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AN EFFICIENT TRAPEZOIDAL SCHEME FOR NUMERICAL CUBATURE WITH HERONIAN MEAN DERIVATIVE

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Abstract

This study focuses on the Heronian mean derivative-based numerical cubature scheme to better evaluate double integrals' infinite limits. The proposed modifications rely on the Trapezoidal-type quadrature and cubature schemes. The aforementioned proposed scheme is important to numerically evaluate the complex double integrals, where the exact value is not available but the approximate values can only be obtained. With regards to higher precision and order of accuracy, the proposed Heronian derivative-based double integral scheme provides efficient results. The discussed scheme, in basic and composite forms, with local and global error terms is presented with necessary proofs with their performance evaluation against conventional Trapezoid rule through some numerical experiments. The consequently observed error distributions of the aforementioned scheme are found to be lower than the conventional Trapezoidal cubature scheme in composite form.

Keywords: Cubature, Double integrals, Heronian mean Derivative-based scheme, Precision, Order of accuracy, Local and global errors, Trapezoid.

I. Introduction

For the multiple integrals and their numerical computation, the nomenclature *cubature* is used whereas the term *quadrature* is usually used for the numerical computation of single integrals. Burden and Faires in [IV] pointed out the numerical computation of areas and volumes for irregular regions that have remained a topic of interest for engineers and scientists, most often such problems are presented in terms of integrals. Numerical integration is used when the functions are complicated; also analytical evaluation of integrals is not possible or is very difficult as well. Different quadrature rules to approximate areas were proposed by mathematicians for single integrals. To evaluate volumes using double integrals which are also important [X] and [XI].

Numerical integration of a family of closed Newton-Cotes quadrature rules was proposed by [XVIII], and obtained two orders of precision as compared to closed Newton-Cotes formulas. In [III] authors modified and then proposed a new numerical integration algorithm, which reduces errors. A new efficient midpoint derivative-based quadrature scheme for the Riemann-Stieltjes integrals of trapezoid-type was developed by Memon et al. in 2020 [X] and efficiently modified [XIX]. A comparison of the polynomial collocation method and uniformly-spaced quadrature rules was attempted by Shaikh [XV] to solve integral equations.

Numerical integration using arithmetic, geometric and harmonic means derivative-based closed Newton-Cotes rules were presented by Ramachandran et al. in 2016 [XIII], the results obtained from presented rules were compared with the existing closed Newton-Cotes quadrature (CNC) rules. A modified four-point quadrature rule was proposed by Shaikh et al. in 2016 [XVI] for numerical integration, to obtain an efficient modification of a method in Zhao et al. [XVIII], instead of 4^{th} order derivative 2^{nd} order derivative was used.

[V], [II] and [I] by Burg in 2012, Bailey and Borwein in 2011 and Babolian et al. in 2005, respectively are the ancient methods and their modifications to approximate integrals. Some new families of open Newton-Cotes rules were presented by Zafar et al in 2013 [XVII] including higher accuracy and derivatives than the classical formulas. Similar work is available in Dehghan and colleagues [VI], [VII], [VIII] in 2005-06, Jain in 2007 [IX], Pal in 2007 [XI], Sastry in 1997 [XIV] and Petrovskaya in 2011 [XII]. Some new and efficient derivative-based schemes for numerical cubature were proposed by Malik et al. in 2020 [X], which improves order of accuracy, degree of precision to a large extent. Also, error terms were derived.

Here, in this research work, a new and efficient modification of the existing closed Newton-Cotes Trapezoidal cubature rule (CNCT) is proposed which involves partial derivatives for evaluation of double integrals, which is the proposed Heronian mean derivative-based Trapezoidal cubature scheme PHeMT. This proposed scheme has a higher order of accuracy and degree of precision also the errors reduced are smaller than the conventional CNCT scheme.

II. General Formulation and Existing Closed Newton-Cotes Cubature Scheme for Double Integrals

The mathematical formulation of the double integrals over finite rectangles as described in this section, with the help of [XI], the Trapezoidal rule in its basic form in two dimensions for cubature. The main contribution of the research work is described in this material.

The double integrals in two dimensions defined over rectangles in the general form are defined as

$$I = \int_c^d \int_a^b f(x,y)dx\,dy \tag{1}$$

the volume of the surface is I here, which is defined by the integrand over the area element. With all limits being finite, we evaluate the double integral over a rectangle *x=a, x=b, y=c, y=d.*

In [XI] the discussion about the existing closed Newton-Cotes cubature scheme (CNCT double integral Scheme) to approximate volume in (1) for double integrals is of the form:

$$CNCT = \int_c^d \int_a^b f(x,y)\,dx\,dy \approx \frac{(b-a)(d-c)}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)] \tag{2}$$

The precision degree of the CNCT scheme defined by (2) is one. Its composite form is not available in [XI]**.**

CNCT-Cn is the composite form of CNCT, theorem 1 defines the global error term, which is the achievement in [X], as it was not available in [XI].

Theorem 1. *Let a, b, c, d be finite real numbers, and f (x, y) along with its second order partial derivatives exist and are continuous in [a,b]×[c, d]. Let {*x_i*, i=0,1,…,n} and {*y_j*, j=0,1,…,n} form uniformly spaced partitions of [a, b] and [c, d] such that b-a = nh and d-c = nk then the CNCT-Cn scheme in composite form for n elements with the global error term is defined as:*

$$CNCT - Cn = \int_c^d \int_a^b f(x,y)\,dx\,dy = \sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} f(x,y)dxdy$$
$$= \sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\frac{hk}{4}\left[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right]$$
$$-\frac{h^2(b-a)(d-c)}{12}f_{xx}\left(\underline{\xi},\underline{\eta}\right)-\frac{k^2(b-a)(d-c)}{12}f_{yy}\left(\underline{\xi},\underline{\eta}\right) \tag{3}$$
$$where\ \underline{\xi} \in (a,b)\ and\ \underline{\eta} \in (c,d)$$

III. Present Work and Proposed Heronian mean Derivative-based Trapezoidal Numerical Cubature Scheme PHeMT

Modifications of CNCT rule in a basic form involving derivatives acquires the form:

$$PHeMT(a{:}b,c{:}d) = \int_c^d \int_a^b f(x,y)\,dx\,dy = CNCT + \sum_{i=1}^{i=3} D_i\alpha_i \tag{4}$$

The coefficients $D_i = h_i(b-a, d-c)$ depends upon limits, we define derivative terms as :

$$\alpha_1 = \sum_{\forall y} f_{xx}(\mu_x, y)\,, \alpha_2 = \sum_{\forall x} f_{yy}(x,\mu_y)\ and\ \alpha_3 = f_{xxyy}(\mu_x,\mu_y).$$

The notation CNCT stands for the existing closed Newton-Cotes cubature scheme and PHeMT for the proposed Heronian mean derivative-based cubature scheme of Trapezoidal-type. In (4), by using heronian mean (HeM), we get a new derivative-based scheme, namely PHeMT for efficient evaluation of numerical cubature. The coefficients D_i's in the basic forms, and the averages concerning x and y in the existing CNCT as well as proposed PHeMT cubature schemes are summarized in Table 1.

Table 1: Coefficients and means used in CNCT and PHeMT cubature schemes

T schemes	D_1	D_2	D_3	μ_x	μ_y
CNCT	Derivative free	Derivative free	Derivative free	Derivative free	Derivative free
PHeMT	$-\frac{(b-a)(d-c)^3}{24}$	$-\frac{(b-a)^3(d-c)}{24}$	$\frac{(b-a)^3(d-c)^3}{144}$	$\frac{a+\sqrt{ab}+b}{3}$	$\frac{c+\sqrt{cd}+d}{3}$

The local error terms of the existing CNCT and proposed PHeMT scheme along with the precision are summarized in Table 2.

Table 2: Local error terms and degrees of precision of CNCT and PHeMT cubature schemes

T schemes	Local Error terms	Precision
CNCT	$-\frac{(b-a)^3(d-c)}{12}f_{xx}(\xi,\eta)-\frac{(b-a)(d-c)^3}{12}f_{yy}(\xi,\eta)$	1
PHeMT	$-\frac{(b-a)^3(d-c)}{72}\left(\sqrt{b}-\sqrt{a}\right)^2 f_{xxx}(\xi,\eta)-\frac{(b-a)(d-c)^3}{72}\left(\sqrt{d}-\sqrt{c}\right)^2 f_{yyy}(\xi,\eta)$	2

Theorems 2-3 shows the proof of precision degree and local error term of the proposed PHeMT cubature scheme respectively

Definition 1. The largest positive integer n for which the exact and approximate results of $f(x,y)=(xy)^n$ are same, is defined as the degree of precision of the cubature scheme.

Theorem 2. *Let a, b, c, d be finite real numbers, and f (x, y) along with its third-order partial derivatives exist and are continuous in [a, b]×[c, d], then the PHeMT scheme in basic form defined in (4) with coefficients in Table 1 has a degree of precision equal to two.*

Proof of Theorem 2.

Over the rectangle [a, b]×[c, d], the exact results of the $f(x,y)=(xy)^n$ with n = 0, 1, 2 and 3 are given as:

$$\int_c^d\int_a^b x^0y^0\,dx\,dy=(b-a)(d-c) \tag{5}$$

$$\int_c^d\int_a^b (xy)\,dx\,dy=\frac{(b^2-a^2)(d^2-c^2)}{4} \tag{6}$$

$$\int_c^d \int_a^b (x^2y^2)\, dx\, dy = \frac{(b^3-a^3)(d^3-c^3)}{9} \quad (7)$$

$$\int_c^d \int_a^b (x^3y^3)\, dx\, dy = \frac{(b^4-a^4)(d^4-c^4)}{16} \quad (8)$$

The approximate evaluation of the integral with the same values of n using the proposed PHeMT scheme with coefficient in Table 1 and general expression (4) is:

$$PHeMT(x^0y^0) = (b-a)(d-c) \quad (9)$$

$$PHeMT(xy) = \frac{(b^2-a^2)(d^2-c^2)}{4} \quad (10)$$

$$PHeMT(x^2y^2) = \frac{(b^3-a^3)(d^3-c^3)}{9} \quad (11)$$

$$PHeMT(x^3y^3) = \frac{(b-a)(d-c)}{36}\begin{bmatrix}\{3(b^3+a^3)-(b-a)^2(a+\sqrt{ab}+b)\}\\ .\{3(d^3+c^3)-(d-c)^2(c+\sqrt{cd}+d)\}\end{bmatrix} \quad (12)$$

Comparison of (5)-(8) with (9)-(12) gives:

For

$$n \le 2, \qquad \int_c^d \left(\int_a^b (xy)^n dx\right) dy - PHeMT(xy)^n = 0. \quad (13)$$

But if

$$n \ge 3, \text{ then } \int_c^d \left(\int_a^b x^4y^4 dx\right) dy - PHeMT(x^4y^4) \ne 0. \quad (14)$$

Hence by using Definition 1 and (13)-(14), the degree of precision of *PHeMT* double integral scheme is two.

Definition 2.

For precision degree M of a cubature scheme, the leading local error term is defined as the difference between exact and approximate results of an integral in the neighborhood of (x_o, y_0) for the (M+1)th order term in Taylor's series development of $f(x, y)$.

Theorem 3. *Let a, b, c, d be finite real numbers, and f (x, y) along with its third-order partial derivatives exist and are continuous in [a, b]×[c, d], then the local error term of the proposed PHeMT scheme in basic form defined in (4) with coefficients in Table 1 is given as:*

$$LE_{PHeMT} = -\frac{(b-a)^3(d-c)}{72}\left(\sqrt{b}-\sqrt{a}\right)^2 f_{xxx}(\xi,\eta) - \frac{(b-a)(d-c)^3}{72}\left(\sqrt{d}-\sqrt{c}\right)^2 f_{yyy}(\xi,\eta) \quad (15)$$

where $\xi \in (a,b)$ *and* $\eta \in (c,d)$

Proof of Theorem 3.

Equations (13)-(14), confirm that the degree of precision of the proposed PHeMT cubature scheme is 2. To obtain leading local error terms, the third-order term in Taylor's series expression [XIV] is selected:

$$\frac{1}{3!}\begin{bmatrix}(x-x_0)^3\frac{\partial^3 f}{\partial x^3}(x_0,y_0)+3(x-x_0)^2(y-y_0)\frac{\partial^3 f}{\partial x^2\partial y}(x_0,y_0)\\ +3(x-x_0)(y-y_0)^2\frac{\partial^3 f}{\partial x\partial y^2}(x_0,y_0)+(y-y_0)^3\frac{\partial^3 f}{\partial y^3}(x_0,y_0)\end{bmatrix} \quad (16)$$

The shape for some $\xi \in (a,b)$ and $\eta \in (c,d)$ in the local error term of the proposed PHeMT scheme is:

$$LE_{PHeMT} = \frac{1}{3!}\left[\int_c^d \int_a^b x^3 dx\,dy - PHeMT(x^3)\right] f_{xxx}(\xi,\eta) + \frac{1}{2}\left[\int_c^d \int_a^b x^2 y dx\,dy - PHeMT(x^2y)\right] f_{yxx}(\xi,\eta)$$

$$+\frac{1}{2}\left[\int_c^d \int_a^b xy^2 dx\,dy - PHeMT(xy^2)\right] f_{yyx}(\xi,\eta) + \frac{1}{3!}\left[\int_c^d \int_a^b y^3 dx\,dy - PHeMT(y^3)\right] f_{yyy}(\xi,\eta) \quad (17)$$

Implementing required integrals in (17), as the precision of PHeMT is 2 the two middle terms vanish, after simplification we finally have:

$$LE_{PHeMT} - \frac{(b-a)^3(d-c)}{72}\left(\sqrt{b}-\sqrt{a}\right)^2 f_{xxx}(\xi,\eta) - \frac{(b-a)(d-c)^3}{72}\left(\sqrt{d}-\sqrt{c}\right)^2 f_{yyy}(\xi,\eta)$$

for some $\xi \in (a,b)$ and $\eta \in (c,d)$.

For the proposed PHeMT scheme in general, the composite form is denoted by *PHeMT-C_n* which is defined in (18).

$$PHeMT - Cn = \int_c^d \int_a^b f(x,y)\,dx\,dy = \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} PT\left(x_i : x_{i+1}, y_j : y_{j+1}\right) \quad (18)$$

With $b - a = n\,h$ and $d - c = n\,k$ partitioning the original square element, the averages are the means of each sub-square-element in n^2 elements.

IV. Numerical Experiments, Results and Discussion

To examine the performance of the proposed PHeMT cubature scheme and existing CNCT scheme. The following two examples from the literature [XI] are selected.

Example 1.

$$\int_0^{\ln 5}\int_0^{1} xe^{xy}\,dxdy$$

Example 2.

$$\int_1^{2}\int_1^{3} \frac{1}{1+x+y}\,dxdy$$

The direct results for examples 1-2 using MATLAB software are 1.485339738238448 and 0.454026674722594, which are used for the computation of absolute errors.

The numerical difference between the true values mentioned above through MATLAB and approximate values from the proposed numerical scheme is the absolute error, the results are presented up to 40 elements for a various number of elements, n = 1,2,3,….to understand the decreasing capability of absolute errors in existing and proposed schemes.

In Fig. 1 n = 1,2,3,…,40, the proposed PHeMT scheme reduced absolute errors as compared to the existing CNCT scheme in Example 1. However, in basic and composite forms, the proposed scheme is an efficient modification of the CNCT rule.

The number of function evaluations in a scheme is termed as computational cost, which is compared in Figure 2, the proposed PHeMT scheme consumed low computational cost as compared to the existing CNCT scheme.

Similarly, the absolute error drop and computational cost for Example 2 are respectively shown in Figs. 3-4. The accelerated error drop and lesser computational costs for both examples demonstrate the efficient performance of PHeMT over the conventional CNCT scheme for double integrals.

V. Conclusion

A successful extension to the composite form of the CNCT from literature for evaluating numerical cubature was discussed. A new and efficient numerical cubature scheme i.e. PHeMT is proposed in basic and generalized composite forms to evaluate double integrals. The degrees of precision and local error terms related to theorems have been proved. The proposed PHeMT scheme through numerical experiments shows more efficiency than the existing CNCT scheme. The decreased absolute errors and low computational cost of the proposed cubature scheme have been the main feature of this research work. The decreased absolute error distributions and low computational cost of the proposed cubature scheme are the main features of this research work.

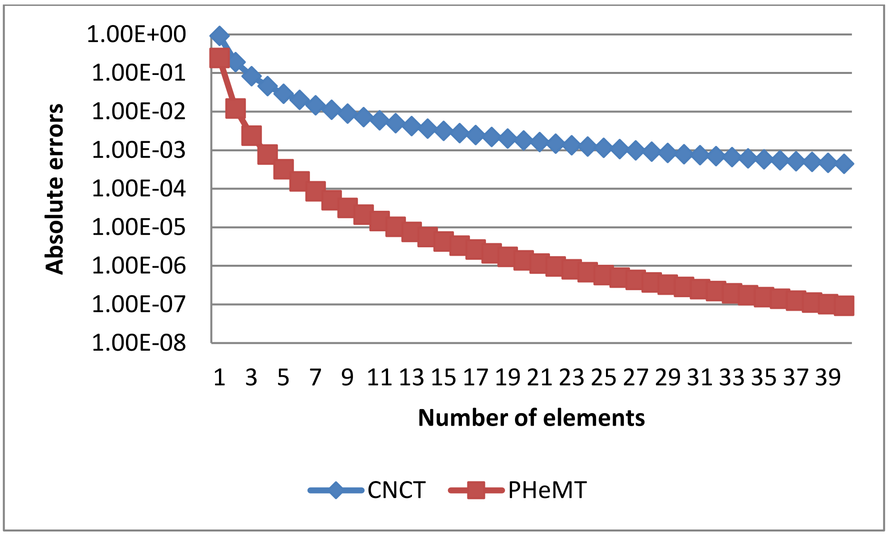

Fig 1. Absolute error distributions versus a number of elements for Example 1.

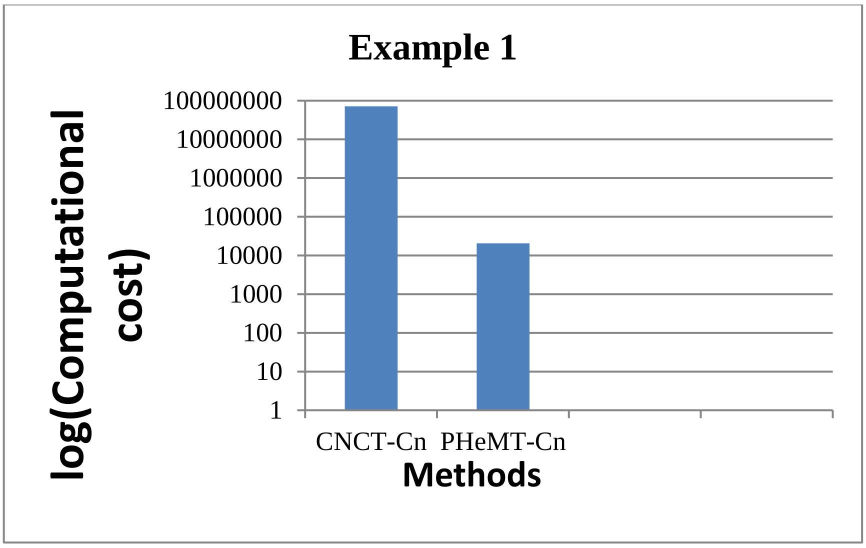

Fig 2. Computational cost (in logarithm scale) to achieve an absolute error of at most 1E-06 from Example 1.

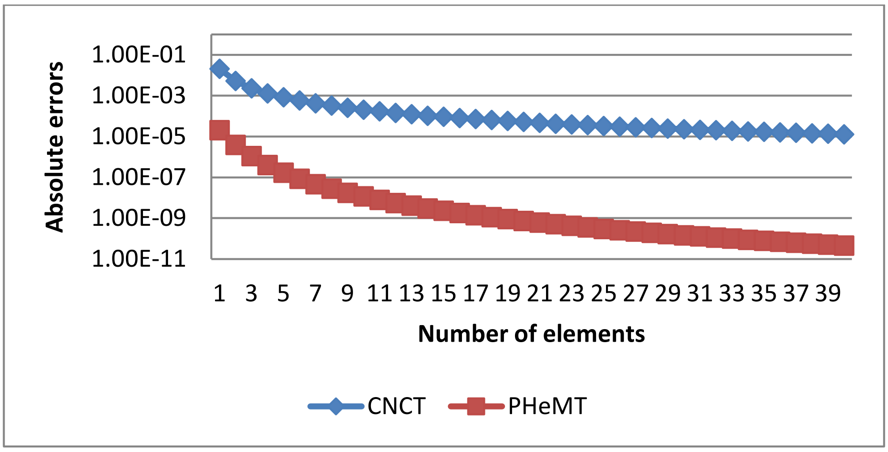

Fig. 3. Absolute error distributions versus a number of elements for Example 2.

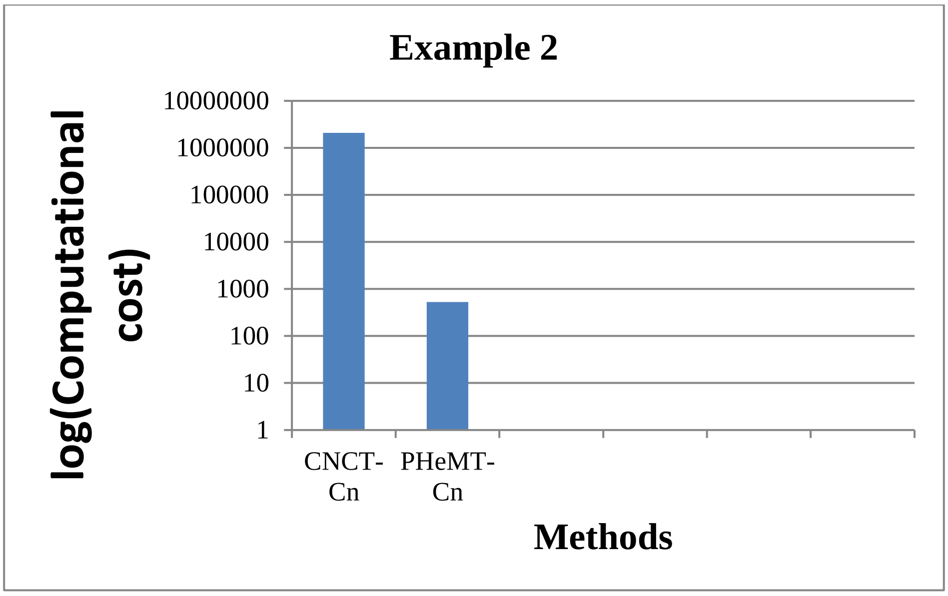

Fig 4. Computational cost (in logarithm scale) to achieve an absolute error of at most 1E-06 from Example 2.

Conflict of Interest:

There is no conflict of interest regarding this article

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