GENERATION OF NEW OPERATIONAL MATRICES FOR DERIVATIVE, INTEGRATION AND PRODUCT BY USING SHIFTED CHEBYSHEV POLYNOMIALS OF TYPE FOUR

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Abstract

While solving the fractional order differential equation the requirement of the higher-order derivative is obvious therefore, this paper gives a definite expression for constructing the operational matrices of derivative which is the direct method to find the derivative of higher-order according to the requirement of the total differential equation. The proposed work expands the Chebyshev polynomial of type four up to six degrees that could help get the accuracy for the numerical solution of a given differential equation. Previously Chebyshev polynomial of the third type has been used by cutting the domain from [-1, 1] to [0, 1]. This study also generates the integrational operational matrix for solving the integral equation as well as an integrodifferential equation by using the Chebyshev polynomial of type four and expand it up to six order and generate the matrix by cutting the domain from [-1, 1] to [0, 1]. This is the first attempt to generate an integrational operational matrix that has never been highlight nor generate by any researcher. Another contribution of this paper is the generation of categorical expressions for the product of two Chebyshev vectors that will help in solving the differential equation of several kinds.

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I. Introduction

There are several kinds of Chebyshev polynomials, especially the first and second Chebyshev polynomials in addition to third and fourth Chebyshev polynomials [VIII]. The properties of the four kinds of polynomials lead to an extended range of results in many areas of Mathematics providing an outgrowth to developing new methods [I, XVIII, XIX]. Chebyshev polynomials are widely used in numerical analysis such as Polynomial approximation [VIII], series expansions, interpolation and quadrature, numerical solution of integral equations[VII, X, XII, XVII], Integro differential equations[II] total differential equations and partial differential equations[XX], spectral methods, and optimal control problem[V, VI]. Chebyshev polynomial has many advantages including Minimax approximation of a function by polynomials[VIII], Fastest convergence of its coefficients than other polynomials[XV] etc. One of the significant techniques used in the numerical computation is the formation of Operational matrices of derivative and Integration of Orthogonal polynomials such as Chebyshev polynomials[XIV], Legendre polynomials, Brenstien polynomials [XVI] etc.

Most literature regarding Chebyshev polynomials contains mainly results of Chebyshev polynomials of the first and second kinds and their numerous uses in different fields [IV, X] However there is very limited literature on Chebyshev polynomials of third and fourth kinds either from theoretical or practical points of view and their uses in various fields [XI, XIII]. If it asked for a summary of these four Chebyshev polynomials then we would say that the first kind is the most important and versatile. However, all four kinds of Chebyshev polynomials have their significant role. Now a day Chebyshev polynomials of a third and fourth kind both are widely used in the numerical analysis[III]. This paper is a useful addition to the literature regarding Chebyshev polynomial of type four.

The proposed study will generate a definite expression for the Operational matrix of Derivative by using Chebyshev polynomial of type four which is the most essential tool for solving the total differential equation. Another effort that has been done for the first time is the introduction of the integrational operational matrix for Integration of shifted Chebyshev polynomials of type four that would solve the integro as well as an integral equation. Also, another important result is the generation of the matrix for the product of Shifted Chebyshev polynomials type four is derived and the result is generalized for computational purposes. The reason behind generating these above-mentioned matrices is, that while solving the differential equation of different types, accuracy is required and could be possible by taking the polynomial of a higher order. To solve these equations

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that involve higher-order polynomial these proposed generated matrices will be a more beneficial tool for solving the equations.

II. Chebyshev polynomial of type four and their properties

Chebyshev polynomials type four are defined by the following relations given by equations (1), (2), (3).

$$\emptyset_n(t) = \frac{\sin(n+\frac{1}{2})\theta}{\sin(\frac{\theta}{2})} \quad n \geq 0 \tag{1}$$

With

$$\emptyset_0(t) = 1 \tag{2}$$
$$\emptyset_1(t) = 2t + 1 \tag{3}$$

And recursive formula defined by the equation (4)

$$\emptyset_n(t) = 2t\emptyset_{n-1}(t) - \emptyset_{n-2}(t) \quad n \geq 2 \tag{4}$$

This is a non-symmetric special case of Jacobi polynomial with $\alpha = \frac{1}{2}, \beta = \frac{-1}{2}$

And the orthogonal property of $\emptyset_n(t)$ in domain [-1, 1] is given by relation (5a)

$$\int_{-1}^{1} \emptyset_n(t)\emptyset_m(t)w(t) = \begin{cases} \pi & m = n \\ 0 & m \neq n \end{cases} \tag{5a}$$

With weight function given by equation (5b)

$$w(t) = \sqrt{\frac{1-t}{1+t}} \quad t \neq -1 \tag{5b}$$

By shifting a domain to [0,1] the shifted Chebyshev polynomial of type four is also orthogonal with the orthogonal property given relation (6a)

$$\int_0^1 \emptyset_n(t) \emptyset_m(t) w(t)dt = \begin{cases} \frac{\pi}{2} & n = m \\ 0 & n \neq m \end{cases} \tag{6a}$$

With weight function given by equation (6b)

$$w(t) = \begin{cases} 0 & x \leq 1 \\ \sqrt{\frac{1-t}{t}} & 0 < x < 1 \\ 0 & x \geq 1 \end{cases} \tag{6b}$$

And recursive formula defined by equation (7)

$$\emptyset_{n+1}(t) = 2(2t-1)\emptyset_n(t) - \emptyset_{n-1}(t) \tag{7}$$

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III. Operational matrix of derivative of shifted Chebyshev polynomial type four

In this section Operational matrix for the derivative of shifted Chebyshev polynomial type, four is formed. To explain the formulation of the derivative matrix Chebyshev polynomials with n=7 are selected. The polynomials are given by equations (8a) to (8g).

$$\emptyset_0(t) = 1 \tag{8a}$$

$$\emptyset_1(t) = 4t - 1 \tag{8b}$$

$$\emptyset_2(t) = 16\,t^2 - 12t + 1 \tag{8c}$$

$$\emptyset_3(t) = 64t^3 - 80t^2 + 24t - 1 \tag{8d}$$

$$\emptyset_4(t) = 256t^4 - 448t^3 + 240t^2 - 40t + 1 \tag{8e}$$

$$\emptyset_5(t) = 1024t^5 - 230t^4 + 1792t^3 - 560t^2 + 60t - 1 \tag{8f}$$

$$\emptyset_6(t) = 4096t^6 - 1126t^5 + 11520t^4 - 5376t^3 + 1120t^2 - 84t + 1 \tag{8g}$$

Let D be the differential operator $D = \frac{d}{dt}$. The respective derivative of the above polynomial is given by equations (9a) to (9g).

$$D\,[\emptyset_0(t)] = 1 \tag{9a}$$
$$D\,[\emptyset_1(t)] = 4(1) \tag{9b}$$
$$D\,[\emptyset_2(t)] = 16\,(2t) - 12(1) \tag{9c}$$
$$D\,[\emptyset_3(t)] = 64(3t^2) - 80(2t) + 24(1) \tag{9d}$$
$$D\,[\emptyset_4(t)] = 256(4t^3) - 448(3t^2) + 240(2t) - 40(1) \tag{9e}$$
$$D\,[\emptyset_5(t)] = 1024(5t^4) - 230(4t^3) + 1792(3t^2) - 560(2t) + 60(1) \tag{9f}$$
$$D\,[\emptyset_6(t)] = 4096(6t^5) - 1126(5t^4 + 11520(4t^{3)} - 5376(3t^2) + 1120(2t) - 84(1) \tag{9g}$$

These derivatives of a polynomial can be represented as lower-order Chebyshev polynomial of type four given by equations (10a) to (10g).

$$D\,[\emptyset_0(t)] = 0 \tag{10a}$$
$$D\,[\emptyset_1(t)] = 4\emptyset_0(t) \tag{10b}$$
$$D\,[\emptyset_2(t)] = -4\emptyset_0(t) + 8\emptyset_1(t) \tag{10c}$$
$$D\,[\emptyset_3(t)] = 8\emptyset_0(t) - 4\emptyset_1(t) + 12\emptyset_2(t) \tag{10d}$$
$$D\,[\emptyset_4(t)] = -8\emptyset_0(t) + 12\emptyset_1(t) - 4\emptyset_2(t) + 16\emptyset_3(t) \tag{10e}$$
$$D\,[\emptyset_5(t)] = 12\emptyset_0(t) - 8\emptyset_1(t) + 16\emptyset_2(t) - 4\emptyset_3(t) + \emptyset_4(t) \tag{10f}$$
$$D\,[\emptyset_6(t)] = -12\emptyset_0(t) + 16\emptyset_1(t) - 8\emptyset_2(t) + 20\emptyset_3(t) - 4\emptyset_4(t) + 24\emptyset_5(t) \tag{10g}$$

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Above derivation in the form of the equation can be written as $D\left[\emptyset_n(t)\right] = \frac{d}{dt}\left[\emptyset_n(t)\right]$

When $\emptyset_n(t) = [\emptyset_0,\ \emptyset_1, \ldots \ldots \emptyset_n]^T$

Also, D is a matrix of order 7×6

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ -4 & 8 & 0 & 0 & 0 & 0 \\ 8 & -4 & 12 & 0 & 0 & 0 \\ -8 & 12 & -4 & 16 & 0 & 0 \\ 12 & -8 & 16 & -4 & 20 & 0 \\ -12 & 16 & -8 & 20 & -4 & 24 \end{bmatrix}$$

To generalized, the above result odd and even entries are formulated separately. The odd entries are

$d_{1,0} = 4$	$d_{2,1} = 8$				
$d_{3,0} = 8$	$d_{4,1} = 12$	$d_{3,2} = 12$	$d_{4,3} = 16$		
$d_{5,0} = 12$	$d_{6,1} = 16$	$d_{5,2} = 16$	$d_{6,3} = 20$	$d_{5,4} = 20$	$d_{6,5} = 24$

$$d_{i,j} = 2(i + j + 1) \qquad\qquad i > j,\ (i + j)\ is\ odd$$

$$d_{i,j} = 0 \qquad\qquad i \leq j$$

The even entries are

$d_{2,0} = -4$				
$d_{4,0} = -8$	$d_{3,1} = -4$	$d_{4,2} = -4$		
$d_{6,0} = -12$	$d_{5.1} = -8$	$d_{6,2} = -8$	$d_{5,3} = --4$	$d_{6,4} = -4$

$$d_{i,j} = 2(j - i) \qquad\qquad i > j,\ (i + j)\ is\ even$$
$$d_{i,j} = 0 \qquad\qquad i \leq j$$

In general, the result can be written in form of equations (11a) to (11c)

$$d_{i,j} = 2(i + j + 1) \qquad\qquad i > j,\ (i + j)\ is\ odd \qquad (11a)$$
$$d_{i,j} = 2(j - i) \qquad\qquad i > j,\ (i + j)\ is\ even \qquad (11b)$$
$$d_{i,j} = 0 \qquad\qquad i \leq j \qquad (11c)$$

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For Higher order derivatives up to the nth order, the result can be summarized as

$$\frac{d^m \emptyset}{dt^m} = D^m \emptyset(t) \tag{12}$$

Theorem 1:

The derivative of Chebyshev polynomial of type four can be written as a linear combination of lower-order Chebyshev polynomial of type four and given by the following relation (13) and (14).

For n even:

$$D[\emptyset_n(t)] = 2\left[\sum_{i=1}^{n-1}(i-n-1)\emptyset_{i-1}(t) \, i \, odd + \sum_{i=2}^{n}(i+n)\emptyset_{i-1}(t) \, i \, even\right] \tag{13}$$

For n odd:

$$D[\emptyset_n(t)] = 2\left[\sum_{i=1}^{n}(i+n)\emptyset_{i-1}(t) \, i \, odd + \sum_{i=2}^{n-1}(i-n-1)\emptyset_{i-1}(t) \, i \, even\right] \tag{14}$$

IV. Operational Matrix of Product of Shifted Chebyshev Polynomial of type four

In this section product of two Chebyshev polynomial of type four $\emptyset\_n(t)$ and $\emptyset\_m(t)$ has been evaluated. To derive the operational matrix for the product of Chebyshev polynomial of type four let n=m= 4.

$$\emptyset_n(t) = [\emptyset_0 \ldots \ldots \ldots \emptyset_{n-1}]^T \quad and \quad \emptyset_m^T(t) = [\emptyset_0 \ldots \ldots \ldots \emptyset_{m-1}]$$

$$\emptyset_n(t).\emptyset_m^T(t) = \begin{pmatrix} \emptyset_0\emptyset_0 & \emptyset_0\emptyset_1 & \emptyset_0\emptyset_2 & \emptyset_0\emptyset_3 \\ \emptyset_1\emptyset_0 & \emptyset_1\emptyset_1 & \emptyset_1\emptyset_2 & \emptyset_1\emptyset_3 \\ \emptyset_2\emptyset_0 & \emptyset_2\emptyset_1 & \emptyset_2\emptyset_2 & \emptyset_2\emptyset_3 \\ \emptyset_3\emptyset_0 & \emptyset_3\emptyset_1 & \emptyset_3\emptyset_2 & \emptyset_3\emptyset_3 \end{pmatrix}$$

These product elements can be expressed in the polynomial form (15a) to (15j).

$$\emptyset_0\emptyset_0 = \emptyset_0 \tag{15a}$$
$$\emptyset_0\emptyset_1 = \emptyset_1 \tag{15b}$$
$$\emptyset_0\emptyset_2 = \emptyset_2 \tag{15c}$$
$$\emptyset_0\emptyset_3 = \emptyset_3 \tag{15d}$$
$$\emptyset_1\emptyset_1 = \emptyset_2 + \emptyset_1 + \emptyset_0 \tag{15e}$$
$$\emptyset_1\emptyset_2 = \emptyset_3 + \emptyset_2 + \emptyset_1 \tag{15f}$$
$$\emptyset_1\emptyset_3 = \emptyset_4 + \emptyset_3 + \emptyset_2 \tag{15g}$$
$$\emptyset_2\emptyset_2 = \emptyset_4 + \emptyset_3 + \emptyset_2 + \emptyset_1 + \emptyset_0 \tag{15h}$$
$$\emptyset_2\emptyset_3 = \emptyset_5 + \emptyset_4 + \emptyset_3 + \emptyset_2 + \emptyset_1 \tag{15i}$$
$$\emptyset_3\emptyset_3 = \emptyset_6 + \emptyset_5 + \emptyset_4 + \emptyset_3 + \emptyset_2 \tag{15j}$$

In general, an implicit formula to compute the product vector of two Chebyshev polynomials of type four $\emptyset\_n \emptyset\_m$ is given by equation (16).

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$$\emptyset_n \emptyset_m = \sum_{k=i}^{j} \emptyset_k \tag{16}$$

Where

$$i = |m - n| \quad and \ j = (m + n)$$

V. Operational Matrix of Integration of Shifted Chebyshev Polynomial of type four

Let I will be an integral operator given as

$$I = \int_a^b f(x)dx$$

Applying integral operator to the polynomials in section 3 for n=5 in domain [0, 1] we have the following results from equation (17a) to (17f).

$$I\emptyset_0 = t \tag{17a}$$

$$I\emptyset_1 = 2t^2 - t \tag{17b}$$

$$I\emptyset_2 = \frac{16t^3}{3} - 6t^2 + t \tag{17c}$$

$$I\emptyset_3 = 16t^4 - \frac{80t^3}{3} + 12t^2 - t \tag{17d}$$

$$I\emptyset_4 = \frac{256t^5}{5} - 112t^4 + 80t^3 - 20t^2 + t \tag{17e}$$

$$I\emptyset_5 = \frac{512t^6}{3} - \frac{2304t^5}{5} + 448t^4 - \frac{560t^3}{3} + 30t^2 - t \tag{17f}$$

These integrals of nth order polynomial can be written as a linear combination of (n+1) th order and lower order Chebyshev polynomial of type four as follow(18a) to (18f)

$$I\emptyset_0 = \frac{1}{4}(\emptyset_0 + \emptyset_1) \tag{18a}$$

$$I\emptyset_1 = \frac{1}{8}(\emptyset_2 + \emptyset_1) \tag{18b}$$

$$I\emptyset_2 = \frac{1}{24}(2\emptyset_3 + \emptyset_2 - 3\emptyset_1 - 2\emptyset_0) \tag{18c}$$

$$I\emptyset_3 = \frac{1}{48}(3\emptyset_4 + \emptyset_3 - \emptyset_2 + ..0.. + 2\emptyset_0) \tag{18d}$$

$$I\emptyset_4 = \frac{1}{80}(4\emptyset_5 + 10\emptyset_4 - 5\emptyset_3 + ..0.. + 2\emptyset_0) \tag{18e}$$

$$I\emptyset_5 = \frac{1}{480}(20\emptyset_6 + 4\emptyset_5 - 24\emptyset_4 + ..0.. + 10\emptyset_0) \tag{18f}$$

The above equations can be written in general equation form in (19)

$$\int_0^t \emptyset_{n-1}(s)ds = P\emptyset_n(t) \tag{19}$$

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Where matrix P is 6×7 matrix given as:

$$P = \begin{bmatrix} 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/8 & 1/8 & 0 & 0 & 0 & 0 \\ -1/12 & -1/8 & 1/24 & 1/8 & 0 & 0 & 0 \\ 1/24 & 0 & -1/12 & 1/48 & 1/16 & 0 & 0 \\ -1/40 & 0 & 0 & -1/16 & 1/8 & 1/20 & 0 \\ 1/48 & 0 & 0 & 0 & -1/20 & 1/120 & 1/24 \end{bmatrix}$$

Theorem 2.

The integral of (n-1) order of Chebyshev polynomial of type four is a linear combination of nth order and lower order Chebyshev polynomial of type four.

The relation can be given by the equation (20a) and (20b*)*

For n even and $n \geq 2$

$$\int_0^t \emptyset_{n-1}(s)ds = \frac{1}{4n}\emptyset_n + \frac{1}{4n(n-1)}\emptyset_{n-1} - \frac{1}{4(n-1)}\emptyset_{n-2} + 0 \dots \dots + \frac{1}{(n+1)^2-1}\emptyset_0 \quad (19a)$$

For n odd and $n \geq 3$

$$\int_0^t \emptyset_{n-1}(s)ds = \frac{1}{4n}\emptyset_n + \frac{1}{2(n-1)}\emptyset_{n-1} - \frac{1}{4(n-1)}\emptyset_{n-2} + 0 \dots \dots + \frac{(-1)^n}{(n+1)^2+4}\emptyset_0 \quad (19b)$$

VI. Conclusions

The proposed research has made an effort towards the accurate methods of solving differential equations. In this paper, some important properties of Chebyshev polynomial of type four have been derived. The Operational matrices for derivatives have been introduced for solving the total differential equation. Previously the Chebyshev polynomial of type three has been used [16] by cutting the domain from [-1, 1] to [0, 1]. In addition to this, integrational operational matrices for the integration of Chebyshev polynomial of type four have also been derived for the first time which is a useful tool for solving integral equations of any type. Formation of the matrix for the product of two Chebyshev polynomial of type four has been done that work as an important tool in solving the differential equation. All the above-mentioned results are applicable in a wide range of computational processes and are useful to find more exact results.

Conflict of Interest:

Authors declared: No conflict of interest regarding this article

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