ON ASYMPTOTIC APPROXIMATIONS OF THE SOLUTION FOR TRANSLATING STRING UNDER EXTERNAL DAMPING

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Abstract

In this paper, a mathematical model for an externally damped axially moving string is studied. This mathematical model is a second order partial differential equation which is a wave-like equation. The String is assumed to be externally damped by the viscous medium such as oil, and there is no restriction on the parametric values of the damping parameter. From a physical point of view, a string is represented as a chain moving in oil in the positive horizontal direction between pair of pulleys. The axial speed of the string is assumed to be constant, positive and small compared to wave-velocity. To approximate the exact solutions of the initial-boundary value problem, the straightforward expansion method has been used to obtain valid approximations. It will be shown that if the damping parameter is neglected then the method breaks down as expected, and if damping is present in the system then the amplitudes of the oscillations are damped out and, solutions are valid and uniform.

Keywords: axially moving string, viscous damping, straightforward expansion method

I. Introduction

Axially translating elastic systems are frequently appearing in industry and technology due to their flexibility and usage. These systems have received great importance for the last few decades. These types of systems are found in many industrial, practical and physical situations as well as applications. If the systems are resisted by wind, storms or by any viscous materials, the systems might dissipate energy; this dissipation of energy is called damping. In industry and applications, these systems are represented as belt systems, band-saw blades, transportation and elevator cables, oil or gas pipelines, and mining hoists. The systems often experience unwanted vibrations and noise due to many external or internal causes. It is important to mention that in some cases vibrations play an important and useful

role. For example, oscillations of the heart, vibration of mobile phone during the silent mode, and vibration of the shaving devices. In most cases, the vibration is undesirable and unwanted phenomena, especially when earthquakes and bomb blasts damage the buildings and structures and, create human discomfort. The severe vibration in the bridges due to wind and heavy traffic is another aspect where the vibration can create the most undesirable phenomena. The vibration can even cause the failure of bridges and the tall buildings. The Tacoma Narrow suspension bridge in the USA collapsed on 7 November 1940 due to 42 miles per hour wind speed is a classic example of a structural failure. As mechanical systems can be damaged due to many causes as mentioned, it is important to understand the methods and techniques to reduce unnecessary noise and vibration in these systems. These days damping devices are frequently used to control the vibrations in several physical and mechanical systems either within the system [VI, VII, IX, XIV, XV, XXI] or at boundary [II, III, IV, V, XXII, XXIII].

The equations of motion for axially moving systems are described by mathematical models such as linear (nonlinear) string equation or by beam equation. It is also possible to control the oscillation amplitudes of the two-dimensional plates by using dampers, see [VIII]. In [XXIV] authors have studied the forced string-like equation with fixed boundary conditions. To solve the initial-boundary value problem Laplace transform method and model analysis have been used. It has been found that the Laplace transform method has computational, advantages over model analysis. In [XI, XVI] authors studied string-like equation subject to time-varying velocity. For asymptotic solution a two timescales perturbation method together with the Laplace transform method has been used. It has been found that Galerkin's truncation method cannot be applied to obtain approximations valid on long timescales. In [I, XVII] authors studied the string-like and beam-like models under the influence of the viscous damping and asymptotic approximation of solution have been obtained by two timescales perturbation method. It has been found that damping has a clear effect on the amplitude of oscillations. In [X, XII] authors studied string-like equations under the influence of boundary damping. A two timescales perturbation method, method of characteristic coordinates and Laplace transform methods have been used for the solution. It has been found that boundary damping has a clear effect on the vibration of oscillations.

In [XVIII] authors have studied the lateral vibrations of a vertically moving string by using a two timescales perturbation method. Vertical systems are variable-coefficient systems in nature since the tension is varying with position and time. In [XIX] authors studied damped string-like equation with fixed boundary and by considering general initial conditions. Asymptotic approximation of solution was obtained by using the Fourier-sine series method together with two timescales perturbation methods. It has been found that truncation of modes is applicable in some parametric values of damping. Numerical solution of vibration equation has been obtained by applying collocation method together with Haar wavelet in [XX].

In this paper, a string-like equation [I] which is a second order partial differential equation is studied with external damping. This external damping refers to continuous resistance of string motion in the vertical direction. To construct the analytic approximations of the exact solution of the initial-boundary value problem

a straightforward expansion method is utilized. It has been shown that a straightforward expansion method applies to damped problems, whereas the method breaks down and becomes non-uniform to un-damped equations. All cases for the damping parameter: critical damped, under damped and over damped have been discussed in detail.

The paper is organized as follows: In Section II, governing equation of motion with associated initial and boundary conditions is given. A Formal asymptotic solution for given initial-boundary value problem is obtained by the straightforward expansion method and is presented in Section III. In Section IV solution of the given model by two timescales perturbation method which is obtained from [I] has been discussed. In Section V, comparison of straightforward expansion and two timescales perturbation method has been discussed through graphs. Finally, some important conclusion has been presented in Section VI.

II. The governing equations of motion

The damped string-like equation representing axially moving belt between pair of pulleys is as given by, see [I]

$$\rho(U_{TT} + 2\bar{V}U_{XT} + \bar{V}^2 U_{XX}) - PU_{XX} + \bar{\delta}(U_T + \bar{V}U_X) = 0, T \geq 0, \tag{1}$$

with the boundary and the initial conditions:

$$U(0,T) = U(L,T) = 0, T \geq 0 \tag{2}$$

$$U(X,0) = F(X), U_T(X,0) = G(X), 0 < X < L \tag{3}$$

where ρ is constant linear mass density, P is the pre-tension, $\bar{\delta}$ is viscous damping parameter, V is the axial velocity, L is a constant distance between a pair of pulleys, $U(X,T)$ is the transversal displacement field variable which is a function of a spatial variable X and a time T, $F(X)$ and $G(X)$ is the initial shape of the string and the initial velocity, respectively. The parameters ρ, V, P and $\bar{\delta}$ are positive constants. The schematic diagram of the axially moving string is described in the following figure Fig. 1

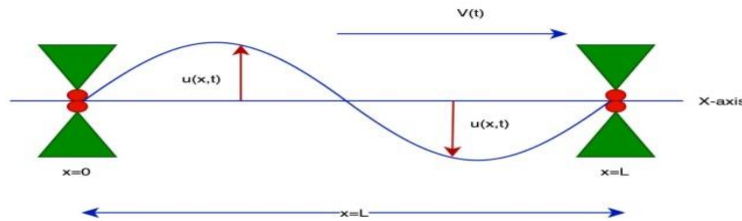
Fig.1: The schematic model of damped axially moving belt

The following dimensionless quantities are used to keep the equations (1)-(3) in the non-dimensional form:

$$u = \frac{U}{L}, x = \frac{X}{L}, t = \frac{cT}{L}, V^* = \frac{\bar{V}}{c}, \delta = \frac{\bar{\delta}L}{\rho c}, f(x) = \frac{F(X)}{L}, g(x) = \frac{G(X)}{c}$$

where $c = \sqrt{\frac{P}{\rho}}$ is a wave speed.

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Thus, Eqs. (1)-(3) into non-dimensional form become:

$$u_{tt} - u_{xx} + \delta u_t = -2V^* u_{xt} - \delta V^* u_x - V^{*2} u_{xx}; \; t > 0, 0 < x < 1 \tag{4}$$

$$u(0,t) = u(1,t) = 0; \; t > 0 \tag{5}$$

$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x); \; 0 < x < 1 \tag{6}$$

It is assumed that $\bar{V}$ is small compared to c, $\left(\bar{V} \ll c \Rightarrow \frac{\bar{V}}{c} \ll 1 \right)$. Therefore, it can be assumed that $V^* = O(\varepsilon)$, which means $V^* = \varepsilon V$ where $0 < \varepsilon \ll 1$. Thus, from Eqs. (4)-(6) it follows that:

$$u_{tt} - u_{xx} + \delta u_t = -2\varepsilon V u_{xt} - \delta \varepsilon V u_x - \varepsilon^2 V^2 u_{xx} \tag{7}$$

$$u(0,t) = u(1,t) = 0 \tag{8}$$

$$u(x,0) = f(x), u_t(x,0) = g(x) \tag{9}$$

where the conditions $0 < x < 1$ and $t > 0$ are omitted in Eqs. (7)-(9), and henceforth.

III. Straight forward expansion method

In Eqs. (7)-(9), it can be observed that the unknown function u does not only depends on x and t, but also on the small parameter ε. Therefore, it is reasonable to expand the unknown function $u(x,t;\varepsilon)$ in powers of ε (asymptotic expansion in a small parameter):

$$u(x,t;\varepsilon) = u_0(x,t) + \varepsilon u_1(x,t) + O(\varepsilon^2) \tag{10}$$

Now, by substituting Eq. (10) and its subsequent derivatives into Eq. (7), it yields (up to $O(\varepsilon)$):

$$\varepsilon^0 \left(u_{0_{tt}} - u_{0_{xx}} + \delta u_{0_t} \right) + \varepsilon^1 \left(u_{1_{xx}} - u_{1_{xx}} + \delta u_{1_t} + 2V_0 u_{0_{xt}} + \delta V_0 u_{0_x} \right) \\ + \varepsilon^2 \cdots \tag{11}$$

Now, by comparing the powers of $\varepsilon^0, \varepsilon^1$ and neglecting the higher order terms, it follows that the $O(1)$-problem is given by

$$u_{0_{tt}} - u_{0_{xx}} + \delta u_{0_t} = 0$$

$$u_0(0,t) = u_0(1,t) = 0 \tag{12}$$

$$u_0(x,0) = f(x), \quad \text{and } u_{0_t}(x,0) = g(x).$$

and, the $O(\varepsilon)$-problem is given by,

$$u_{1_{tt}} - u_{1_{xx}} + \delta u_{1_t} = -2V_0 u_{0_{xt}} - \delta V_0 u_{0_x}$$
$$u_1(0,t) = 0, u_1(1,t) = 0 \tag{13}$$
$$u_1(x,0) = f(x), \quad \text{and } u_{1_t}(x,0) = g(x).$$

The Solution of the $O(1)$-problem (12)

To solve the $O(1)$-problem, Bernoulli's method of separation of variables can be used. Since the partial differential equation and the boundary conditions are linear and homogenous, the following form of product solutions is assumed:

$$u_0(x,t) = \phi(x)\psi(t) \tag{14}$$

By substituting Eq. (14) into the first two equations in Eq. (12), the space-dependent (Sturm-Liouville eigenvalue problem) and the time-dependent ordinary differential equation, respectively, are as given:

$$\phi''(x) + \lambda\phi(x) = 0 \tag{15}$$
$$\phi(0) = \phi(1) = 0 \tag{16}$$
$$\ddot{\psi}(t) + \delta\dot{\psi}(t) + \lambda\psi(t) = 0 \tag{17}$$

where for convenience a separation constant $-\lambda$ is used. For a complete overview of Bernoulli's method reader is referred to [XIII].

Analysis of the time-dependent equation

By assuming the nontrivial solutions: $\psi(t) = e^{\alpha t}$ for Eq. (17), the characteristic equation is

$$\alpha^2 + \delta\alpha + \lambda = 0 \tag{18}$$

where α is to be determined. By using the quadratic formula, the roots of the characteristic equation are as given by

$$\alpha_{1,2} = -\frac{\delta}{2} \pm \sqrt{\frac{\delta^2}{4} - \lambda} \tag{19}$$

Note that the nature of the roots depends on the nature of the discriminant. In this regard, the following are the three cases for the discriminant.

Case 1. *Critical Damping*

If $\frac{\delta^2}{4} - \lambda = 0,\ \Rightarrow \delta^2 = 4\lambda,\ \Rightarrow \delta = \pm 2\sqrt{\lambda},\ \Rightarrow \delta = 2\sqrt{\lambda}$, for $\lambda > 0$.

Note that $-2\sqrt{\lambda}$ is discarded, since δ is positive. and, since δ is positive therefore λ must also be positive. In this case, the roots are real and repeated, therefore the time-dependent solution is given as follows

$$\psi(t) = (c_1 + c_2 t)e^{-\frac{\delta}{2}t} \tag{20}$$

For $\delta = 0$, it can be observed that the time-dependent solution increases linearly without bound for time $t > 0$. Since the time-dependent equation was expected to produce oscillatory solutions, therefore, physically this solution is uninteresting and can be neglected.

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Case 2. *Under Damping*

If $\frac{\delta^2}{4} - \lambda < 0, \Rightarrow 0 < \delta < 2\sqrt{\lambda}$, for $\lambda > 0$.

In this case, the roots are complex with real part negative, that is:

$$\alpha_1 = -\frac{\delta}{2} + i\sqrt{-\frac{\delta^2}{4} + \lambda} \quad \alpha_2 = -\frac{\delta}{2} - i\sqrt{-\frac{\delta^2}{4} + \lambda} \tag{21}$$

Therefore, the time-dependent solution is as given by

$$T(t) = e^{-\frac{\delta}{2}t}(c_1\cos(\omega_n t) + c_2\sin(\omega_n t)) \tag{22}$$

where $\omega_n = \sqrt{-\frac{\delta^2}{4} + \lambda}$. It can be observed that for $\delta = 0$, the solution in Eq. (22)

is oscillatory, which damps out by the damping coefficient $\Gamma = e^{-\frac{\delta}{2}t}$ when $\delta > 0$. For t to be sufficiently large, it can easily be observed that $T(t) \equiv 0$. It means that the time-dependent part is bounded for large time t. This time dependent solution is what was expected.

Case 3. *Over Damping*

If $\frac{\delta^2}{4} - \lambda > 0 \Rightarrow \delta > 2\sqrt{\lambda}$ for $\lambda > 0$.

In this case, the roots are real and distinct, that is

$$\alpha_1 = -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} - \lambda}, \quad \alpha_2 = -\frac{\delta}{2} - \sqrt{\frac{\delta^2}{4} - \lambda} \tag{23}$$

Therefore, the time-depending part has the solution as given by

$$\psi(t) = e^{-\frac{\delta}{2}t}(c_1 e^{\omega_n t} + c_2 e^{-\omega_n t}) \tag{24}$$

where $\omega_n = \sqrt{\frac{\delta^2}{4} - \lambda}$. Note that for $\delta = 0$, the solution in Eq. (24) is growing exponentially in time t, which is uninteresting and physically impossible. Therefore, this case can be neglected.

Analysis of the space-dependent equation

The space dependent part given in Eq. (15)-(16) is a well-known Sturm-Liouville eigenvalue problem, and has the positive eigenvalues $\lambda_n = (n\pi)^2, n = 1,2,3,\cdots$, and the corresponding eigenfunctions as given by

$$\phi_n(x) = \sin(\sqrt{\lambda_n}x) = \sin(n\pi x), n = 1,2,3\cdots. \tag{25}$$

General Solution of the $O(1)$-Problem $(0 < \delta < 2\sqrt{\lambda_n}$, for $\lambda_n = (n\pi)^2 > 0, n = 1,2,3,\cdots)$

By using the superposition principle, from the product solutions (14) with Eqs. (22) and (25), it yields

$$u_0(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\delta}{2}t} \left[A_{n0} \cos\left(\sqrt{\left((n\pi)^2 - \frac{\delta^2}{4}\right)} t \right) \right.$$
$$\left. + B_{n0} \sin\left(\sqrt{\left((n\pi)^2 - \frac{\delta^2}{4}\right)} t \right) \right] \sin(n\pi x) \tag{26}$$

where A_{n0} and B_{n0} are given by using the orthogonality properties of the eigenfunctions:

$$A_{n0} = 2 \int_0^1 f(x) \sin(n\pi x) dx \tag{27}$$

$$\sqrt{\left((n\pi)^2 - \frac{\delta^2}{4}\right)} B_{n0} = \frac{\delta}{2} A_{n0} + 2 \int_0^1 g(x) \sin(n\pi x) dx \tag{28}$$

and where, the orthogonality property of the eigenfunctions is as given by

$$< \sin(n\pi x), \sin(m\pi x) > = \int_0^1 \sin(n\pi x) \sin(m\pi x) \, dx = \begin{cases} \frac{1}{2}, & n = m \\ 0, & n \neq m \end{cases} \tag{29}$$

The Solution of the $O(\varepsilon)$-problem (13)

To solve the $O(\varepsilon)$-problem (13) an eigenfunction expansion method will be used. The solutions will only be sought for the case $\frac{\delta^2}{4} - \lambda_n < 0$, $\lambda_n = n^2\pi^2 > 0, n = 1,2,3,\cdots$, since this is the only case that yields the oscillatory solutions of the time-dependent equation. By substituting the solution of the $O(1)$-problem (26) and the required derivatives $u_{0_{xt}}$ and u_{0_x} in the partial differential equation in Eq. (13) it follows that:

$$u_{1_{tt}} - u_{1_{xx}} + \delta u_{1_t}$$
$$= \sum_{n=1}^{\infty} 2e^{-\frac{\delta}{2}t} (A_{n0} \sin(\omega_n t) \tag{30}$$
$$- B_{n0} \cos(\omega_n t)) V_0 n\pi \omega_n \cos(n\pi x)$$

where $\omega_n = \sqrt{\left(n^2\pi^2 - \frac{\delta^2}{4}\right)}$, and where A_{n0} and B_{n0} are given by Eqs. (27) and (28), respectively. Now, expanding the solution $u_1(x,t)$ into the eigenfunction-expansion as given by

$$u_1(x,t) = \sum_{n=1}^{\infty} p_n(t) \sin(n\pi x) \tag{31}$$

By substituting Eq. (31) and the required derivatives $u_{1_{tt}}$, $u_{1_{xx}}$ and u_{1_t} into Eq. (30), it follows that:

$$\sum_{n=1}^{\infty} (\ddot{p}_n(t) + \delta\, \dot{p}_n(t) + n^2\pi^2\, p_n(t)) \sin(n\pi x)$$

$$= \sum_{n=1}^{\infty} 2e^{-\frac{\delta}{2}t}(A_{n0}\sin(\omega_n t)$$

$$- B_{n0}\cos(\omega_n t))Vn\pi\omega_n\cos(n\pi x)$$

(32)

By multiplying Eq. (32) on both sides by $\sin(m\pi x)$ and then integrating so-obtained equation from $x = 0$ to $x = 1$ it follows that:

$$\sum_{n=1}^{\infty} (\ddot{p}_n(t) + \delta\, \dot{p}_n(t) + n^2\pi^2\, p_n(t)) \int_0^1 \sin(n\pi x)\sin(m\pi x)dx$$

$$= \sum_{n=1}^{\infty} e^{-\frac{\delta}{2}t}(A_{n0}\sin(\omega_n t) - B_{n0}\cos(\omega_n t))2V\omega_n n\pi e^{-\frac{\delta}{2}t}$$

$$\times \int_0^1 \cos(n\pi x)\sin(m\pi x)\,dx$$

(33)

The orthogonality relation of cosine and sine is as given by

$$< \cos(n\pi x), \sin(m\pi x) >= \int_0^1 \cos(n\pi x)\sin(m\pi x)\,dx = \begin{cases} 0, & n = m \\ \theta_{nm}, & n \neq m \end{cases}$$

(34)

where θ_{nm} is defined by

$$\theta_{nm} = \int_0^1 \cos(n\pi x)\sin(m\pi x)\,dx$$

(35)

Now, by using Eqs. (29), (34) and (35) into Eq. (33), it gives

$$\ddot{p}_n(t) + \delta\, \dot{p}_n(t) + n^2\pi^2\, p_n(t)$$

$$= \sum_{n=1,n\neq m}^{\infty} [(A_{n0}\sin(\omega_n t)$$

$$- B_{n0}\cos(\omega_n t))]4V\omega_n n\pi e^{-\frac{\delta}{2}t}\theta_{nm}$$

(36)

Since, the homogeneous solution of the Eq. (36) can easily be determined, and is given as

$$p_{n_h}(t) = e^{-\frac{\delta}{2}t}(A_{n1}\cos(\omega_n t) + B_{n1}\sin(\omega_n t))$$

(37)

where A_{n1} and B_{n1} are constants of integration and they can be obtained from the $O(\varepsilon^2)$-problem. To construct the particular integral of Eq. (36), the method of undetermined coefficients can be used. The assumption for the particular solution for Eq. (36) is:

$$p_{n_p}(t) = e^{-\frac{\delta}{2}t}(E_m\cos(\omega_n t) + F_m\sin(\omega_n t))$$

(38)

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where E_m and F_m are obtained as follows

$$E_m = -\sum_{n=1, n\neq m}^{\infty} \frac{B_{n0} 4V\omega_n n\pi\theta_{nm}}{m^2\pi^2 - \omega_n^2} \tag{39}$$

$$F_m = \sum_{n=1, n\neq m}^{\infty} \frac{A_{n0} 4V\omega_n n\pi\theta_{nm}}{m^2\pi^2 - \omega_n^2} \tag{40}$$

Thus, $u_1(x,t)$ is completely known and is given by

$$u_1(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\delta}{2}t} \left[(A_{n1}\cos(\omega_n t) + B_{n1}\sin(\omega_n t)) \right.$$
$$- \sum_{n=1, n\neq m}^{\infty} \frac{4V\omega_n n\pi\theta_{nm}}{m^2\pi^2 - \omega_n^2} (B_{n0}\cos(\omega_n t) \tag{41}$$
$$\left. - A_{n0}\sin(\omega_n t)) \right] \sin(n\pi x)$$

where $\omega_n = \sqrt{\left(n^2\pi^2 - \frac{\delta^2}{4}\right)}$. At this moment we are not interested in higher order approximations, therefore it is reasonable to assume that $A_{n1} \equiv 0$ and $B_{n1} \equiv 0$. Thus, Eq. (41) can be expressed as

$$u_1(x,t) = -\sum_{n=1}^{\infty} e^{-\frac{\delta}{2}t} \left[\sum_{n=1, n\neq m}^{\infty} \frac{4V\omega_n n\pi\theta_{nm}}{m^2\pi^2 - \omega_n^2} (B_{n0}\cos(\omega_n t) \right.$$
$$\left. - A_{n0}\sin(\omega_n t)) \right] \sin(n\pi x) \tag{42}$$

Hence, the complete solution of the initial-boundary value problem (7)-(9), is as given by

$$u(x,t;\varepsilon) = u_0(x,t) + \varepsilon u_1(x,t)$$
$$= e^{-\frac{\delta}{2}t} \sum_{n=1}^{\infty} \left[(A_{n0}\cos(\omega_n t) + B_{n0}\sin(\omega_n t)) \right.$$
$$- \varepsilon \sum_{n=1, n\neq m}^{\infty} \frac{4V\omega_n n\pi\theta_{nm}}{m^2\pi^2 - \omega_n^2} (B_{n0}\cos(\omega_n t) \tag{43}$$
$$\left. - A_{n0}\sin(\omega_n t)) \right] \sin(n\pi x)$$

where A_{n0} and B_{n0} are given by Eqs. (27) and (28), respectively, and where $\omega_n = \sqrt{\left(\frac{\delta^2}{4} - (n\pi)^2\right)}$.

II. Two timescales perturbation method

In this Section, a short introduction to a two timescales perturbation method and direct computations already presented in literature are taken for comparison of the

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two methods, see [I]. In Eq. (7), if it is further assumed that $\delta^* = \varepsilon\delta$, and introduce fast timescale $t_1 = t$ and slow timescale $t_2 = \varepsilon t$, then the unknown function $u(x, t; \varepsilon)$ is expressed as $w(x, t_1, t_2; \varepsilon)$. It is usually assumed that the function $u(x, t; \varepsilon) = w(x, t_1, t_2; \varepsilon)$ can be approximated by the powers of ε in the asymptotic expansion, as given by

$$w(x, t_1, t_2; \varepsilon) = w_0(x, t_1, t_2) + \varepsilon w_0(x, t_1, t_2) + \varepsilon^2 \cdots \tag{44}$$

and that all the w_j's for $j = 0, 1, 2, \cdots$, are found in such a way that no secular or unbounded terms arise. It is also assumed that the unknown functions w_j are of $O(1)$. For details of the mathematical model and application of a two timescales perturbation method, the reader is suggested to see [I]. The complete solution of the initial-boundary value problem (7)-(9) by two timescales perturbation method is as given by

$$
\begin{aligned}
u(x, t; \varepsilon) = w(x, t_1, t_2; \varepsilon) &= w_0(x, t_1, t_2) + \varepsilon w_1(x, t_1, t_2) \\
&= e^{-\frac{\delta}{2}\varepsilon t} \sum_{n=1}^{\infty} \Bigg[(A_{n0}(0)\cos(n\pi t) + B_{n0}(0)\sin(n\pi t)) \\
&\quad - \varepsilon \sum_{n=1, n\neq m}^{\infty} \frac{4Vn^2\pi^2\theta_{nm}}{m^2\pi^2 - n^2\pi^2} (B_{n0}(0)\cos(n\pi t) \\
&\quad - A_{n0}(0)\sin(n\pi t)) \Bigg] \sin(n\pi x)
\end{aligned}
\tag{45}
$$

where $A_{n0}(0)$ is same as A_{n0} in Eq. (27) and $B_{n0}(0)$ is given by:

$$n\pi B_{n0}(0) = 2\int_0^1 g(x)\sin(n\pi x)\,dx \tag{46}$$

III. Results and discussion

In this Section, the comparison of the approximations of the exact solutions of the initial-boundary value problem (4)-(6) with results available in literature [I] will be presented. In Section III, the initial-boundary value problem has been approximated by using the straightforward expansion method with the assumptions that $V^* = O(\varepsilon)$ and $\delta^* = O(1)$, whereas in Section IV, see [I], the initial-boundary value problem has been approximated by using the two timescales perturbation method with the assumptions that $V^* = O(\varepsilon)$ and $\delta^* = O(\varepsilon)$. The comparison of the approximations of the exact solutions (43) and (45) are presented in the following four figures for some fixed given initial conditions, given values of a small parameter ε, damping parameter δ, axial velocity V, spatial coordinate x, and fixed number of modes n and m.
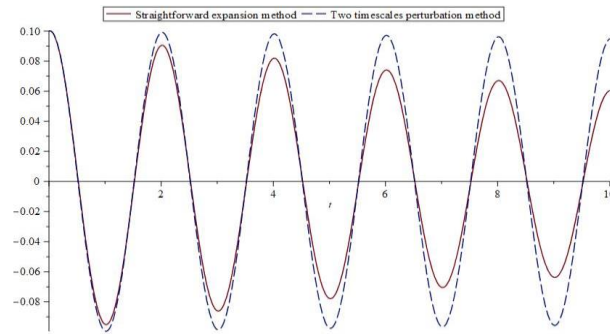
Fig. 2 Comparison of (43) and (45) for
$\delta = 0.1, \varepsilon = 0.5, V = 1, n = 1, m = 2, x = 0.5, f(x)$
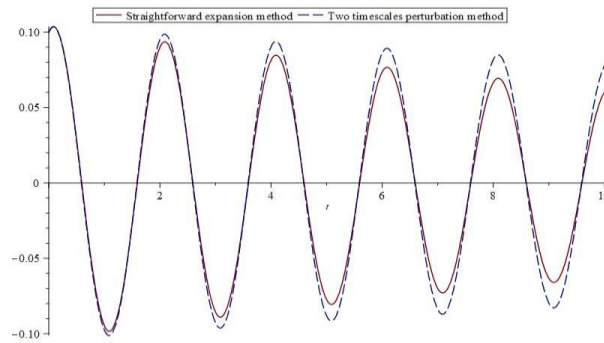$= 0.1\, sin(\pi x), g(x) = 0.1\, cos(\pi x)$

Fig. 3 Comparison of (43) and (45) for
$\delta = 0.1, \varepsilon = 0.5, V = 1, n = 1, m = 2, x = 0.5, f(x)$
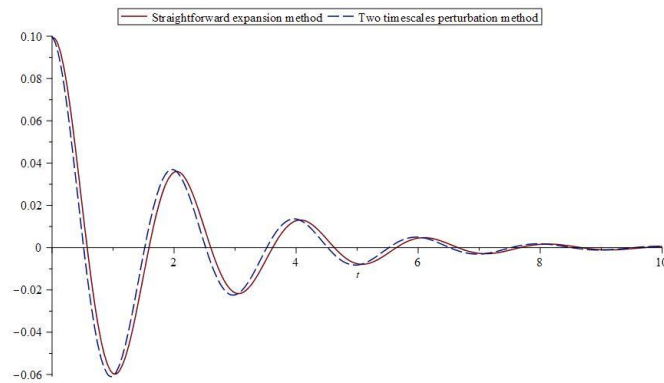$= 0.1\, sin(\pi x), g(x) = 0.1\, cos(\pi x)$

Fig. 4 Comparison of (43) and (45)
$\delta = 1, \varepsilon = 0.5, V = 1, n = 1, m = 2, x = 0.5, f(x)$
$= 0.1\, sin(\pi x), g(x) = 0.1\, cos(\pi x)$
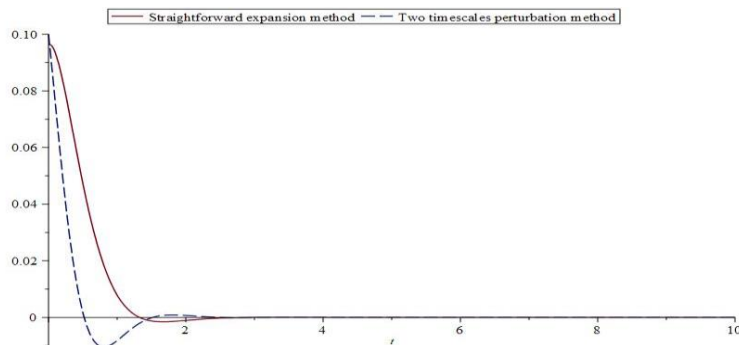
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Fig. 5 Comparison of (43) and (45) for
$\delta = 5, \varepsilon = 0.5, V = 1, n = 1, m = 2, x = 0.5, f(x) = 0.1\ sin(\pi x), g(x) = 0.1\ cos(\pi x)$

In Figs. 2 and 3, it can be observed that as time t increases the dashed curve is going away from the solid curve. This occurs due to the influence of a damping parameter. A damping parameter $\Gamma = e^{-\frac{\delta}{2}}$ damps out the vibrations much fast than a damping parameter $\Gamma = e^{-\frac{\varepsilon\delta}{2}}$. A damping parameter in Eq. (43) is fast varying whereas a damping parameter in Eq. (45) is slowly varying. Both solutions are of great physical importance. The solution in Eq. (43) can be interpreted as the motion of an axially moving string in some medium filled with water or oil, or some other viscous fluid with a sufficiently good amount of density, whereas the solution in Eq. (45) can be interpreted as the motion of an axially moving string in some medium filled with gas or air, or some less viscous medium. In Fig. 4, the solutions are in good agreement with each other, this occurs because that the parametric value of a damping parameter is chosen sufficiently large i.e., $\delta = 1$. It should be noted that the choice $\delta = 1$ does not violate the assumption that $u(x, t)$ has an asymptotic expansion, since in the same figure $\varepsilon = 0.1$ has been chosen. Interestingly, in Fig. 5, it looks like an over damping situation. Both solutions little oscillate and then quickly die away around $t = 2$.

VI. Conclusion

In this paper, string-like equation under the influence of external damping has been studied. Two ends of the string are kept fixed and general initial conditions are considered. The axial speed of the string is assumed to be constant, positive and small compared to wave-velocity. To approximate the exact solutions of the initial-boundary value problem, the straightforward expansion method has been used to obtain valid approximations. Also, to this the obtained solution of the given initial-boundary value problem has been compared to the solution obtained by applying two timescales perturbation method. It has been found that the damping factor has a clear effect on amplitudes of oscillations.

Conflict of Interest:

There is no conflict of interest regarding this article

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