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EFFICIENT DERIVATIVE-BASED SIMPSON'S 1/3-TYPE SCHEME USING CENTROIDAL MEAN FOR RIEMANN-STIELTJES INTEGRAL

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Abstract

In this paper, a new efficient derivative-based quadrature scheme of Simpson's 1/3-type is proposed using the centroidal mean for the approximation of Riemann-Stieltjes integral (RS-integral). Theorems are proved related to the basic form, composite form, local and global errors of the new scheme for the RS-integral. The reduction of the new proposed scheme is verified using g(t) = t for Riemann integral. The theoretical results of new proposed scheme have been proved by experimental work using programming in MATLAB against existing schemes. The order of accuracy, computational cost and average CPU time (in seconds) of the new proposed scheme are determined. The results obtained show the effectiveness of the proposed scheme compared to the existing schemes.

Keywords: Quadrature rule, Riemann-Stieltjes integral, Centroidal Mean, Simpson's 1/3 rule, Composite form, Local error, Global error, Cost-effectiveness, Time-efficiency

I. Introduction

From science and engineering, the nonlinear models arise quite frequently. Such models demand numerical solution due to complexity of equations [XIX],[XX],[VII],[XXII]. The numerical computation of a definite integral is the important problem in numerical integration and this numerical value is known as the area under the curve which is applied in many engineering applications. Definite integral $I(f) = \int_a^b f(x)dx$

cannot be integrated analytically for integrand $f(x) = e^{x^2}$ or $\sin x^2$. These integrals can be integrated numerically by numerical methods. The methods of numerical integration is known as quadrature methods. The Riemann-Stieltjes RS-integral is a variation of the Riemann integral. Let $f(x)$ and $\alpha(x)$ be two bounded functions on $[a, b]$ and $\alpha(x)$ is monotonically increasing on $[a, b]$, the RS-integral is defined [IV] as:

$$RS(f(x):\alpha;a,b) = \int_a^b f(x)d\alpha(x) \tag{1}$$

where $f(x)$ is integrand and $\alpha(x)$ is an integrator.

RS-integrals have been used in various areas of mathematics. For instance, Statistics and theory of Probability, Complex analysis, Functional analysis, the theory of the Operator, etc.

Several works have been focused on the improvement of quadrature rules for the Riemann integral in the literature as in [II], [XVIII]. Moreover, it is extended for the approximation of integral equations in [XXI], and used to improve the reluctance motors in [X]. However, few works have been indicated on the approximation of quadrature rules for the RS-integral as in [XI] Hadamard inequality has been derived for the RS-integral in Trapezoid-type. Also in [XII], the relative convexity concept has been used to derive some inequalities for the approximation of RS-integral using the midpoint and Simpson rules. Authors in [XXIII] used Midpoint derivative approach in order to improve the closed Newton-Cotes quadrature schemes for the approximation of Riemann integral.

Many strategies have been applied to improve Newton-Cotes formulas numerically.

Authors in [XIV],[XV],[XVI],[XVII] used different means at the functional derivative for the improvement of closed Newton-Cotes quadrature schemes. In [V],[VI] cubature schemes using derivatives were proposed.

Initially, the midpoint derivative-based quadrature scheme of trapezoid-type was introduced for the RS-integral in [XXV] without numerical verification. Later, the composite form of trapezoidal rule has been presented for the RS-integral in [XXIV] without verification of theoretical results. Recently, the new efficient derivative-based and derivative-free quadrature schemes have been presented in [VIII],[IX] for the RS-integral with verification of theoretical results.

In this research paper, a new efficient derivative-based Simpson's 1/3-type scheme is presented using the centroidal mean for the RS-integral. The basic and composite forms with error terms are described in theorems. The theoretical results of the new scheme have been verified by experimental work which shows cost efficiency, time efficiency and rapid convergence.

II. Existing Trapezoid-type Schemes for the RS-Integral

Some basic existing schemes of Trapezoid-type can be presented in T [XI], ZT[XXV], MZT [VIII], for the RS-integral and defined in (2)-(4) as:

$$T \approx \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right)f(b) \tag{2}$$

$$ZT \approx \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right)f(b)$$
$$+\left(\int_a^b\int_a^t g(x)dxdt - \frac{b-a}{2}\int_a^b g(t)dt\right)f''(c_{ZT}) \quad (3)$$

$$where, c_{ZT} = \frac{(-2b^2+a^2-ab)\int_a^b g(t)dt+6b\int_a^b\int_a^t g(x)dxdt-6\int_a^b\int_a^t\int_a^y g(x)dxdydt}{6\int_a^b\int_a^t g(x)dxdt-3(b-a)\int_a^b g(t)dt}.$$

$$MZT \approx \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right)f(b)$$
$$+\left(\int_a^b\int_a^t g(x)dxdt - \frac{b-a}{2}\int_a^b g(t)dt\right)f''(c_{MZT}) \quad (4)$$

$$where, c_{MZT} = \frac{(-2b^2+a^2+ab)\int_a^b g(t)dt+6b\int_a^b\int_a^t g(x)dxdt-6\int_a^b\int_a^t\int_a^y g(x)dxdydt}{6\int_a^b\int_a^t g(x)dxdt-3(b-a)\int_a^b g(t)dt}.$$

The composite forms of the CT, ZCT and MZCT schemes are described in (5)-(7) as:

$$CT \approx \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) \quad (5)$$

$$ZCT \approx \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) + \sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{h}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right]f''\left(c_{ZT,k}\right) \quad (6)$$

$$\int_a^b f(t)dg \approx MZCT = \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) + \sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{h}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right]f''(c_k) \quad (7)$$

Where,

$$\boldsymbol{c_{ZT,k}} = \frac{(-2x_k^2+x_{k-1}^2-x_{k-1}x_k)\int_{x_{k-1}}^{x_k} g(t)dt+6b\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt}{6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-\frac{3(x_k-x_{k-1})}{n}\int_{x_{k-1}}^{x_k} g(t)dt}$$

$$\boldsymbol{c_{MZT,k}} = \frac{(-2x_k^2+x_{k-1}^2+x_{k-1}x_k)\int_{x_{k-1}}^{x_k} g(t)dt+6b\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt}{6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-\frac{3(x_k-x_{k-1})}{n}\int_{x_{k-1}}^{x_k} g(t)dt}.$$

III. Proposed Scheme using Centroidal Mean for the Riemann-Stieltjes Integral

The basic centroidal mean derivative-based Simpson's 1/3-type (CMS13) rule for the Riemann integral is defined in (8) as:

$$\int_a^b f(x)dx \approx CMS13 = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right) + \frac{(b-a)^7}{17280(a+b)} f^{(6)}(\xi), \quad (8)$$

The precision of this rule is 4.

The proposed scheme i.e. CMS13 is described in its basic form for the RS-integral on the basis of (8) in Theorem 1.

Theorem 1. Let f'(t)and g(t) be continuous on [a, b] and g(t) be increasing there. Then the proposed derivative-based Simpson's 1/3 scheme using the centroidal mean for the RS-integral with local error term $R_{CMS13}[f]$ can be described as:

$$\int_a^b f(t)dg = CMS13 + R_{CMS13}[f] = \begin{pmatrix} \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt \\ -\frac{1}{b-a}\int_a^b g(t)dt - g(a)\end{pmatrix} f(a)$$
$$+\left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right) f\left(\frac{a+b}{2}\right)$$
$$+\left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right) f(b)$$
$$+\begin{pmatrix} \frac{-(b-a)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdydt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydz\,dt \end{pmatrix} f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)$$
$$+\begin{pmatrix} \frac{(a-b)^2(39a^3+91a^2b+61ab^2+49b^3)}{2880(a+b)}\int_a^b g(t)dt \\ -\frac{(-19a^4-44a^3b+30a^2b^2+4ab^3+29b^4}{288(a+b)}\int_a^b\int_a^t g(x)dxdt \\ +\frac{b(4a^2+ab+b^2)}{6(a+b)}\int_a^b\int_a^t\int_a^y g(x)dxdydt \\ \frac{(b^2+ab-2a^2)^2}{3(a+b)}\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt \\ -b\int_a^b\int_a^t\int_a^w\int_a^z\int_a^y g(x)dxdydz\,dwdt \end{pmatrix} f^{(6)}(\xi)g'(\eta), \quad (9)$$

$where \xi, \eta \in (a,b).$

Proof of Theorem 1.

To derive the centroidal mean derivative-based Simpson's 1/3 scheme for the RS-integral, we search numbers: a_0, b_0, c_0, d_0 so that:

$$\int_a^b f(t)dg \approx a_0 f(a) + b_0 f\left(\frac{a+b}{2}\right) + c_0 f(b) + d_0 f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right) \quad (10)$$

is exact for $f(t) = 1, t, t^2, t^3, t^4$. That is,

$$\int_a^b 1dg = a_0 + b_0 + c_0$$

$$\int_a^b tdg = a_0 a + b_0\left(\frac{a+b}{2}\right) + c_0 b$$

$$\int_a^b t^2 dg = a_0 a^2 + b_0\left(\frac{a+b}{2}\right)^2 + c_0 b^2$$

$$\int_a^b t^3 dg = a_0 a^3 + b_0 \left(\frac{a+b}{2}\right)^3 + c_0 b^3$$

$$\int_a^b t^4 dg = a_0 a^4 + b_0 \left(\frac{a+b}{2}\right)^4 + c_0 b^4 + 24 d_0$$

By using integration by parts of the RS-integral, as in [XXV], we have the following system of equations (11)-(15).

$$a_0 + b_0 + c_0 = g(b) - g(a) \tag{11}$$

$$a_0 a + b_0 \left(\frac{a+b}{2}\right) + c_0 b = bg(b) - ag(a) - \int_a^b g(t)dt \tag{12}$$

$$a_0 a^2 + b_0 \left(\frac{a+b}{2}\right)^2 + c_0 b^2 = b^2 g(b) - a^2 g(a)$$
$$-2b \int_a^b g(t)dt + 2 \int_a^b \int_a^t g(x)dx\,dt \tag{13}$$

$$a_0 a^3 + b_0 \left(\frac{a+b}{2}\right)^3 + c_0 b^3 = b^3 g(b) - a^3 g(a) - 3b^2 \int_a^b g(t)dt$$
$$+6b \int_a^b \int_a^t g(x)dxdt - 6 \int_a^b \int_a^t \int_a^y g(x)dxdt \tag{14}$$

$$a_0 a^4 + b_0 \left(\frac{a+b}{2}\right)^4 + c_0 b^4 + 24 d_0 = b^4 g(b) - a^4 g(a) - 4b^3 \int_a^b g(t)dt$$
$$+12b^2 \int_a^b \int_a^t g(x)dxdt - 24b \int_a^b \int_a^t \int_a^y g(x)dxdydt$$
$$+24 \int_a^b \int_a^t \int_a^z \int_a^y g(x)dxdydz\,dt \tag{15}$$

The system of linear equations (11)-(15) can be described with the coefficient matrix as:

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ a & \frac{a+b}{2} & b & 0 \\ a^2 & \left(\frac{a+b}{2}\right)^2 & b^2 & 0 \\ a^3 & \left(\frac{a+b}{2}\right)^3 & b^3 & 0 \\ a^4 & \left(\frac{a+b}{2}\right)^4 & b^4 & 24 \end{bmatrix}$$

The reduced row echelon form of *M* is:

$$M \overset{R}{\approx} M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As in M_R, rank(M) = 4. To check the linearly independent rows, take a = -1 and b = 1 in matrix M then

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 24 \end{bmatrix}$$

This shows that the first three and fifth are linearly independent rows whereas fourth row is linearly dependent. To find the coefficients a_0, b_0, c_0 and d_0, we solve equations (11), (12), (13) and (15) simultaneously, to have:

$$a_0 = \frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a),$$

$$b_0 = \frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt,$$

$$c_0 = g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt,$$

$$d_0 = -\frac{(a-b)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{(17b^2-10ab-7a^2)}{48}\int_a^b \int_a^t g(x)dxdt$$
$$-b\int_a^b \int_a^t \int_a^y g(x)dxdy\,dt + \int_a^b \int_a^t \int_a^z \int_a^y g(x)\,dxdy\,dzdt$$

Putting the values of coefficients a_0, b_0, c_0 and d_0 in (10),we have:

$$\int_a^b f(t)dg \approx CMS13 = \begin{pmatrix} \frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt \\ -\frac{1}{b-a}\int_a^b g(t)dt - g(a) \end{pmatrix} f(a)$$
$$+\left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt\right) f\left(\frac{a+b}{2}\right)$$
$$+\left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b \int_a^t g(x)dxdt\right) f(b)$$
$$+\begin{pmatrix} \frac{-(b-a)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b \int_a^t g(x)dxdt \\ -b\int_a^b \int_a^t \int_a^y g(x)dxdydt + \int_a^b \int_a^t \int_a^z \int_a^y g(x)dxdydz\,dt \end{pmatrix} f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)$$

Now we derive the local error term of the proposed CMS13 scheme.

Since the precision of the proposed CMS13 scheme is 4, we take $f(t) = \frac{t^5}{5!}$ to find the leading error term defined as:

$$R_{CMS13}[f] = \frac{1}{5!}\int_a^b t^5 dg - CMS13(t^5; g; a, b) \quad (16)$$

We learn, from [XXV], that:

$$\frac{1}{5!}\int_a^b t^5 dg = \frac{1}{120}(b^5 g(b) - a^5 g(a)) - \frac{b^4}{24}\int_a^b g(t)dt + \frac{b^3}{6}\int_a^b \int_a^t g(x)dxdt$$
$$-\frac{b^2}{2}\int_a^b \int_a^t \int_a^y g(x)dxdydt + b\int_a^b \int_a^t \int_a^z \int_a^y g(x)dxdydzdt$$
$$-\int_a^b \int_a^t \int_a^w \int_a^z \int_a^y g(x)\,dxdydzdwdt \quad (17)$$

By Theorem 1 and scheme (9), we have:

$$CMS13(t^5; g; a, b) = \left(\frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)\frac{a^5}{5!}$$
$$+\left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right)\frac{(a+b)^5}{2^5.5!}$$
$$+\left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right)\frac{b^5}{5!}$$
$$+\begin{pmatrix}\frac{-(a-b)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdydt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydz\,dt\end{pmatrix}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right) \quad (18)$$

We use (17) and (18) in (16) to get:

$$R_{CMS13}[f] = \left(\frac{(a-b)^2(39a^3+91a^2b+61ab^2+49b^3)}{2880(a+b)}\int_a^b g(t)dt\right.$$
$$-\frac{(-19a^4-44a^3b+30a^2b^2+4ab^3+29b^4}{288(a+b)}\int_a^b\int_a^t g(x)dxdt$$
$$+\frac{b(4a^2+ab+b^2)}{6(a+b)}\int_a^b\int_a^t\int_a^y g(x)dxdydt + \frac{(b^2+ab-2a^2)^2}{3(a+b)}\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt$$
$$\left.-b\int_a^b\int_a^t\int_a^w\int_a^z\int_a^y g(x)dxdydz\,dwdt\right)f^{(6)}(\xi)g'(\eta), \quad (19)$$

which is the same as (9), and the precision of this scheme is 4.

Theorem 2. When g(t) = t, the proposed CMS13 scheme with the error term (9) for the RS-integrals is reduced to the corresponding CMS13 scheme [XVII], i.e. (8) for the Riemann integrals.

Proof of Theorem 2.

By Theorem 1, we have:

$$\int_a^b f(t)dg = \int_a^b f(t)dt = \left(\frac{4}{(b-a)^2}\int_a^b\int_a^t xdxdt - \frac{1}{b-a}\int_a^b tdt - g(a)\right)f(a)$$
$$+\left(\frac{4}{b-a}\int_a^b tdt - \frac{8}{(b-a)^2}\int_a^b\int_a^t xdxdt\right)f\left(\frac{a+b}{2}\right)$$
$$+\left(g(b) - \frac{3}{b-a}\int_a^b tdt + \frac{4}{(b-a)^2}\int_a^b\int_a^t xdxdt\right)f(b)$$
$$+\begin{pmatrix}\frac{-(b-a)^2(3a+5b)}{96}\int_a^b tdt + \frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t xdxdt \\ -b\int_a^b\int_a^t\int_a^y xdxdydt + \int_a^b\int_a^t\int_a^z\int_a^y xdxdydz\,dt\end{pmatrix}f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)$$
$$+\left(\frac{(a-b)^2(39a^3+91a^2b+61ab^2+49b^3)}{2880(a+b)}\int_a^b tdt\right.$$
$$-\frac{(-19a^4-44a^3b+30a^2b^2+4ab^3+29b^4}{288(a+b)}\int_a^b\int_a^t xdxdt$$
$$+\frac{b(4a^2+ab+b^2)}{6(a+b)}\int_a^b\int_a^t\int_a^y xdxdydt + \frac{(b^2+ab-2a^2)^2}{3(a+b)}\int_a^b\int_a^t\int_a^z\int_a^y xdxdydzdt$$
$$\left.-b\int_a^b\int_a^t\int_a^w\int_a^z\int_a^y xdxdydz\,dwdt\right)f^{(6)}(\xi)g'(\eta) \quad (20)$$

It is obvious to get:

$$\int_a^b tdt = \frac{b^2-a^2}{2},$$
$$\int_a^b\int_a^t xdxdt = \frac{b^3}{6} - \frac{a^2b}{2} + \frac{a^3}{3},$$

$\int_a^b \int_a^t \int_a^y xdxdy\,dt = \frac{b^4}{24} - \frac{a^2b^2}{4} + \frac{a^3b}{3} - \frac{a^4}{8}$,

$\int_a^b \int_a^t \int_a^z \int_a^y x\,dxdydz\,dt = \frac{b^5}{120} - \frac{a^4b}{8} + \frac{a^3b^2}{6} - \frac{a^2b^3}{12} + \frac{a^5}{30}$,

$\int_a^b \int_a^t \int_a^w \int_a^z \int_a^y xdxdydzdwdt = \frac{b^6}{720} + \frac{a^5b}{30} - \frac{a^4b^2}{16} + \frac{a^3b^3}{18} - \frac{a^2b^4}{48} - \frac{a^6}{144}$,

And, finally using these in (20) we get:

$$\int_a^b f(t)\,dt = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right) + \frac{(b-a)^7}{17280(a+b)} f^{(6)}(\xi), \quad (21)$$

where $\xi \in (a,b)$.

This illustrates that the proposed CMS13 rule is reducible to the classical Riemann integral form (8) in terms of the centroidal mean.

Now, the proposed composite centroidal mean derivative-based Simpson's 1/3 scheme for the RS-integral is derived by dividing the interval into small subintervals and applying integration rule to each subinterval, and the results are showcased in Theorem 3.

Theorem 3. Let f'(t) and g(t) be continuous on [a, b] and g(t) be increasing there. Let the interval [a, b] be subdivided into 2n subintervals [x_k, x_{k+1}] with width h = $\frac{b-a}{n}$ by using the equally spaced nodes $x_k = a+kh$, where k = 0, 1, …, n. The composite Centroidal mean Simpson's 1/3 scheme to 2n subintervals for the RS-integral can be described as

$$\int_a^b f(t)dg \approx CMCS13 = \left[\frac{4n^2}{(b-a)^2}\int_a^{x_1}\int_a^t g(x)dxdt - \frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right] f(a)$$
$$+ \frac{4n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt - \frac{2n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right] f\left(\frac{x_{k-1}+x_k}{2}\right)$$
$$+ \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\frac{4n}{b-a}\left(\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt + \int_{x_k}^{x_{k+1}}\int_{x_k}^{t} g(x)dxdt\right) - \left(3\int_{x_{k-1}}^{x_k} g(t)dt + \int_{x_k}^{x_{k+1}} g(t)dt\right)\right] f(x_k)$$
$$+ \sum_{k=1}^{n}\left[\begin{array}{c} \frac{-h^2}{96}(3x_{k-1} + 5x_k)\int_{x_{k-1}}^{x_k} g(t)dt \\ + \frac{17x_k{}^2 - 10x_{k-1}x_k - 7x_{k-1}{}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt \\ - x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdy\,dt \\ + \int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y} g(x)dxdy\,dzdt \end{array}\right] f^{(4)}\left(\frac{2(x_{k-1}{}^2 + x_{k-1}x_k + x_k{}^2)}{3(x_{k-1}+x_k)}\right)$$
$$+ \left[g(b) - \frac{3n}{b-a}\int_{x_{n-1}}^{b} g(t)dt + \frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right] f(b) \qquad (22)$$

Proof of Theorem 3.

The proposed basic form CMS13 scheme for the RS-integral is given in (9). Applying the proposed CMS13 rule over each subinterval, we have:

$$\int_a^b f(t)dg \approx \left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt - \frac{1}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt - g(a)\right] f(a)$$

$$+\left[\frac{4}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt - \frac{8}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt\right] f\left(\frac{a+x_1}{2}\right)$$

$$+\left[g(x_1) - \frac{3}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt + \frac{4}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt\right] f(x_1)$$

$$+\begin{pmatrix} \frac{-\left(\frac{b-a}{n}\right)^2(3a+5x_1)}{96}\int_a^{x_1} g(t)dt + \frac{17{x_1}^2-10ax_1-7a^2}{48}\int_a^{x_1}\int_a^t g(x)dxdt \\ -x_1\int_a^{x_1}\int_a^t\int_a^y g(x)dxdy\,dt + \int_a^{x_1}\int_a^t\int_a^z\int_a^y g(x)dxdy\,dzdt \end{pmatrix} f^{(4)}\left(\frac{2(a^2+ax_1+{x_1}^2)}{3(a+x_1)}\right)$$

$$+\left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt - \frac{1}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt - g(x_1)\right] f(x_1)$$

$$+\left[g(x_2) - \frac{3}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt + \frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt\right] f(x_2) + \left[\frac{4}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt - \right.$$

$$\left.\frac{8}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt\right] f\left(\frac{x_1+x_2}{2}\right)$$

$$+\left[g(x_2) - \frac{3}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt + \frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt\right] f(x_2)$$

$$+\begin{pmatrix} \frac{-\left(\frac{b-a}{n}\right)^2(3x_1+5x_2)}{96}\int_{x_1}^{x_2} g(t)dt + \frac{17{x_2}^2-10x_1x_2-7{x_1}^2}{48}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt \\ -x_2\int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^y g(x)dxdy\,dt + \int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^z\int_{x_1}^y g(x)dxdy\,dzdt \end{pmatrix} f^{(4)}\left(\frac{2({x_1}^2+x_1x_2+{x_2}^2)}{3(x_1+x_2)}\right)$$

$$+\ldots+\left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt - \frac{1}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k} g(t)dt - g(x_{k-1})\right] f(x_{k-1})$$

$$+\left[\frac{4}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k} g(t)dt - \frac{8}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt\right] f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\left[g(x_k) - \frac{3}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k} g(t)dt + \frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt\right] f(x_k)$$

$$+\begin{pmatrix} \frac{-\left(\frac{b-a}{n}\right)^2(3x_{k-1}+5x_k)}{96}\int_{x_{k-1}}^{x_k} g(t)dt \\ +\frac{17{x_k}^2-10x_{k-1}x_k-7{x_{k-1}}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt \\ -x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdy\,dt \\ +\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^z\int_{x_{k-1}}^y g(x)dxdy\,dzdt \end{pmatrix} f^{(4)}\left(\frac{2({x_{k-1}}^2+x_{k-1}x_k+{x_k}^2)}{3(x_{k-1}+x_k)}\right)$$

$$+\ldots+\left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt-\frac{1}{\frac{b-a}{n}}\int_{x_{n-1}}^{b} g(t)dt-g(x_{n-1})\right]f(x_{n-1})$$

$$+\left[\frac{4}{\frac{b-a}{n}}\int_{x_{n-1}}^{b} g(t)dt-\frac{8}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{n-1}+b}{2}\right)$$

$$+\left[g(b)-\frac{3}{\frac{b-a}{n}}\int_{x_{n-1}}^{b} g(t)dt+\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right]f(b)$$

$$+\begin{pmatrix}\frac{-\left(\frac{b-a}{n}\right)^2(3x_{n-1}+5b)}{96}\int_{x_{n-1}}^{b} g(t)dt+\frac{17b^2-10x_{n-1}b-7x_{n-1}{}^2}{48}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\\-b\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdy\,dt+\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{z}\int_{x_{n-1}}^{y} g(x)dxdy\,dzdt\end{pmatrix}f^{(4)}\left(\frac{2(x_{n-1}{}^2+x_{n-1}b+b^2)}{3(x_{n-1}+b)}\right)$$

$$=\left[\frac{4n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t} g(x)dxdt-\frac{n}{b-a}\int_{a}^{x_1} g(t)dt-g(a)\right]f(a)$$

$$+\left[\frac{4n}{b-a}\int_{a}^{x_1} g(t)dt-\frac{8n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t} g(x)dxdt\right]f\left(\frac{a+x_1}{2}\right)$$

$$+\left[\frac{4n}{b-a}\int_{x_1}^{x_2} g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt\right]f\left(\frac{x_1+x_2}{2}\right)$$

$$+\ldots+\left[\frac{4n}{b-a}\int_{x_{k-1}}^{x_k} g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\ldots+\left[\frac{4n}{b-a}\int_{x_{n-1}}^{b} g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{n-1}+b}{2}\right)$$

$$+\left[\frac{4n^2}{(b-a)^2}\left(\int_{a}^{x_1}\int_{a}^{t} g(x)dxdt+\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt\right)-\frac{n}{b-a}\left(3\int_{a}^{x_1} g(t)dt+\int_{x_1}^{x_2} g(t)dt\right)\right]f(x_1)$$

$$+\left[\frac{4n^2}{(b-a)^2}\left(\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt+\int_{x_2}^{x_3}\int_{x_2}^{t} g(x)dxdt\right)-\frac{n}{b-a}\left(3\int_{x_1}^{x_2} g(t)dt+\int_{x_2}^{x_3} g(t)dt\right)\right]f(x_2)\ +\ldots+\left[\frac{4n^2}{(b-a)^2}\left(\int_{x_{n-2}}^{x_{n-1}}\int_{x_{n-2}}^{t} g(x)dxdt+\int_{x_{n-1}}^{x_n}\int_{x_{n-1}}^{t} g(x)dxdt\right)-\frac{n}{b-a}\left(3\int_{x_{n-.2}}^{x_{n-1}} g(t)dt+\int_{x_{n-1}}^{x_n} g(t)dt\right)\right]f(x_{n-1})$$

$$+\left[g(b)-\frac{3n}{b-a}\int_{x_{n-1}}^{b} g(t)dt+\frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right]f(b)$$

$$+\begin{pmatrix}\frac{-h^2(3x_{n-1}+5b)}{96}\int_{x_{n-1}}^{b} g(t)dt+\frac{17b^2-10x_{n-1}b-7x_{n-1}{}^2}{48}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\\-b\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdy\,dt+\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{z}\int_{x_{n-1}}^{y} g(x)dxdy\,dzdt\end{pmatrix}f^{(4)}\left(\frac{2(x_{n-1}{}^2+x_{n-1}b+b^2)}{3(x_{n-1}+b)}\right)$$

$$=\left[\frac{4n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t} g(x)dxdt-\frac{n}{b-a}\int_{a}^{x_1} g(t)dt-g(a)\right]f(a)$$

$$+\frac{4n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt-\frac{2n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\frac{4n}{b-a}\left(\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt+\int_{x_k}^{x_{k+1}}\int_{x_k}^{t} g(x)dxdt\right)-\left(3\int_{x_{k-1}}^{x_k} g(t)dt+\int_{x_k}^{x_{k+1}} g(t)dt\right)\right]f(x_k)$$

$$+\left[g(b)-\frac{3n}{b-a}\int_{x_{n-1}}^{b}g(t)dt+\frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\right]f(b)$$

$$+\sum_{k=1}^{n}\left(\begin{array}{l}\frac{-h^2(3x_{k-1}+5x_k)}{96}\int_{x_{k-1}}^{x_k}g(t)dt+\frac{17{x_k}^2-10x_{k-1}x_k-7{x_{k-1}}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt\\-x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y}g(x)dxdy\,dt+\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y}g(x)dxdy\,dzdt\end{array}\right)f^{(4)}\left(\frac{2({x_{n-1}}^2+x_{n-1}b+b^2)}{3(x_{n-1}+b)}\right)$$

So the proposed composite derivative-based Simpson's 1/3 scheme using the centroidal mean for the RS-integral is proved.

It is noted that the global error terms of the proposed CMCS13 scheme cannot be defined in classical form.

IV. Results and Discussion

Theoretical results did not verify in [XI],[XII],[XXV], by experimental works on quadrature schemes for RS-integral. In this research paper, experimental works have been performed on quadrature schemes for RS-integral to verify the theoretical results. Three different numerical problems have been tested using MATLAB software for each scheme taken from [VIII], [IX], [III], [XIII], [I] etc. All the results are observed in Intel (R) Core (TM) Laptop with RAM 8.00GB and a processing speed of 1.00GHz-1.61GHz.. Double-precision arithmetic is used for numerical results.

Example 1. $\int_{3.5}^{4.5} sin\,5\,xd(cos\,x)$ = 0.227676016130689

Example 2. $\int_{5}^{6} sin\,x\,d(x^3)$ = -59.655908136641912

Example 3. $\int_{5}^{6} e^x d\,sin\,x$ = 187.4269314248657

The absolute error and computational order of accuracy (COC) formulae have been obtained from [IX].

In Table 1, the error plots of the proposed CMCS13 scheme have been matched with other existing schemes: CT, ZCT and MZCT under similar conditions, and it is noted that the proposed CMCS13 scheme has the smallest error for all examples. When the number of strips is increased, it is noticed from Figs. 1-3 through the line plots that errors in the proposed scheme are reduced rapidly as compared to other schemes.

The observed COC has been verified for all discussed methods for the RS-integral except the ZCT scheme due to errors highlighted in [VIII], the order oscillates and did not converge to 4, as in Tables 2-4.

In Table 5, the total evaluation of functions, derivatives and integrations are used per strip are mentioned for the discussed methods, which are required to compute the computational costs. Figs. 4-9 represent the total computational cost and the average CPU time in seconds for three different integrals using CT, ZCT, MZCT and CMCS13 schemes. The numerical results show that the proposed scheme computed less cost and smaller average CPU time to achieve the error 10^{-5} as compared to existing schemes for all test problems.

Table 1: Absolute error comparison by CMCS13 and others for Examples 1-3.

Quadrature variants	Example 1 (m=20)	Example 2 (m=100)	Example 3 (m=20)
CT	1.1862E-03	4.9713E-04	3.9042E-02
ZCT	1.6698 E-03	5.5959 E-04	3.9042E-02
MZCT	1.8552 E-06	1,2428E-09	2.4399E-06
CMCS13	1.5474e-07	8.3806e-11	3.8453e-07

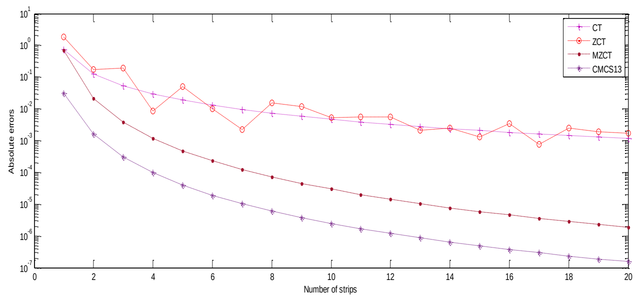

Fig 1. Comparison of error drops by all methods for Example 1

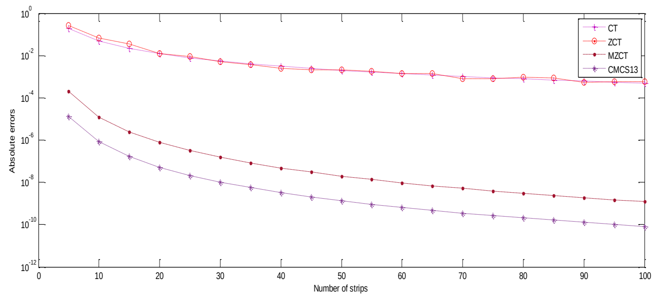

Fig 2. Comparison of error drops by all methods for Example 2

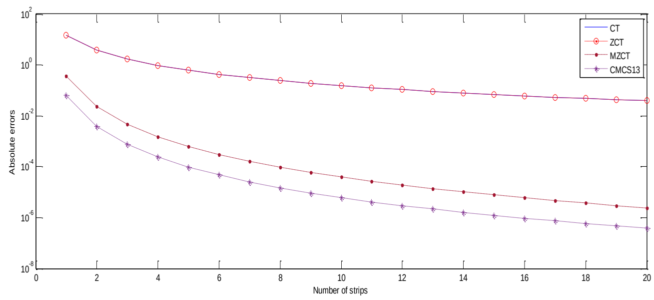

Fig 3. Comparison of error drops by all methods for Example 3

Table 2: Comparison of COC in all methods for Example 1

Number of strips (m)	CT	ZCT	MZCT	CMCS13
1	NA	NA	NA	NA
2	2.5728	3.3788	5.0724	4.2210
4	2.0420	4.3106	4.1474	4.0528
8	2.0091	-0.8154	4.0348	4.0131
16	2.0022	2.1713	4.0086	4.0032
32	2.0006	3.2118	4.0021	4.0008

Table 3: Comparison of COC in all methods for Example 2

Number of strips (m)	CT	ZCT	MZCT	CMCS13
5	NA	NA	NA	NA
10	2.0018	1.9243	4.0025	4.0176
20	2.0004	2.4487	4.0007	4.0044
40	2.0001	2.3985	4.0001	4.0011
80	2.0001	2.3985	4.0000	4.0003

Table 4: Comparison of COC in all methods for Example 3

Number of strips (m)	CT	ZCT	MZCT	CMCS13
1	NA	NA	NA	NA
2	1.9319	1.9319	3.8984	4.0266
4	1.9844	1.9844	3.9761	4.0069
8	1.9962	1.9962	3.9941	4.0018
16	1.9990	1.9990	3.9985	4.0004
32	1.9998	1.9998	3.9985	4.0001

Table 5: Computational cost in quadrature variants for m strips.

Quadrature Variants	Total evaluations
CT	2m+3 [XI]
ZCT	5m+3 [XI]
MZCT	5m+3 [XI]
Proposed CMCS13	7m+3

Figs. 4-6 represent a computational cost to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

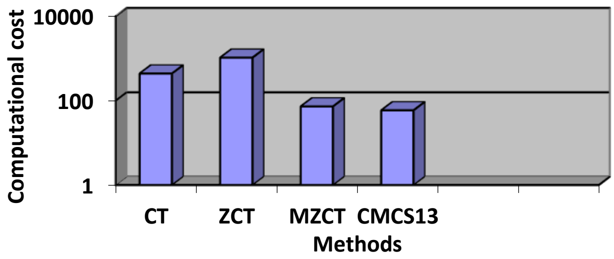

Fig 4. Total computational cost by quadrature variants for Example 1

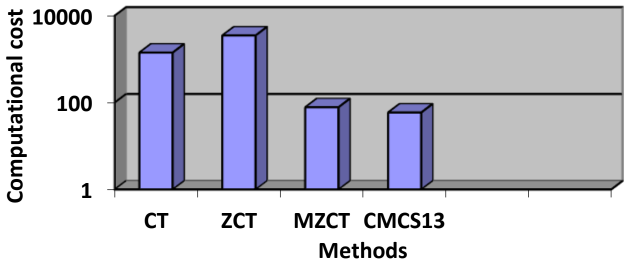

Fig 5. Total computational cost by quadrature variants for Example 2

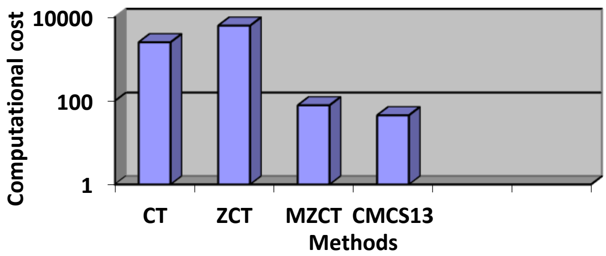

Fig 6. Total computational cost by quadrature variants for Example 3

Figs. 7-9 represent average CPU time to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

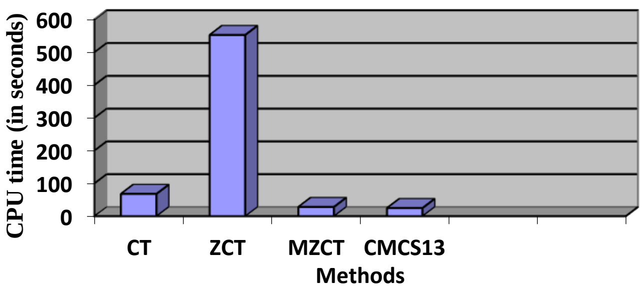

Fig 7. Average CPU usage time by quadrature variants for Example 1

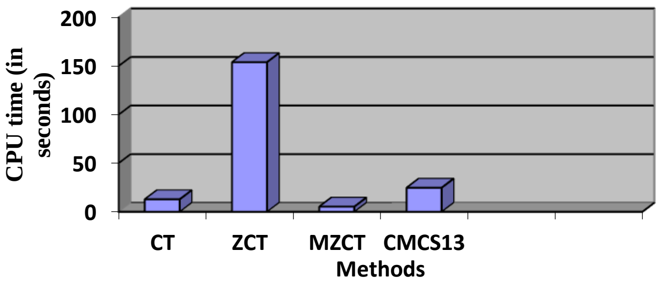

Fig 8. Average CPU usage time by quadrature variants for Example 2

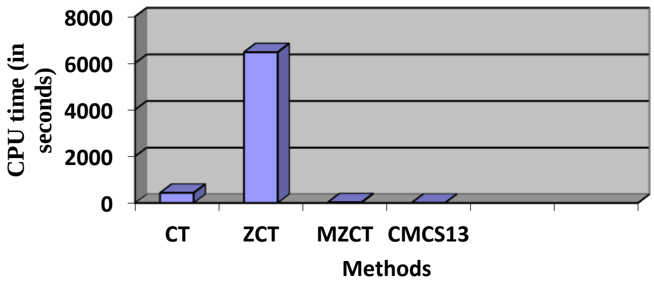

Fig 9. Average CPU usage time by quadrature variants for Example 3

V. Conclusion

A new efficient derivative-based quadrature scheme of Simpson's 1/3-type was derived using the centroidal mean for the RS-integral. The local and global error terms were derived in theorems. Three different numerical problems were analyzed from the literature to check the performance of the proposed scheme against few other existing schemes. The error drops, observed orders of accuracy and computations performance in terms of evaluations and CPU usage show the dominance of the proposed scheme over other discussed schemes for the evaluation for the RS-integral numerically.

Conflict of Interest:

The authors declare that there is no conflict of interest regarding this article

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