b\int_a^t g(x)dxdt - \frac{b-a}{2}\int_a^b g(t)dt\right)f''(c_{MZT}) \quad (4)$$

where

$$c_{MZT} = \frac{(-2b^2+a^2+ab)\int_a^b g(t)dt + 6b\int_a^b\int_a^t g(x)dxdt - 6\int_a^b\int_a^t\int_a^y g(x)dxdydt}{6\int_a^b\int_a^t g(x)dxdt - 3(b-a)\int_a^b g(t)dt}.$$

$$HeMS13 \approx \left(\frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)f(a)$$
$$+\left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right)f\left(\frac{a+b}{2}\right)$$
$$+\left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right)f(b)$$
$$+\begin{pmatrix}\frac{-(b-a)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdydt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydz\,dt\end{pmatrix} f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right) \quad (5)$$

$$CMS13 \approx \left(\frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)f(a)$$
$$+\left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right)f\left(\frac{a+b}{2}\right)$$
$$+\left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right)f(b)$$
$$+\begin{pmatrix}\frac{-(b-a)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdydt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydz\,dt\end{pmatrix} f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right) \quad (6)$$

The composite forms of the CT, ZCT, MZCT, HeMCS13 and CMCS13 schemes are described in (7)-(11) as:

$$CT \approx \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) \quad (7)$$

$$ZCT \approx \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) + \sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{h}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right]f''(c_{ZT,k}) \quad (8)$$

$$\int_a^b f(t)dg \approx MZCT = \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) + \sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{h}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right]f''(c_k) \quad (9)$$

Where,

$$\boldsymbol{c_{ZT,k}} = \frac{(-2x_k^2+x_{k-1}^2-x_{k-1}x_k)\int_{x_{k-1}}^{x_k} g(t)dt+6b\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt}{6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-\frac{3(x_k-x_{k-1})}{n}\int_{x_{k-1}}^{x_k} g(t)dt}$$

$$\boldsymbol{c_{MZT,k}} = \frac{(-2x_k^2+x_{k-1}^2+x_{k-1}x_k)\int_{x_{k-1}}^{x_k} g(t)dt+6b\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt}{6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-\frac{3(x_k-x_{k-1})}{n}\int_{x_{k-1}}^{x_k} g(t)dt}.$$

$$\begin{aligned}\int_a^b f(t)dg \approx HeMCS13 &= \left[\frac{4n^2}{(b-a)^2}\int_a^{x_1}\int_a^t g(x)dxdt - \frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a)\\
&+\frac{4n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt - \frac{2n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right] f\left(\frac{x_{k-1}+x_k}{2}\right)\\
&+\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\frac{4n}{b-a}\left(\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt + \int_{x_k}^{x_{k+1}}\int_{x_k}^{t} g(x)dxdt\right) - \left(3\int_{x_{k-1}}^{x_k} g(t)dt + \int_{x_k}^{x_{k+1}} g(t)dt\right)\right] f(x_k)\\
&+\sum_{k=1}^{n}\begin{bmatrix}\frac{-h^2}{96}(3x_{k-1}+5x_k)\int_{x_{k-1}}^{x_k} g(t)dt\\ +\frac{17{x_k}^2-10x_{k-1}x_k-7{x_{k-1}}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\\ -x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdy\,dt\\ +\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y} g(x)dxdy\,dzdt\end{bmatrix} f^{(4)}\left(\frac{x_{k-1}+\sqrt{x_{k-1}x_k}+x_k}{3}\right)+\\
&\left[g(b) - \frac{3n}{b-a}\int_{x_{n-1}}^{b} g(t)dt + \frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right] f(b)\end{aligned} \tag{10}$$

$$\begin{aligned}\int_a^b f(t)dg \approx CMCS13 &= \left[\frac{4n^2}{(b-a)^2}\int_a^{x_1}\int_a^t g(x)dxdt - \frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a)\\
&+\frac{4n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt - \frac{2n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right] f\left(\frac{x_{k-1}+x_k}{2}\right)\\
&+\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\frac{4n}{b-a}\left(\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt + \int_{x_k}^{x_{k+1}}\int_{x_k}^{t} g(x)dxdt\right) - \left(3\int_{x_{k-1}}^{x_k} g(t)dt + \int_{x_k}^{x_{k+1}} g(t)dt\right)\right] f(x_k)\\
&+\sum_{k=1}^{n}\begin{bmatrix}\frac{-h^2}{96}(3x_{k-1}+5x_k)\int_{x_{k-1}}^{x_k} g(t)dt\\ +\frac{17{x_k}^2-10x_{k-1}x_k-7{x_{k-1}}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\\ -x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdy\,dt\\ \int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y} g(x)dxdy\,dzdt\end{bmatrix} f^{(4)}\left(\frac{2({x_{k-1}}^2+x_{k-1}x_k+{x_k}^2)}{3(x_{k-1}+x_k)}\right)+\\
&\left[g(b) - \frac{3n}{b-a}\int_{x_{n-1}}^{b} g(t)dt + \frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right] f(b)\end{aligned} \tag{11}$$

III. Proposed a new Harmonic Mean Derivative-Based Simpson's 1/3-type Scheme for the Riemann-Stieltjes Integral

The basic form of harmonic mean derivative-based Simpson's 1/3-type (HMS13) scheme for the Riemann integral is described in (12) as:

$$\int_a^b f(x)dx = HMS13 = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{2ab}{a+b}\right) - \frac{(b-a)^7}{5760(a+b)} f^{(6)}(\xi), \tag{12}$$

The precision of this rule is 4.

Based on (12), the proposed scheme, i.e. HMS13 for the RS-integral, in basic form is derived in Theorem 1.

Theorem 1. Let f'(t)and g(t) be continuous on [a, b] and g(t) be increasing there. Then the proposed harmonic mean derivative-based Simpson's 1/3 scheme for the RS-integral with local error term $R_{HMS13}[f]$ can be described as:

$$\begin{aligned}\int_a^b f(t)dg = HMS13 + R_{HMS13}[f] = &\left(\frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a)\right) f(a) \\ &+ \left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right) f\left(\frac{a+b}{2}\right) \\ &+ \left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right) f(b) \\ &+ \begin{pmatrix} \frac{-(b-a)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdydt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydz\,dt \end{pmatrix} f^{(4)}\left(\frac{2ab}{a+b}\right) \\ &+ \begin{pmatrix} \frac{(a-b)^2(-7a^3+37a^2b+67ab^2-17b^3)}{960(a+b)}\int_a^b g(t)dt \\ + \frac{13b^4+20a^3b+30a^2b^2-60ab^3-3a^4}{96(a+b)}\int_a^b\int_a^t g(x)dxdt \\ + \frac{b^2(3a-b)}{2(a+b)}\int_a^b\int_a^t\int_a^y g(x)dxdydt + \frac{b(b-a)}{(a+b)}\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt \\ - \int_a^b\int_a^t\int_a^w\int_a^z\int_a^y g(x)dxdydz\,dwdt \end{pmatrix} f^{(6)}(\xi)g'(\eta), \end{aligned} \tag{13}$$

where $\xi, \eta \in (a,b)$.

Proof of Theorem 1.

To derive the harmonic mean derivative-based Simpson's 1/3 scheme for the RS-integral, we search numbers: a_0, b_0, c_0, d_0 so that:

$$\int_a^b f(t)dg \approx a_0 f(a) + b_0 f\left(\frac{a+b}{2}\right) + c_0 f(b) + d_0 f^{(4)}\left(\frac{2ab}{a+b}\right) \tag{14}$$

is exact for $f(t) = 1, t, t^2, t^3, t^4$. That is,

$$\int_a^b 1dg = a_0 + b_0 + c_0$$

$$\int_a^b t dg = a_0 a + b_0 \left(\frac{a+b}{2}\right) + c_0 b$$

$$\int_a^b t^2 dg = a_0 a^2 + b_0 \left(\frac{a+b}{2}\right)^2 + c_0 b^2$$

$$\int_a^b t^3 dg = a_0 a^3 + b_0 \left(\frac{a+b}{2}\right)^3 + c_0 b^3$$

$$\int_a^b t^4 dg = a_0 a^4 + b_0 \left(\frac{a+b}{2}\right)^4 + c_0 b^4 + 24 d_0$$

By using integration by parts of the RS-integral, as in [XVIII], we have the following system of equations (15)-(19).

$$a_0 + b_0 + c_0 = g(b) - g(a) \tag{15}$$

$$a_0 a + b_0 \left(\frac{a+b}{2}\right) + c_0 b = bg(b) - ag(a) - \int_a^b g(t)dt \tag{16}$$

$$a_0 a^2 + b_0 \left(\frac{a+b}{2}\right)^2 + c_0 b^2 = b^2 g(b) - a^2 g(a) - 2b \int_a^b g(t)dt + 2\int_a^b \int_a^t g(x)dx\, dt \tag{17}$$

$$a_0 a^3 + b_0 \left(\frac{a+b}{2}\right)^3 + c_0 b^3 = b^3 g(b) - a^3 g(a) - 3b^2 \int_a^b g(t)dt$$
$$+6b \int_a^b \int_a^t g(x)dxdt - 6\int_a^b \int_a^t \int_a^y g(x)dxdt \tag{18}$$

$$a_0 a^4 + b_0 \left(\frac{a+b}{2}\right)^4 + c_0 b^4 + 24 d_0 = b^4 g(b) - a^4 g(a) - 4b^3 \int_a^b g(t)dt$$
$$+12b^2 \int_a^b \int_a^t g(x)dxdt - 24b \int_a^b \int_a^t \int_a^y g(x)dxdydt$$
$$+24 \int_a^b \int_a^t \int_a^z \int_a^y g(x)dxdydz\, dt \tag{19}$$

The system of linear equations (15)-(19) can be described with the coefficient matrix as:

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ a & \frac{a+b}{2} & b & 0 \\ a^2 & \left(\frac{a+b}{2}\right)^2 & b^2 & 0 \\ a^3 & \left(\frac{a+b}{2}\right)^3 & b^3 & 0 \\ a^4 & \left(\frac{a+b}{2}\right)^4 & b^4 & 24 \end{bmatrix}$$

The reduced row echelon form of *M* is:

$$M \overset{R}{\approx} M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As in M_R, rank(M) = 4. To check the linearly independent rows, take a = -1 and b = 1 in matrix M then

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 24 \end{bmatrix}$$

This shows that the first three and fifth are linearly independent rows whereas the fourth row is linearly dependent. To find the coefficients a_0, b_0, c_0 and d_0, we solve equations (15), (16), (17) and (19) simultaneously, to have:

$$a_0 = \frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a),$$

$$b_0 = \frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt,$$

$$c_0 = g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt,$$

$$d_0 = -\frac{(a-b)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{(17b^2-10ab-7a^2)}{48}\int_a^b \int_a^t g(x)dxdt$$
$$-b\int_a^b \int_a^t \int_a^y g(x)dxdy\,dt + \int_a^b \int_a^t \int_a^z \int_a^y g(x)\,dxdy\,dzdt$$

Putting the values of coefficients a_0, b_0, c_0 and d_0 in (14),we have:

$$\int_a^b f(t)dg \approx HMS13 = \left(\frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)f(a)$$
$$+\left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt\right)f\left(\frac{a+b}{2}\right)$$
$$+\left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt\right)f(b)$$
$$+\left(\begin{matrix}\frac{-(b-a)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b \int_a^t g(x)dxdt \\ -b\int_a^b \int_a^t \int_a^y g(x)dxdydt + \int_a^b \int_a^t \int_a^z \int_a^y g(x)dxdydz\,dt\end{matrix}\right)f^{(4)}\left(\frac{2ab}{a+b}\right)$$

Now we derive the local error term of the proposed HMS13 scheme.

Since the precision of the proposed HMS13 scheme is 4, we take $f(t) = \frac{t^5}{5!}$ to find the leading error term defined as:

$$R_{HMS13}[f] = \frac{1}{5!}\int_a^b t^5 dg - HMS13(t^5; g; a, b) \tag{20}$$

We learn, from [XVIII], that:

$$\frac{1}{5!}\int_a^b t^5 dg = \frac{1}{120}(b^5 g(b) - a^5 g(a)) - \frac{b^4}{24}\int_a^b g(t)dt + \frac{b^3}{6}\int_a^b \int_a^t g(x)dxdt$$
$$-\frac{b^2}{2}\int_a^b \int_a^t \int_a^y g(x)dxdydt + b\int_a^b \int_a^t \int_a^z \int_a^y g(x)dxdydzdt$$
$$-\int_a^b \int_a^t \int_a^w \int_a^z \int_a^y g(x)\,dxdydzdwdt \tag{21}$$

By Theorem 1 and scheme (13), we have:

$$HMS13(t^5; g; a, b) = \left(\frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)\frac{a^5}{5!}$$
$$+\left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right)\frac{(a+b)^5}{2^5.5!}$$
$$+\left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right)\frac{b^5}{5!}$$
$$+\begin{pmatrix}\frac{-(a-b)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdydt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydz\,dt\end{pmatrix}\left(\frac{2ab}{a+b}\right) \quad (22)$$

We use (21) and (22) in (20) to get:

$$R_{HMS13}[f] = \left(\frac{(a-b)^2(-7a^3+37a^2b+67ab^2-17b^3)}{960(a+b)}\int_a^b g(t)dt\right.$$
$$+\frac{13b^4+20a^3b+30a^2b^2-60ab^3-3a^4}{96(a+b)}\int_a^b\int_a^t g(x)dxdt$$
$$+\frac{b^2(3a-b)}{2(a+b)}\int_a^b\int_a^t\int_a^y g(x)dxdydt + \frac{b(b-a)}{(a+b)}\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt$$
$$\left.-\int_a^b\int_a^t\int_a^w\int_a^z\int_a^y g(x)dxdydz\,dwdt\right)f^{(6)}(\xi)g'(\eta), \quad (23)$$

which is same as (13), and the precision of this scheme is 4.

Theorem 2. When g(t) = t, the proposed HMS13 scheme with the error term (13) for the RS-integrals is reduced to the corresponding HMS13 scheme [XIV], i.e. (12) for the Riemann integrals.

Proof of Theorem 2.
By Theorem 1, we have:

$$\int_a^b f(t)dg = \int_a^b f(t)dt = \left(\frac{4}{(b-a)^2}\int_a^b\int_a^t xdxdt - \frac{1}{b-a}\int_a^b tdt - g(a)\right)f(a)$$
$$+\left(\frac{4}{b-a}\int_a^b tdt - \frac{8}{(b-a)^2}\int_a^b\int_a^t xdxdt\right)f\left(\frac{a+b}{2}\right)$$
$$+\left(g(b) - \frac{3}{b-a}\int_a^b tdt + \frac{4}{(b-a)^2}\int_a^b\int_a^t xdxdt\right)f(b)$$
$$+\begin{pmatrix}\frac{-(b-a)^2(3a+5b)}{96}\int_a^b tdt + \frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t xdxdt \\ -b\int_a^b\int_a^t\int_a^y xdxdydt + \int_a^b\int_a^t\int_a^z\int_a^y xdxdydz\,dt\end{pmatrix}f^{(4)}\left(\frac{2ab}{a+b}\right)$$
$$+\left(\frac{(a-b)^2(-7a^3+37a^2b+67ab^2-17b^3)}{960(a+b)}\int_a^b g(t)dt\right.$$
$$+\frac{13b^4+20a^3b+30a^2b^2-60ab^3-3a^4}{96(a+b)}\int_a^b\int_a^t g(x)dxdt$$
$$+\frac{b^2(3a-b)}{2(a+b)}\int_a^b\int_a^t\int_a^y g(x)dxdydt + \frac{b(b-a)}{(a+b)}\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt$$
$$\left.-\int_a^b\int_a^t\int_a^w\int_a^z\int_a^y g(x)dxdydz\,dwdt\right)f^{(6)}(\xi)g'(\eta) \quad (24)$$

It is obvious to get:

$$\int_a^b t dt = \frac{b^2-a^2}{2},$$

$$\int_a^b \int_a^t x dx dt = \frac{b^3}{6} - \frac{a^2 b}{2} + \frac{a^3}{3},$$

$$\int_a^b \int_a^t \int_a^y x dx dy\, dt = \frac{b^4}{24} - \frac{a^2 b^2}{4} + \frac{a^3 b}{3} - \frac{a^4}{8},$$

$$\int_a^b \int_a^t \int_a^z \int_a^y x\, dx dy dz\, dt = \frac{b^5}{120} - \frac{a^4 b}{8} + \frac{a^3 b^2}{6} - \frac{a^2 b^3}{12} + \frac{a^5}{30},$$

$$\int_a^b \int_a^t \int_a^w \int_a^z \int_a^y x dx dy dz dw dt = \frac{b^6}{720} + \frac{a^5 b}{30} - \frac{a^4 b^2}{16} + \frac{a^3 b^3}{18} - \frac{a^2 b^4}{48} - \frac{a^6}{144},$$

And, finally using these in (24) we get:

$$\int_a^b f(t)\, dt = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{2ab}{a+b}\right) - \frac{(b-a)^7}{5760(a+b)} f^{(6)}(\xi), \quad (25)$$

where $\xi \in (a,b)$.

This shows that the proposed HMS13 rule is reducible to the classical Riemann integral form (12) in terms of harmonic mean.

Now, the proposed harmonic mean derivative-based Simpson's 1/3 scheme for the RS-integral is derived in composite form by dividing the interval into small subintervals and applying integration rule to each subinterval, the composite of the proposed scheme is described in Theorem 3.

Theorem 3. Let f'(t) and g(t) be continuous on [a, b] and g(t) be increasing there. Let the interval [a, b] be subdivided into 2n subintervals $[x_k, x_{k+1}]$ with width h = $\frac{b-a}{n}$ by using the equally spaced nodes $x_k = a+kh$, where k = 0, 1, …, n. The composite Harmonic mean Simpson's 1/3 scheme to 2n subintervals for the RS-integral can be described as

$$\int_a^b f(t) dg \approx HMCS13 = \left[\frac{4n^2}{(b-a)^2}\int_a^{x_1}\int_a^t g(x)dxdt - \frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right] f(a)$$
$$+ \frac{4n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt - \frac{2n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right] f\left(\frac{x_{k-1}+x_k}{2}\right)$$
$$+ \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\frac{4n}{b-a}\left(\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt + \int_{x_k}^{x_{k+1}}\int_{x_k}^{t} g(x)dxdt\right) - \left(3\int_{x_{k-1}}^{x_k} g(t)dt + \int_{x_k}^{x_{k+1}} g(t)dt\right)\right] f(x_k)$$
$$+ \sum_{k=1}^{n}\begin{bmatrix} \frac{-h^2}{96}(3x_{k-1} + 5x_k)\int_{x_{k-1}}^{x_k} g(t)dt \\ + \frac{17{x_k}^2 - 10x_{k-1}x_k - 7{x_{k-1}}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt \\ -x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdy\, dt \\ \int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y} g(x)dxdy\, dzdt \end{bmatrix} f^{(4)}\left(\frac{2x_{k-1}x_k}{x_{k-1}+x_k}\right)$$
$$+ \left[g(b) - \frac{3n}{b-a}\int_{x_{n-1}}^{b} g(t)dt + \frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right] f(b) \quad (26)$$

Proof of Theorem 3.

The proposed basic form HMS13 scheme for the RS-integral is given in (13). Applying the proposed HMS13 rule over each subinterval, we have:

$$\int_a^b f(t)dg \approx \left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt - \frac{1}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt - g(a)\right]f(a)$$

$$+\left[\frac{4}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt - \frac{8}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt\right]f\left(\frac{a+x_1}{2}\right)$$

$$+\left[g(x_1) - \frac{3}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt + \frac{4}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt\right]f(x_1)$$

$$+\left(\begin{array}{l}\frac{-\left(\frac{b-a}{n}\right)^2(3a+5x_1)}{96}\int_a^{x_1} g(t)dt + \frac{17{x_1}^2-10ax_1-7a^2}{48}\int_a^{x_1}\int_a^t g(x)dxdt \\ -x_1\int_a^{x_1}\int_a^t\int_a^y g(x)dxdy\,dt + \int_a^{x_1}\int_a^t\int_a^z\int_a^y g(x)dxdy\,dzdt\end{array}\right)f^{(4)}\left(\frac{2ax_1}{a+x_1}\right)$$

$$+\left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt - \frac{1}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt - g(x_1)\right]f(x_1)$$

$$+\left[g(x_2) - \frac{3}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt + \frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt\right]f(x_2) + \left[\frac{4}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt - \right.$$

$$\left.\frac{8}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt\right]f\left(\frac{x_1+x_2}{2}\right)$$

$$+\left[g(x_2) - \frac{3}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt + \frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt\right]f(x_2)$$

$$+\left(\begin{array}{l}\frac{-\left(\frac{b-a}{n}\right)^2(3x_1+5x_2)}{96}\int_{x_1}^{x_2} g(t)dt + \frac{17{x_2}^2-10x_1x_2-7{x_1}^2}{48}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt \\ -x_2\int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^y g(x)dxdy\,dt + \int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^z\int_{x_1}^y g(x)dxdy\,dzdt\end{array}\right)f^{(4)}\left(\frac{2x_1x_2}{x_1+x_2}\right)$$

$$+\ldots+\left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt - \frac{1}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k} g(t)dt - g(x_{k-1})\right]f(x_{k-1})$$

$$+\left[\frac{4}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k} g(t)dt - \frac{8}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\left[g(x_k) - \frac{3}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k} g(t)dt + \frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt\right]f(x_k)$$

$$+\left(\begin{array}{l}\frac{-\left(\frac{b-a}{n}\right)^2(3x_{k-1}+5x_k)}{96}\int_{x_{k-1}}^{x_k} g(t)dt + \frac{17{x_k}^2-10x_{k-1}x_k-7{x_{k-1}}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt \\ -x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdy\,dt + \int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^z\int_{x_{k-1}}^y g(x)dxdy\,dzdt\end{array}\right)f^{(4)}\left(\frac{2x_{k-1}x_k}{x_{k-1}+x_k}\right)$$

$$+\ldots+\left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt-\frac{1}{\frac{b-a}{n}}\int_{x_{n-1}}^{b}g(t)dt-g(x_{n-1})\right]f(x_{n-1})$$

$$+\left[\frac{4}{\frac{b-a}{n}}\int_{x_{n-1}}^{b}g(t)dt-\frac{8}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\right]f\left(\frac{x_{n-1}+b}{2}\right)$$

$$+\left[g(b)-\frac{3}{\frac{b-a}{n}}\int_{x_{n-1}}^{b}g(t)dt+\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\right]f(b)$$

$$+\begin{pmatrix}\frac{-\left(\frac{b-a}{n}\right)^2(3x_{n-1}+5b)}{96}\int_{x_{n-1}}^{b}g(t)dt+\frac{17b^2-10x_{n-1}b-7x_{n-1}{}^2}{48}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\\-b\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y}g(x)dxdy\,dt+\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{z}\int_{x_{n-1}}^{y}g(x)dxdy\,dzdt\end{pmatrix}f^{(4)}\left(\frac{2x_{n-1}b}{x_{n-1}+b}\right)$$

$$=\left[\frac{4n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t}g(x)dxdt-\frac{n}{b-a}\int_{a}^{x_1}g(t)dt-g(a)\right]f(a)$$

$$+\left[\frac{4n}{b-a}\int_{a}^{x_1}g(t)dt-\frac{8n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t}g(x)dxdt\right]f\left(\frac{a+x_1}{2}\right)$$

$$+\left[\frac{4n}{b-a}\int_{x_1}^{x_2}g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_1}^{x_2}\int_{x_1}^{t}g(x)dxdt\right]f\left(\frac{x_1+x_2}{2}\right)$$

$$+\ldots+\left[\frac{4n}{b-a}\int_{x_{k-1}}^{x_k}g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\ldots+\left[\frac{4n}{b-a}\int_{x_{n-1}}^{b}g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\right]f\left(\frac{x_{n-1}+b}{2}\right)$$

$$+\left[\frac{4n^2}{(b-a)^2}\left(\int_{a}^{x_1}\int_{a}^{t}g(x)dxdt+\int_{x_1}^{x_2}\int_{x_1}^{t}g(x)dxdt\right)-\frac{n}{b-a}\left(3\int_{a}^{x_1}g(t)dt+\right.\right.$$
$$\left.\left.\int_{x_1}^{x_2}g(t)dt\right)\right]f(x_1)$$

$$+\left[\frac{4n^2}{(b-a)^2}\left(\int_{x_1}^{x_2}\int_{x_1}^{t}g(x)dxdt+\int_{x_2}^{x_3}\int_{x_2}^{t}g(x)dxdt\right)-\frac{n}{b-a}\left(3\int_{x_1}^{x_2}g(t)dt+\right.\right.$$
$$\left.\left.\int_{x_2}^{x_3}g(t)dt\right)\right]f(x_2)+\ldots+\left[\frac{4n^2}{(b-a)^2}\left(\int_{x_{n-2}}^{x_{n-1}}\int_{x_{n-2}}^{t}g(x)dxdt+\int_{x_{n-1}}^{x_n}\int_{x_{n-1}}^{t}g(x)dxdt\right)-\right.$$
$$\left.\frac{n}{b-a}\left(3\int_{x_{n-.2}}^{x_{n-1}}g(t)dt+\int_{x_{n-1}}^{x_n}g(t)dt\right)\right]f(x_{n-1})$$

$$+\left[g(b)-\frac{3n}{b-a}\int_{x_{n-1}}^{b}g(t)dt+\frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\right]f(b)$$

$$+\begin{pmatrix}\frac{-h^2(3x_{n-1}+5b)}{96}\int_{x_{n-1}}^{b}g(t)dt+\frac{17b^2-10x_{n-1}b-7x_{n-1}{}^2}{48}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\\-b\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y}g(x)dxdy\,dt+\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{z}\int_{x_{n-1}}^{y}g(x)dxdy\,dzdt\end{pmatrix}f^{(4)}\left(\frac{2x_{n-1}b}{x_{n-1}+b}\right)$$

$$=\left[\frac{4n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t}g(x)dxdt-\frac{n}{b-a}\int_{a}^{x_1}g(t)dt-g(a)\right]f(a)$$

$$+\frac{4n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}g(t)dt-\frac{2n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\frac{4n}{b-a}\left(\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt+\int_{x_k}^{x_{k+1}}\int_{x_k}^{t}g(x)dxdt\right)-\left(3\int_{x_{k-1}}^{x_k}g(t)dt+\right.\right.$$
$$\left.\left.\int_{x_k}^{x_{k+1}}g(t)dt\right)\right]f(x_k)$$

$$+\left[g(b)-\frac{3n}{b-a}\int_{x_{n-1}}^{b} g(t)dt+\frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right]f(b)$$

$$+\sum_{k=1}^{n}\begin{pmatrix}\frac{-h^2(3x_{k-1}+5x_k)}{96}\int_{x_{k-1}}^{x_k} g(t)dt+\frac{17{x_k}^2-10x_{k-1}x_k-7{x_{k-1}}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt \\ -x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdy\,dt+\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y} g(x)dxdy\,dzdt\end{pmatrix} f^{(4)}\left(\frac{2x_{n-1}b}{x_{n-1}+b}\right)$$

So the proposed harmonic mean derivative-based Simpson's 1/3 scheme for the RS-integral in composite form is proved.

It is noted that the global error terms of the proposed HMCS13 scheme cannot be defined in classical form.

IV. Results and Discussion

Some authors were not conducted experimental works on quadrature schemes for RS-integral in [X],[XI], [XVIII]. However in [VI], [VII], [VIII], [IX], experimental works have been conducted on quadrature schemes for RS-integral. Also in this research paper, experimental works have been performed on quadrature schemes for RS-integral and compared with existing schemes. Three numerical problems have been taken for each scheme taken from [VI], [VIII], [IX], etc, which were determined using MATLAB software. All the results are noted in Intel (R) Core (TM) Laptop with RAM 8.00GB and a processing speed of 1.00GHz-1.61GHz.. Double-precision arithmetic is used for numerical results.

Example 4.1. $\int_{3.5}^{4.5} \sin 5\,x d(\cos x)$ = 0.227676016130689

Example 4.2. $\int_{5}^{6} \sin x\, d(x^3)$ = -59.655908136641912

Example 4.3. $\int_{5}^{6} e^{x} d\,\sin x$ = 187.4269314248657

The absolute error and computational order of accuracy (COC) formulae are taken from [VII].

In Table 1, the absolute error drops have been compared for the proposed HMCS13 scheme and other existing schemes: CT, ZCT, MZCT, HeMCS13 and CMCS13 under similar conditions, and it is observed that the proposed HMCS13 scheme has the smallest error for all examples. When the number of strips is increased, it is observed from Figs. 1-3 through the line plots of decreasing error distributions that errors in the proposed scheme reduce rapidly in the comparison of other schemes.

Using the COC formula, the observed COC have been determined for the used methods, and are listed in Tables 2-4 for Examples 1-3, respectively versus several strips. The numbers in Tables 2-4 verify the theoretical accuracy of the discussed methods, including the proposed HMCS13 scheme for the RS-integral. The order of accuracy of the proposed HMCS13 scheme is 4 which is the same as the MZCT scheme, but the error reduction is rapid for the former. The CT scheme shows the order of accuracy of 2, whereas for the ZCT scheme, due to the issues and mistakes highlighted in [VIII], the order oscillates and doesn't converge to 4.

In Table 5, the total evaluations required per strip are summarized for the discussed methods, which are necessary to compute the computational costs. In Table 6, we list the total computational cost and the average CPU usage in seconds for the three integrals mentioned in Examples 1-3 using CT, ZCT, MZCT, HeMCS13, CMCS13 and

HMCS13 schemes. It is observed from numerical results that proposed scheme took less cost to achieve the error 10^{-5} as compared to existing schemes for all test problems, and the similar performance is obvious from Table 6 regarding the smaller average CPU time to achieve the error of 10^{-5} for the proposed method against others for Examples 1-3.

Table 1: Absolute error comparison by HMCS13 and other schemes for Examples 1-3.

Quadrature variants	Example 1 (m=20)	Example 2 (m=100)	Example 3 (m=20)
CT	1.1862E-03	4.9713E-04	3.9042E-02
ZCT	1.6698 E-03	5.5959 E-04	3.9042E-02
MZCT	1.8552 E-06	1,2428E-09	2.4399E-06
HeMCS13	1.5471e-07	8.3806e-11	3.8454e-07
CMCS13	1.5474e-07	8.3806e-11	3.8453e-07
HMCS13	1.5466e-07	8.3806e-11	3.8453e-07

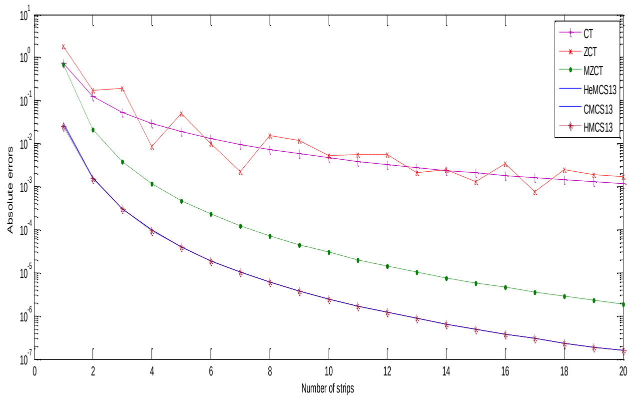

Fig 1. Comparison of error drops by all methods for Example 1

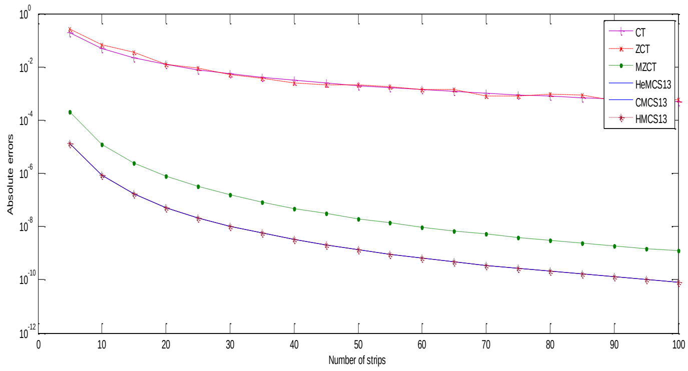

Fig 2. Comparison of error drops by all methods for Example 2

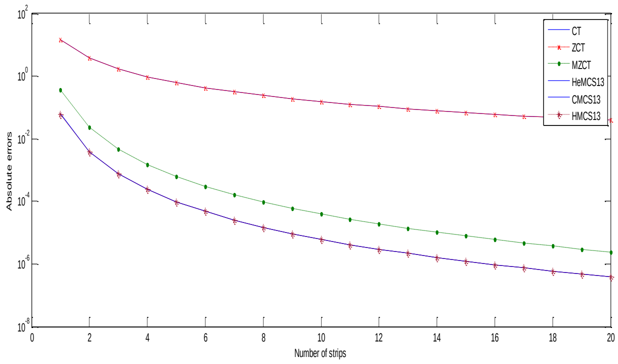

Fig 3. Comparison of error drops by all methods for Example 3

Table 2: Comparison of COC in all methods for Example 1

Number of strips (m)	CT	ZCT	MZCT	HeMCS13	CMCS13	HMCS13
1	NA	NA	NA	NA	NA	NA
2	2.5728	3.3788	5.0724	4.1456	4.2210	3.9992
4	2.0420	4.3106	4.1474	4.0337	4.0528	4.0006
8	2.0091	-0.8154	4.0348	4.0083	4.0131	4.0002
16	2.0022	2.1713	4.0086	4.0021	4.0000	4.0000

Table 3: Comparison of COC in all methods for Example 2

Number of strips (m)	CT	ZCT	MZCT	HeMCS13	CMCS13	HMCS13
5	NA	NA	NA	NA	NA	NA
10	2.0018	1.9243	4.0025	4.0264	4.0176	4.0408
20	2.0004	2.4487	4.0007	4.0066	4.0044	4.0103
40	2.0001	2.3985	4.0001	4.0017	4.0011	4.0026
80	2.0001	2.3985	4.0000	4.0004	4.0003	4.0006

Table 4: Comparison of COC in all methods for Example 3

Number of strips (m)	CT	ZCT	MZCT	HeMCS13	CMCS13	HMCS13
1	NA	NA	NA	NA	NA	NA
2	1.9319	1.9319	3.8984	4.0187	4.0266	4.0054
4	1.9844	1.9844	3.9761	4.0004	4.0069	4.0016
8	1.9962	1.9962	3.9941	4.0004	4.0018	4.0004
16	1.9990	1.9990	3.9985	4.0003	4.0004	4.0001

Table 5: Computational cost in quadrature variants form strips.

Quadrature Variants	Total evaluations
CT	2m+3 [VII]
ZCT	5m+3 [VII]
MZCT	5m+3 [VII]
HeMCS13	7m+3 [VIII]
CMCS13	7m+3 [IX]
Proposed HMCS13	7m+3

Table 6: Computational cost and CPU time comparison to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

Quadrature Variants	Computational cost			CPU time (in seconds)		
	Example 1	Example 2	Example 3	Example 1	Example 2	Example 3
CT	439	1415	2503	68.04	12.82	432.30
ZCT	1043	3503	6253	552.60	153.98	6469.38
MZCT	73	78	78	28.01	5.26	27.35
HeMCS13	59	45	66	26.65	5.99	27.82
CMCS13	59	45	66	24.74	5.88	27.48
HMCS13	59	45	66	25.72	7.13	19.85

Figs. 4-6 represent a computational cost to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

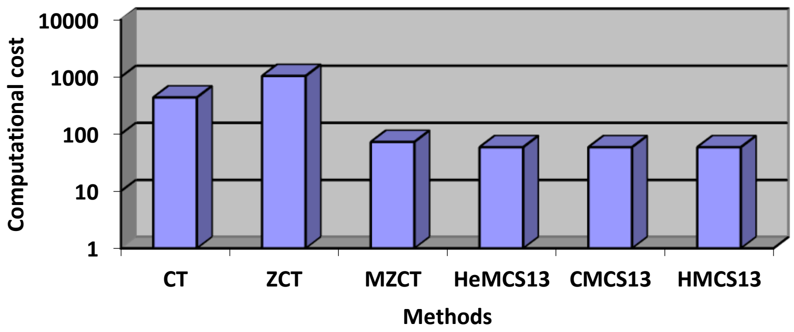

Fig 4. Total computational cost by quadrature variants for Example 1

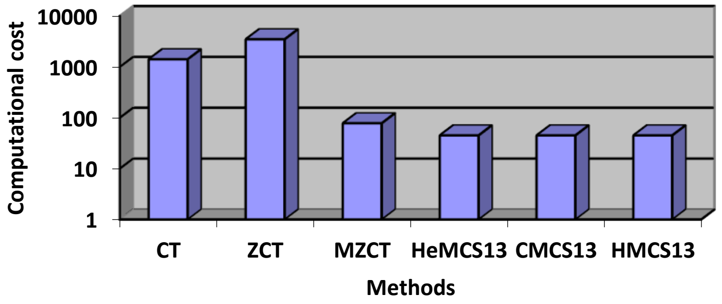

Fig 5. Total computational cost by quadrature variants for Example 2

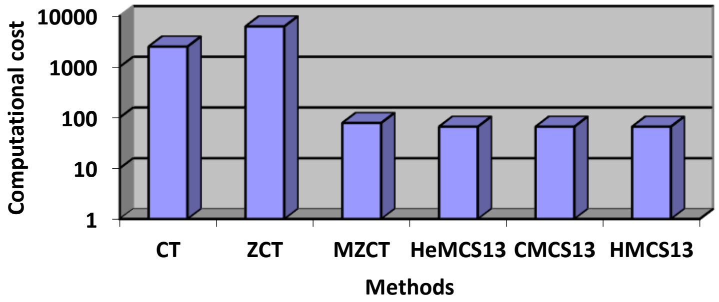

Fig 6. Total computational cost by quadrature variants for Example 3

Figs. 7-9 represent average CPU time to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

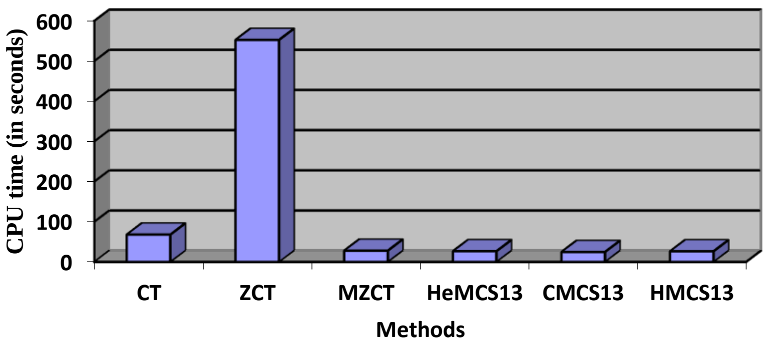

Fig 7. Average CPU usage time by quadrature variants for Example 1

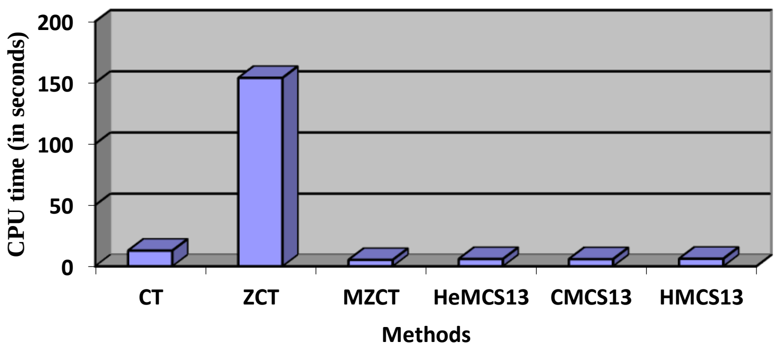

Fig 8. Average CPU usage time by quadrature variants for Example 2

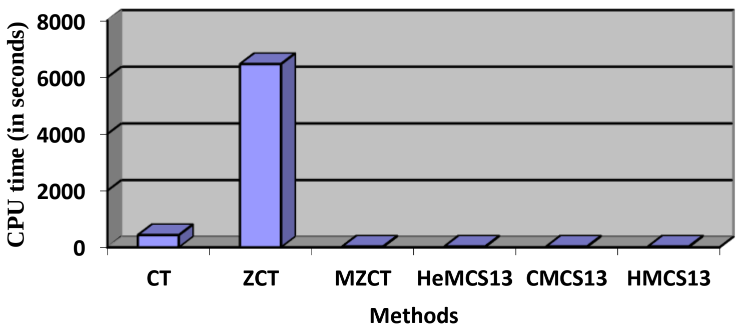

Fig 9. Average CPU usage time by quadrature variants for Example 3

V. Conclusion

A new harmonic mean derivative-based quadrature scheme of Simpson's 1/3 type was proposed for efficient approximations of the RS-integral and extended for higher strips in a composite sense. The theorems regarding the local and global error terms were proved. Three numerical problems were tested from the literature to discuss the performance of the proposed scheme against few other existing schemes. The error drops, observed orders of accuracy and computations performance in terms of evaluations and CPU usage show the dominance of the proposed scheme over other discussed schemes for the evaluation for the RS-integral numerically.

Conflict of Interest:

There is no conflict of interest regarding this article

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