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HERONIAN MEAN DERIVATIVE-BASED SIMPSON'S-TYPE SCHEME FOR RIEMANN-STIELTJES INTEGRAL

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Abstract

In this paper, a new heronian mean derivative-based quadrature scheme of Simpson's 1/3-type is proposed for the approximation of the Riemann-Stieltjes integral (RS-integral). Theorems are proved related to the basic form, composite form, local and global errors of the new scheme for the RS-integral. The reduction of the new proposed scheme is verified using g(t) = t for Riemann integral. The theoretical results of the new proposed scheme have been proved by experimental work using programming in MATLAB against existing schemes. The order of accuracy, computational cost and average CPU time (in seconds) of the new proposed scheme are determined. The results obtained show the effectiveness of the proposed scheme compared to the existing schemes.

Keywords : Quadrature rule, Riemann-Stieltjes, Simpson's 1/3 rule, Composite form, Local error, Global error, Cost-effectiveness, Time-efficiency, Heronian Mean.

I. Introduction

Mathematical models are often nonlinear, as the physical phenomenon in nature is not always simple and linear. For example nonlinear equations [XIX],[XX] and their solvers [VII],[XXII], etc. The approximate calculation of a definite integral is the fundamental problem in numerical integration and this approximate value is known as the area under the curve used in engineering applications. Indefinite integral $I(f) = \int_a^b f(x)dx$, the analytic solution is not always available for integrand $f(x) = e^{x^2}$ or $\sin x^2$. Such definite integrals can also be evaluated with numerical methods. The numerical methods of integration is known as quadrature schemes. A lot of work has been done in the literature to improve the approximations of quadrature rules for the Riemann integral, and some studies have pointed to the approximation of the RS-integral. The Riemann-Stieltjes RS-integral is a modification of the Riemann integral. Suppose $f(x)$ is bounded function in $[a, b]$, and another function $\alpha\ (x)$ which is monotonically increasing in $[a, b]$, the RS-integral is defined [IV] as:

$$RS(f(x):\alpha;a,b)=\int_a^b f(x)d\alpha(x) \tag{1}$$

where $f(x)$ is integrand and $\alpha(x)$ is an integrator.

RS-integrals are applied in several fields of mathematics. For instance, Statistics and theory of Probability, Complex analysis, Functional analysis, the theory of the Operator, and other fields. As mentioned above, quadrature approximations for the Riemann integral are extremely available in literature as in [II], [XVIII]. Further, it is used for integral equations in [XXI], and utilized in reluctance motors [X]. In contrast, only some works in the past used numerical approximation for the RS-integrals. Initially, Trapezoid-type approximation with Hadamard inequality can be used in [XI] for the RS-integral. Further, applied the relative convexity concept in [XII] in which some important inequalities were developed for the approximation of RS-integral using the midpoint and Simpson rules. Authors in [XXIII] introduced a new family of closed Newton-Cotes quadrature schemes with Midpoint derivative for the Riemann integral.

There are so many approaches used to improve Newton-Cotes formulas numerically. Authors in [XIV],[XV],[XVI],[XVII] derived derivative-based closed Newton-Cotes quadrature schemes using different means such as geometric, harmonic, heronian and centroidal at the evaluation of function derivative. Similarly, in [V],[VI] cubature schemes using derivatives were proposed.

The first time in [XXV], the midpoint derivative-based trapezoid scheme for the RS-integral was presented. Afterward in [XXIV], authors proposed the composite form of trapezoidal rule for the RS-integral without numerical experiments. Recently, Memon *et al.* [VIII],[IX] proposed efficient derivative-based and derivative-free quadrature schemes for the RS-integral with numerical verifications.

In this research work, a new heronian mean derivative-based Simpson's 1/3-type scheme is proposed for the approximation of the RS-integral. The theorems in basic and composite forms and related error analysis are described. The present research verified the validity of the new scheme by numerical experiments from literature to show cost efficiency, time efficiency and rapid convergence.

II. Some Existing Schemes for the RS-Integral

Some existing schemes: T [XI], ZT [XXV], MZT [VIII], can be described for the RS-integral in basic form in (2)-(4) as:

$$T\approx\left(\frac{1}{b-a}\int_a^b g(t)dt-g(a)\right)f(a)+\left(g(b)-\frac{1}{b-a}\int_a^b g(t)dt\right)f(b) \tag{2}$$

$$ZT\approx\left(\frac{1}{b-a}\int_a^b g(t)dt-g(a)\right)f(a)+\left(g(b)-\frac{1}{b-a}\int_a^b g(t)dt\right)f(b)$$
$$+\left(\int_a^b\int_a^t g(x)dxdt-\frac{b-a}{2}\int_a^b g(t)dt\right)f''(c_{ZT}) \tag{3}$$

$$where, c_{ZT} = \frac{(-2b^2+a^2-ab)\int_a^b g(t)dt + 6b\int_a^b\int_a^t g(x)dxdt - 6\int_a^b\int_a^t\int_a^y g(x)dxdydt}{6\int_a^b\int_a^t g(x)dxdt - 3(b-a)\int_a^b g(t)dt}.$$

$$MZT \approx \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right)f(b) + \left(\int_a^b\int_a^t g(x)dxdt - \frac{b-a}{2}\int_a^b g(t)dt\right)f''(c_{MZT}) \quad (4)$$

$$where, c_{MZT} = \frac{(-2b^2+a^2+ab)\int_a^b g(t)dt + 6b\int_a^b\int_a^t g(x)dxdt - 6\int_a^b\int_a^t\int_a^y g(x)dxdydt}{6\int_a^b\int_a^t g(x)dxdt - 3(b-a)\int_a^b g(t)dt}.$$

The composite forms of the CT, ZCT and MZCT schemes are described in (5)-(7) as:

$$CT \approx \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) \quad (5)$$

$$ZCT \approx \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) + \sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{h}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right]f''(c_{ZT,k}) \quad (6)$$

$$\int_a^b f(t)dg \approx MZCT = \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) + \sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{h}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right]f''(c_k) \quad (7)$$

Where,

$$\boldsymbol{c_{ZT,k}} = \frac{(-2x_k^2+x_{k-1}^2-x_{k-1}x_k)\int_{x_{k-1}}^{x_k} g(t)dt + 6b\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - 6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt}{6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{3(x_k-x_{k-1})}{n}\int_{x_{k-1}}^{x_k} g(t)dt}$$

$$\boldsymbol{c_{MZT,k}} = \frac{(-2x_k^2+x_{k-1}^2+x_{k-1}x_k)\int_{x_{k-1}}^{x_k} g(t)dt + 6b\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - 6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt}{6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{3(x_k-x_{k-1})}{n}\int_{x_{k-1}}^{x_k} g(t)dt}.$$

III. Proposed Heronian Mean Derivative-Based Simpson's 1/3-type Scheme for the Riemann-Stieltjes Integral

We base the proposed heronian mean derivative-based Simpson's 1/3-type schemes for the RS-integral approximation on the well-known closed Newton-Cotes heronian mean derivative-based quadrature rule – the Simpson's 1/3 (HeMS13) rule – which is defined in basic form in (8).

$$\int_a^b f(x)dx \approx HeMS13 = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right) - \frac{(b-a)^5\left(\sqrt{b}-\sqrt{a}\right)^2}{17280} f^{(5)}(\xi), \quad (8)$$

The Simpson's 1/3 rule for the Riemann integral as in (8) provides an exact answer for all polynomials whose degree is four or less, so its precision is 4.

Based on (8), the proposed scheme, i.e. HeMS13 for the RS-integral, in basic form is derived in Theorem 1.

Theorem 1. Let f'(t)and g(t) be continuous on [a, b] and g(t) be increasing there. Then the proposed heronian mean derivative-based Simpson's 1/3 scheme for the RS-integral can be described as:

$$\int_a^b f(t)dg \approx HeMS13 = \begin{pmatrix} \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt \\ -\frac{1}{b-a}\int_a^b g(t)dt - g(a) \end{pmatrix} f(a)$$
$$+\left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right) f\left(\frac{a+b}{2}\right)$$
$$+\left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right) f(b)$$
$$+\begin{pmatrix} \frac{-(b-a)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdydt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydz\,dt \end{pmatrix} f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right), (9)$$

Proof of Theorem 1.

To derive the heronian mean derivative-based Simpson's 1/3 scheme for the RS-integral, we search numbers: a_0, b_0, c_0, d_0 so that:

$$\int_a^b f(t)dg \approx a_0 f(a) + b_0 f\left(\frac{a+b}{2}\right) + c_0 f(b) + d_0 f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right) \quad (10)$$

is exact for $f(t) = 1, t, t^2, t^3, t^4$. That is,

$$\int_a^b 1dg = a_0 + b_0 + c_0$$

$$\int_a^b tdg = a_0 a + b_0\left(\frac{a+b}{2}\right) + c_0 b$$

$$\int_a^b t^2 dg = a_0 a^2 + b_0\left(\frac{a+b}{2}\right)^2 + c_0 b^2$$

$$\int_a^b t^3 dg = a_0 a^3 + b_0\left(\frac{a+b}{2}\right)^3 + c_0 b^3$$

$$\int_a^b t^4 dg = a_0 a^4 + b_0\left(\frac{a+b}{2}\right)^4 + c_0 b^4 + 24 d_0$$

By using integration by parts of the RS-integral, as in [XXV], we have the following system of equations (11)-(15).

$$a_0 + b_0 + c_0 = g(b) - g(a) \quad (11)$$

$$a_0 a + b_0\left(\frac{a+b}{2}\right) + c_0 b = bg(b) - ag(a) - \int_a^b g(t)dt \quad (12)$$

$$a_0 a^2 + b_0\left(\frac{a+b}{2}\right)^2 + c_0 b^2 = b^2 g(b) - a^2 g(a) - 2b\int_a^b g(t)dt + 2\int_a^b\int_a^t g(x)dx\,dt \quad (13)$$

$$a_0a^3 + b_0\left(\frac{a+b}{2}\right)^3 + c_0b^3 = b^3g(b) - a^3g(a) - 3b^2\int_a^b g(t)dt$$
$$+6b\int_a^b\int_a^t g(x)dxdt - 6\int_a^b\int_a^t\int_a^y g(x)dxdt \qquad (14)$$

$$a_0a^4 + b_0\left(\frac{a+b}{2}\right)^4 + c_0b^4 + 24d_0 = b^4g(b) - a^4g(a) - 4b^3\int_a^b g(t)dt$$
$$+12b^2\int_a^b\int_a^t g(x)dxdt - 24b\int_a^b\int_a^t\int_a^y g(x)dxdydt$$
$$+24\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydz\,dt \qquad (15)$$

The system of linear equations (11)-(15) can be described with the coefficient matrix as:

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ a & \frac{a+b}{2} & b & 0 \\ a^2 & \left(\frac{a+b}{2}\right)^2 & b^2 & 0 \\ a^3 & \left(\frac{a+b}{2}\right)^3 & b^3 & 0 \\ a^4 & \left(\frac{a+b}{2}\right)^4 & b^4 & 24 \end{bmatrix}$$

The reduced row echelon form of *M* is:

$$M \overset{R}{\approx} M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As in M_R, rank(M) = 4. In order to check the linearly independent rows, take a = -1 and b = 1 in matrix *M* then

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 24 \end{bmatrix}$$

This shows that the first three and fifth are linearly independent rows whereas the fourth row is linearly dependent. To find the coefficients a_0, b_0, c_0 and d_0, we solve equations (11), (12), (13) and (15) simultaneously, to have:

$$a_0 = \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a),$$

$$b_0 = \frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt,$$

$$c_0 = g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt,$$

$$d_0 = -\frac{(a-b)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{(17b^2-10ab-7a^2)}{48}\int_a^b\int_a^t g(x)dxdt$$
$$-b\int_a^b\int_a^t\int_a^y g(x)dxdy\,dt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)\,dxdy\,dzdt$$

Putting the values of coefficients a_0, b_0, c_0 and d_0 in (10), we have:

$$\int_a^b f(t)dg \approx HeMS13 = \begin{pmatrix}\frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \\ \frac{1}{b-a}\int_a^b g(t)dt - g(a)\end{pmatrix} f(a)$$
$$+\left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right) f\left(\frac{a+b}{2}\right)$$
$$+\left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right) f(b)$$
$$+\begin{pmatrix}\frac{-(b-a)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdydt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydz\,dt\end{pmatrix} f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right)$$

which is the same as (9), and the precision of this scheme is 4.

The local error term of the proposed heronian mean derivative-based Simpson's 1/3 scheme is described in Theorem 2.

Theorem 2. Let f'(t)and g(t) be continuous on [a, b] and g(t) be increasing there. Then the proposed HeMS13 scheme for the RS-integral with local error term can be defined as:

$$\int_a^b f(x)dg = HeMS13 + R_{HeMS13}[f]$$

with HeMS13 as in (9), and error term $R_{HeMS13}[f]$ as:

$$R_{HeMS13}[f] = \left(\frac{(a-b)^2(9a^2+32ab+30a\sqrt{ab}+50b\sqrt{ab}-b^2)}{2880}\int_a^b g(t)dt\right.$$
$$+\frac{(a-b)(5a^2+24ab+14a\sqrt{ab}+34b\sqrt{ab}-5b^2}{288}\int_a^b\int_a^t g(x)dxdt$$
$$-\frac{b(2a+2\sqrt{ab}-b)}{6}\int_a^b\int_a^t\int_a^y g(x)dxdydt + \frac{2b-\sqrt{ab}-a}{3}\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt$$
$$\left.-\int_a^b\int_a^t\int_a^w\int_a^z\int_a^y g(x)dxdydz\,dwdt\right) f^{(5)}(\xi)g'(\eta), \quad (16)$$

$where \xi, \eta \in (a, b).$

Proof of Theorem 2.

Since the precision of the proposed HeMS13 scheme is 4, we take $f(t) = \frac{t^5}{5!}$ to find the leading error term defined as:

$$R_{HeMS13}[f] = \frac{1}{5!}\int_a^b t^5 dg - HeMS13(t^5; g; a, b) \quad (17)$$

We learn, from [XXV], that:

$$\frac{1}{5!}\int_a^b t^5 dg = \frac{1}{120}(b^5 g(b) - a^5 g(a)) - \frac{b^4}{24}\int_a^b g(t)dt + \frac{b^3}{6}\int_a^b\int_a^t g(x)dxdt$$
$$-\frac{b^2}{2}\int_a^b\int_a^t\int_a^y g(x)dxdydt + b\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt$$
$$-\int_a^b\int_a^t\int_a^w\int_a^z\int_a^y g(x)\,dxdydzdwdt \quad (18)$$

By Theorem 1 and scheme (9), we have:

$$HeMS13(t^5;g;a,b)=\left(\frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt-\frac{1}{b-a}\int_a^b g(t)dt-g(a)\right)\frac{a^5}{5!}$$
$$+\left(\frac{4}{b-a}\int_a^b g(t)dt-\frac{8}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right)\frac{(a+b)^5}{2^5.5!}$$
$$+\left(g(b)-\frac{3}{b-a}\int_a^b g(t)dt+\frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right)\frac{b^5}{5!}$$
$$+\begin{pmatrix}\frac{-(a-b)^2(3a+5b)}{96}\int_a^b g(t)dt+\frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t g(x)dxdt\\ -b\int_a^b\int_a^t\int_a^y g(x)dxdydt+\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydz\,dt\end{pmatrix}\left(\frac{a+\sqrt{ab}+b}{3}\right)\quad(19)$$

We use (18) and (19) in (17) to get:

$$R_{HeMS13}[f]=\left(\frac{(a-b)^2(9a^2+32ab+30a\sqrt{ab}+50b\sqrt{ab}-b^2)}{2880}\int_a^b g(t)dt\right.$$
$$+\frac{(a-b)(5a^2+24ab+14a\sqrt{ab}+34b\sqrt{ab}-5b^2}{288}\int_a^b\int_a^t g(x)dxdt$$
$$-\frac{b(2a+2\sqrt{ab}-b)}{6}\int_a^b\int_a^t\int_a^y g(x)dxdydt+\frac{2b-\sqrt{ab}-a}{3}\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt$$
$$\left.-\int_a^b\int_a^t\int_a^w\int_a^z\int_a^y g(x)dxdydz\,dwdt\right)f^{(5)}(\xi)g'(\eta),\quad(20)$$

This is the required local error term of the proposed HeMS13 scheme for RS-integral.

Theorem 3. When g(t) = t, the proposed HeMS13 scheme (9) with the error term (16) for the RS-integrals is reduced to the corresponding HeMS13 scheme [XVI], i.e. (8) for the Riemann integrals.

Proof of Theorem 3.

By Theorem 2, we have:

$$\int_a^b f(t)dg=\int_a^b f(t)dt=\left(\frac{4}{(b-a)^2}\int_a^b\int_a^t xdxdt-\frac{1}{b-a}\int_a^b tdt-g(a)\right)f(a)$$
$$+\left(\frac{4}{b-a}\int_a^b tdt-\frac{8}{(b-a)^2}\int_a^b\int_a^t xdxdt\right)f\left(\frac{a+b}{2}\right)$$
$$+\left(g(b)-\frac{3}{b-a}\int_a^b tdt+\frac{4}{(b-a)^2}\int_a^b\int_a^t xdxdt\right)f(b)$$
$$+\begin{pmatrix}\frac{-(b-a)^2(3a+5b)}{96}\int_a^b tdt+\frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t xdxdt\\ -b\int_a^b\int_a^t\int_a^y xdxdydt+\int_a^b\int_a^t\int_a^z\int_a^y xdxdydz\,dt\end{pmatrix}f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right)$$
$$+\left(\frac{(a-b)^2(9a^2+32ab+30a\sqrt{ab}+50b\sqrt{ab}-b^2)}{2880}\int_a^b tdt\right.$$
$$+\frac{(a-b)(5a^2+24ab+14a\sqrt{ab}+34b\sqrt{ab}-5b^2}{288}\int_a^b\int_a^t xdxdt$$
$$-\frac{b(2a+2\sqrt{ab}-b)}{6}\int_a^b\int_a^t\int_a^y g(x)dxdydt+\frac{2b-\sqrt{ab}-a}{3}\int_a^b\int_a^t\int_a^z\int_a^y xdxdydzdt$$
$$\left.-\int_a^b\int_a^t\int_a^w\int_a^z\int_a^y xdxdydz\,dwdt\right)f^{(5)}(\xi)g'(\eta)\quad(21)$$

It is obvious to get:

$$\int_a^b tdt=\frac{b^2-a^2}{2},$$
$$\int_a^b\int_a^t xdxdt=\frac{b^3}{6}-\frac{a^2b}{2}+\frac{a^3}{3},$$

$$\int_a^b \int_a^t \int_a^y x dx dy\, dt = \frac{b^4}{24} - \frac{a^2b^2}{4} + \frac{a^3b}{3} - \frac{a^4}{8},$$
$$\int_a^b \int_a^t \int_a^z \int_a^y x\, dx dy dz\, dt = \frac{b^5}{120} - \frac{a^4b}{8} + \frac{a^3b^2}{6} - \frac{a^2b^3}{12} + \frac{a^5}{30},$$
$$\int_a^b \int_a^t \int_a^w \int_a^z \int_a^y x dx dy dz dw dt = \frac{b^6}{720} + \frac{a^5b}{30} - \frac{a^4b^2}{16} + \frac{a^3b^3}{18} - \frac{a^2b^4}{48} - \frac{a^6}{144},$$

And, finally using these in (21) we get:

$$\int_a^b f(t)\,dt = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a+\sqrt{ab}+b}{3}\right) - \frac{(b-a)^5\left(\sqrt{b}-\sqrt{a}\right)^2}{17280} f^{(6)}(\xi), \qquad (22)$$

where $\xi \in (a, b)$.

This illustrates that the proposed HeMS13 rule is reducible to the classical Riemann integral form (8) in terms of heronian mean.

Now, the proposed composite Heronian mean derivative-based Simpson's 1/3 scheme for the RS-integral is derived by dividing the interval into small subintervals and applying integration rule to each subinterval, and the results are showcased in Theorem 4.

Theorem 4. Let f'(t) and g(t) be continuous on [a, b] and g(t) be increasing there. Let the interval [a, b] be subdivided into 2n subintervals $[x_k, x_{k+1}]$ with width $h = \frac{b-a}{n}$ by using the equally spaced nodes $x_k = a+kh$, where k = 0, 1, …, n. The composite Heronian mean Simpson's 1/3 scheme to 2n subintervals for the RSI can be described as

$$\int_a^b f(t)dg \approx HeMCS13 = \left[\frac{4n^2}{(b-a)^2}\int_a^{x_1}\int_a^t g(x)dxdt - \frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right] f(a)$$
$$+ \frac{4n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt - \frac{2n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right] f\left(\frac{x_{k-1}+x_k}{2}\right)$$
$$+ \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\frac{4n}{b-a}\left(\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt + \int_{x_k}^{x_{k+1}}\int_{x_k}^{t} g(x)dxdt\right) - \left(3\int_{x_{k-1}}^{x_k} g(t)dt + \int_{x_k}^{x_{k+1}} g(t)dt\right)\right] f(x_k)$$
$$+ \sum_{k=1}^{n}\left[\begin{array}{l}\frac{-h^2}{96}(3x_{k-1}+5x_k)\int_{x_{k-1}}^{x_k} g(t)dt + \frac{17{x_k}^2 - 10x_{k-1}x_k - 7{x_{k-1}}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt \\ -x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdy\,dt + \int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y} g(x)dxdy\,dzdt\end{array}\right] f^{(4)}\left(\frac{x_{k-1}+\sqrt{x_{k-1}x_k}+x_k}{3}\right)$$
$$+ \left[g(b) - \frac{3n}{b-a}\int_{x_{n-1}}^{b} g(t)dt + \frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right] f(b) \qquad (23)$$

Proof of Theorem 4.

The proposed basic form HeMS13 scheme for the RS-integral is given in (9). Applying the proposed HeMS13 rule over each subinterval, we have:

$$\int_a^b f(t)dg \approx \left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt - \frac{1}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt - g(a)\right] f(a)$$
$$+ \left[\frac{4}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt - \frac{8}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt\right] f\left(\frac{a+x_1}{2}\right)$$

$$+\left[g(x_1)-\frac{3}{\frac{b-a}{n}}\int_a^{x_1}g(t)dt+\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt\right]f(x_1)$$

$$+\begin{pmatrix}\frac{-\left(\frac{b-a}{n}\right)^2(3a+5x_1)}{96}\int_a^{x_1}g(t)dt+\frac{17{x_1}^2-10ax_1-7a^2}{48}\int_a^{x_1}\int_a^t g(x)dxdt\\ -x_1\int_a^{x_1}\int_a^t\int_a^y g(x)dxdy\,dt+\int_a^{x_1}\int_a^t\int_a^z\int_a^y g(x)dxdy\,dzdt\end{pmatrix}f^{(4)}\left(\frac{a+\sqrt{ax_1}+x_1}{3}\right)$$

$$+\left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt-\frac{1}{\frac{b-a}{n}}\int_{x_1}^{x_2}g(t)dt-g(x_1)\right]f(x_1)$$

$$+\left[g(x_2)-\frac{3}{\frac{b-a}{n}}\int_{x_1}^{x_2}g(t)dt+\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt\right]f(x_2)\;+\left[\frac{4}{\frac{b-a}{n}}\int_{x_1}^{x_2}g(t)dt-\right.$$

$$\left.\frac{8}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt\right]f\left(\frac{x_1+x_2}{2}\right)$$

$$+\left[g(x_2)-\frac{3}{\frac{b-a}{n}}\int_{x_1}^{x_2}g(t)dt+\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt\right]f(x_2)$$

$$+\begin{pmatrix}\frac{-\left(\frac{b-a}{n}\right)^2(3x_1+5x_2)}{96}\int_{x_1}^{x_2}g(t)dt+\frac{17{x_2}^2-10x_1x_2-7{x_1}^2}{48}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt\\ -x_2\int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^y g(x)dxdy\,dt+\int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^z\int_{x_1}^y g(x)dxdy\,dzdt\end{pmatrix}f^{(4)}\left(\frac{x_1+\sqrt{x_1x_2}+x_2}{3}\right)$$

$$+\ldots+\left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt-\frac{1}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k}g(t)dt-g(x_{k-1})\right]f(x_{k-1})$$

$$+\left[\frac{4}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k}g(t)dt-\frac{8}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\left[g(x_k)-\frac{3}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k}g(t)dt+\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt\right]f(x_k)$$

$$+\begin{pmatrix}\frac{-\left(\frac{b-a}{n}\right)^2(3x_{k-1}+5x_k)}{96}\int_{x_{k-1}}^{x_k}g(t)dt\\ +\frac{17{x_k}^2-10x_{k-1}x_k-7{x_{k-1}}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt\\ -x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdy\,dt\\ +\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^z\int_{x_{k-1}}^y g(x)dxdy\,dzdt\end{pmatrix}f^{(4)}\left(\frac{x_{k-1}+\sqrt{x_{k-1}x_k}+x_k}{3}\right)$$

$$+\ldots+\left[\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt-\frac{1}{\frac{b-a}{n}}\int_{x_{n-1}}^{b} g(t)dt-g(x_{n-1})\right]f(x_{n-1})$$

$$+\left[\frac{4}{\frac{b-a}{n}}\int_{x_{n-1}}^{b} g(t)dt-\frac{8}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{n-1}+b}{2}\right)$$

$$+\left[g(b)-\frac{3}{\frac{b-a}{n}}\int_{x_{n-1}}^{b} g(t)dt+\frac{4}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right]f(b)$$

$$+\begin{pmatrix}\frac{-\left(\frac{b-a}{n}\right)^2(3x_{n-1}+5b)}{96}\int_{x_{n-1}}^{b} g(t)dt+\frac{17b^2-10x_{n-1}b-7x_{n-1}{}^2}{48}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\\ -b\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdy\,dt+\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{z}\int_{x_{n-1}}^{y} g(x)dxdy\,dzdt\end{pmatrix}f^{(4)}\left(\frac{x_{n-1}+\sqrt{x_{n-1}b}+b}{3}\right)$$

$$=\left[\frac{4n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t} g(x)dxdt-\frac{n}{b-a}\int_{a}^{x_1} g(t)dt-g(a)\right]f(a)$$

$$+\left[\frac{4n}{b-a}\int_{a}^{x_1} g(t)dt-\frac{8n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t} g(x)dxdt\right]f\left(\frac{a+x_1}{2}\right)$$

$$+\left[\frac{4n}{b-a}\int_{x_1}^{x_2} g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt\right]f\left(\frac{x_1+x_2}{2}\right)$$

$$+\ldots+\left[\frac{4n}{b-a}\int_{x_{k-1}}^{x_k} g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\ldots+\left[\frac{4n}{b-a}\int_{x_{n-1}}^{b} g(t)dt-\frac{8n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{n-1}+b}{2}\right)$$

$$+\left[\frac{4n^2}{(b-a)^2}\left(\int_{a}^{x_1}\int_{a}^{t} g(x)dxdt+\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt\right)-\frac{n}{b-a}\left(3\int_{a}^{x_1} g(t)dt+\int_{x_1}^{x_2} g(t)dt\right)\right]f(x_1)$$

$$+\left[\frac{4n^2}{(b-a)^2}\left(\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt+\int_{x_2}^{x_3}\int_{x_2}^{t} g(x)dxdt\right)-\frac{n}{b-a}\left(3\int_{x_1}^{x_2} g(t)dt+\int_{x_2}^{x_3} g(t)dt\right)\right]f(x_2)+\ldots+\left[\frac{4n^2}{(b-a)^2}\left(\int_{x_{n-2}}^{x_{n-1}}\int_{x_{n-2}}^{t} g(x)dxdt+\int_{x_{n-1}}^{x_n}\int_{x_{n-1}}^{t} g(x)dxdt\right)-\frac{n}{b-a}\left(3\int_{x_{n-.2}}^{x_{n-1}} g(t)dt+\int_{x_{n-1}}^{x_n} g(t)dt\right)\right]f(x_{n-1})$$

$$+\left[g(b)-\frac{3n}{b-a}\int_{x_{n-1}}^{b} g(t)dt+\frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\right]f(b)$$

$$+\begin{pmatrix}\frac{-h^2(3x_{n-1}+5b)}{96}\int_{x_{n-1}}^{b} g(t)dt+\frac{17b^2-10x_{n-1}b-7x_{n-1}{}^2}{48}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt\\ -b\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdy\,dt+\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{z}\int_{x_{n-1}}^{y} g(x)dxdy\,dzdt\end{pmatrix}f^{(4)}\left(\frac{x_{n-1}+\sqrt{x_{n-1}b}+b}{3}\right)$$

$$=\left[\frac{4n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t} g(x)dxdt-\frac{n}{b-a}\int_{a}^{x_1} g(t)dt-g(a)\right]f(a)$$

$$+\frac{4n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt-\frac{2n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\frac{4n}{b-a}\left(\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt+\int_{x_k}^{x_{k+1}}\int_{x_k}^{t} g(x)dxdt\right)-\left(3\int_{x_{k-1}}^{x_k} g(t)dt+\int_{x_k}^{x_{k+1}} g(t)dt\right)\right]f(x_k)$$

$$+\left[g(b)-\frac{3n}{b-a}\int_{x_{n-1}}^{b}g(t)dt+\frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}g(x)dxdt\right]f(b)$$

$$+\sum_{k=1}^{n}\begin{pmatrix}\frac{-h^2(3x_{k-1}+5x_k)}{96}\int_{x_{k-1}}^{x_k}g(t)dt+\frac{17{x_k}^2-10x_{k-1}x_k-7{x_{k-1}}^2}{48}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt\\ -x_k\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y}g(x)dxdy\,dt+\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y}g(x)dxdy\,dzdt\end{pmatrix}f^{(4)}\left(\frac{x_{n-1}+\sqrt{x_{n-1}b}+b}{3}\right)$$

So the proposed composite heronian mean derivative-based Simpson’s 1/3 scheme for the RS-integral is proved.

It is noted that the global error terms of the proposed HeMCS13 scheme cannot be defined in classical form.

IV. Results and Discussion

Some authors [XI],[XII],[XXV], in previous studies did not perform experimental works for their theoretical works on quadrature schemes for RS-integral. In this study, experimental works have been performed on quadrature schemes for RS-integral which confirm the validity of theoretical results. Three numerical problems have been solved for each scheme taken from [VIII], [IX], [III], [XIII], [I] etc, which were determined using MATLAB software. All the results are noted in Intel (R) Core (TM) Laptop with RAM 8.00GB and a processing speed of 1.00GHz-1.61GHz.. Double-precision arithmetic is used for numerical results.

Example 1. $\int_{3.5}^{4.5} sin\,5\,xd(cos\,x)$ = 0.227676016130689

Example 2. $\int_{5}^{6} sin\,x\,d(x^3)$ = -59.655908136641912

Example 3. $\int_{5}^{6} e^x d\,sin\,x$ = 187.4269314248657

The absolute error and computational order of accuracy (COC) formulae are taken from [IX]. In Table 1, the absolute error drops have been compared for the proposed HeMCS13 scheme and other existing schemes: CT, ZCT and MZCT under similar conditions, and it is observed that the proposed HeMCS13 scheme has the smallest error for all examples. When the number of strips is increased, it is observed from Figs. 1-3 through the line plots of decreasing error distributions that errors in the proposed scheme reduce rapidly in the comparison of other schemes.

Using the COC formula, the observed COC have been determined for the used methods, and are listed in Tables 2-4 for Examples 1-3, respectively versus several strips. The numbers in Tables 2-4 verify the theoretical accuracy of the discussed methods, including the proposed HeMCS13 scheme for the RS-integral. The order of accuracy of the proposed HeMCS13 scheme is 4 which is the same as the MZCT scheme, but the error reduction is rapid for the former. The CT scheme shows the order of accuracy of 2, whereas for the ZCT scheme, due to the issues and mistakes highlighted in [VIII], the order oscillates and doesn’t converge to 4.

In Table 5, the total evaluations required per strip are summarized for the discussed methods, which are necessary to compute the computational costs. In Table 6, we list the total computational cost and the average CPU usage in seconds for the three integrals mentioned in Examples 1-3 using CT, ZCT, MZCT and HeMCS13 schemes. It is observed from numerical results that the proposed scheme took less cost to achieve the error 10^{-5} as compared to existing schemes for all test problems, and the similar performance is obvious from Table 6 regarding the smaller average CPU time to achieve the error 10^{-5} for the proposed method against others for Examples 1-3.

Table 1: Absolute error comparison by HeMCS13 and other schemes for Examples 1-3.

Quadrature variants	Example 1 (m=20)	Example 2 (m=100)	Example 3 (m=20)
CT	1.1862E-03	4.9713E-04	3.9042E-02
ZCT	1.6698 E-03	5.5959 E-04	3.9042E-02
MZCT	1.8552 E-06	1,2428E-09	2.4399E-06
HeMCS13	1.5471e-07	8.3806e-11	3.8454e-07

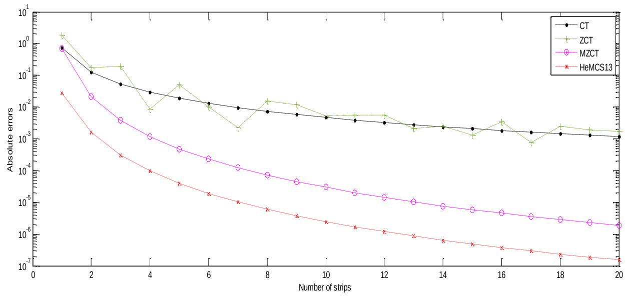

Fig 1. Comparison of error drops by all methods for Example 1

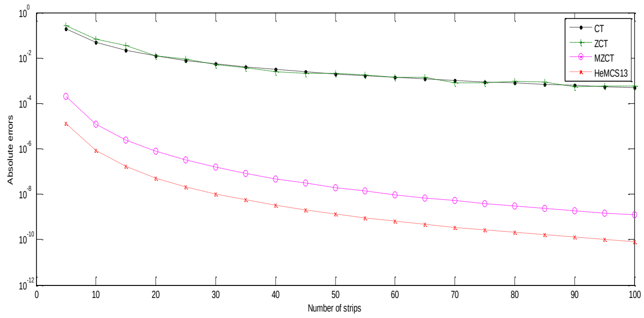

Fig 2. Comparison of error drops by all methods for Example 2

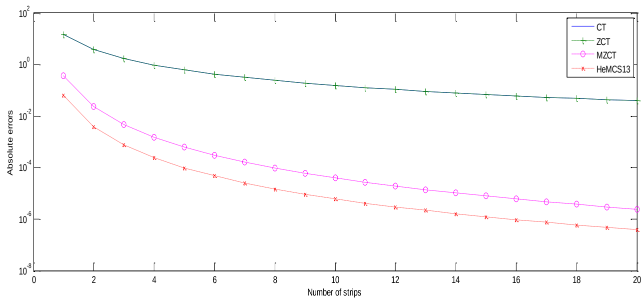

Fig 3. Comparison of error drops by all methods for Example 3

Table 2: Comparison of COC in all methods for Example 1

Number of strips (m)	CT	ZCT	MZCT	HeMCS13
1	NA	NA	NA	NA
2	2.5728	3.3788	5.0724	4.1456
4	2.0420	4.3106	4.1474	4.0337
8	2.0091	-0.8154	4.0348	4.0083
16	2.0022	2.1713	4.0086	4.0021
32	2.0006	3.2118	4.0021	4.0005

Table 3: Comparison of COC in all methods for Example 2

Number of strips (m)	CT	ZCT	MZCT	HeMCS13
5	NA	NA	NA	NA
10	2.0018	1.9243	4.0025	4.0264
20	2.0004	2.4487	4.0007	4.0066
40	2.0001	2.3985	4.0001	4.0017
80	2.0001	2.3985	4.0000	4.0004

Table 4: Comparison of COC in all methods for Example 3

Number of strips (m)	CT	ZCT	MZCT	HeMCS13
1	NA	NA	NA	NA
2	1.9319	1.9319	3.8984	4.0187
4	1.9844	1.9844	3.9761	4.0049
8	1.9962	1.9962	3.9941	4.0012
16	1.9990	1.9990	3.9985	4.0003
32	1.9998	1.9998	3.9985	4.0000

Table 5: Computational cost in quadrature variants form strips.

Quadrature Variants	Total evaluations
CT	2m+3 [XI]
ZCT	5m+3 [XI]
MZCT	5m+3 [XI]
Proposed HeMCS13	7m+3

Table 6: Computational cost and CPU time comparison to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

Quadrature Variants	Computational cost			CPU time (in seconds)		
	Example 1	Example 2	Example 3	Example 1	Example 2	Example 3
CT	439	1415	2503	68.04	12.82	432.30
ZCT	1043	3503	6253	552.60	153.98	6469.38
MZCT	73	78	78	28.01	5.26	27.35
HeMCS13	59	45	66	26.65	5.99	27.82

V. Conclusion

A new heronian mean derivative-based quadrature scheme of Simpson's 1/3 type was proposed for efficient approximations of the RS-integral and extended for higher strips in a composite sense. The theorems regarding the local and global error terms were proved. Three numerical problems were tested from the literature to discuss the performance of the proposed scheme against few other existing schemes. The error drops, observed orders of accuracy and computations performance in terms of evaluations and CPU usage show the dominance of the proposed scheme over other discussed schemes for the evaluation for the RS-integral numerically.

Conflict of Interest:

No conflict of interest regarding this article

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