JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 Vol.-16, No.-2, February (2021) pp 87-101 ISSN (Print) 0973-8975

MODIFIED QUADRATURE ITERATED METHODS OF BOOLE RULE AND WEDDLE RULE FOR SOLVING NON-LINEAR EQUATIONS

Umair Khalid Qureshi[1], Sanaullah Jamali[2], Zubair Ahmed Kalhoro[3] Abdul Ghafoor Shaikh[4]

[1]Department of Business Administration, Shaheed Benazir Bhutto University, Sanghar, Sindh, Pakistan

[2]Department of Mathematics, University of Sindh Laar, Badin, Sindh, Pakistan

[3]Institute of Mathematics and Computer Science, University of Sindh Jamshoro, Sindh, Pakistan

[4]Department of Basic Sciences & Related Studies, QUEST, Nawabshah, Sindh, Pakistan

Corresponding Author: **Umair Khalid Qureshi**

E-mail: umair.khalid_sng@sbbusba.edu.pk

https://doi.org/10.26782/jmcms.2021.02.00008

(Received: December 23, 2020; Accepted: February 7, 2021)

Abstract

This article is presented a modified quadrature iterated methods of Boole rule and Weddle rule for solving non-linear equations which arise in applied sciences and engineering. The proposed methods are converged quadratically and the idea of developed research comes from Boole rule and Weddle rule. Few examples are demonstrated to justify the proposed method as the assessment of the newton raphson method, steffensen method, trapezoidal method, and quadrature method. Numerical results and graphical representations of modified quadrature iterated methods are examined with C++ and EXCEL. The observation from numerical results that the proposed modified quadrature iterated methods are performance good and well executed as the comparison of existing methods for solving non-linear equations.

Keywords: Boole Rule and Weddle Rule, convergence criteria, existing methods, graph, results

I. Introduction

Numerical integration is a powerful tool for solving areas under the curve in numerical analysis, which arises in many fields of applied sciences and engineering. Quadrature rule is a process of computing the numerical integration with the help of the newton–Cotes formula. Newton–Cotes formula is developed by Isaac Newton and Roger Cotes. There are five basic quadrature rules, which are commonly used for solving numerical integration such as a trapezoidal rule, Simson 1/3 rule, Simson 3/8 rule, Boole rule and Weddle rule. Recently, many researchers and scientists have taken an interest and developed many techniques for solving numerical integration with the help of quadrature rules [IX], [II], [XII], [X] and [XIII]. Furthermore, now a day's estimating a single root of nonlinear application equations is also an important topic in the research field, therefore numerous iterated method has been developed for estimating a single root of nonlinear equations by using newton Raphson method and quadrature rule in reference [VII] the method developed by trapezoidal rule and newton Raphson method,

$$y_n = x_n - \frac{f(x_n)}{f`(x_n)}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f`(x_n)+f`(y_n)}$$

Likewise, in reference [IV] the method developed by Simson rule and newton Raphson method, such as

$$y_n = x_n - \frac{f(x_n)}{f`(x_n)}$$

$$x_{n+1} = y_n - \frac{6f(y_n)}{f`(y_n)+4f`\left(\frac{x_n+y_n}{2}\right)+f`(x_n)}$$

Similarly, this research is developed a modified quadrature iterated method to find a root of the nonlinear equations with the help of quadrature rule and numerical technique. These proposed methods are based on Boole's rule and Weddle's rule, and which are converged quadratically. The idea of modified quadrature iterated methods comes from [VII] and [VIII]. Modified quadrature iterated methods compare with the newton Raphson method, Steffensen method, quadrature method and trapezoidal method [I], [III], [VII] and [VIII]. From, numerical fallouts and graphical representation are shown to justify the proposed methods as the assessment of existing methods for estimating a single root of nonlinear equations.

II. Modified Quadrature Iterated Methods

a) Boole's rule iterated method

This segment developed Boole's iterated method for solving nonlinear equations with the help of Boole's rule and Numerical Technique. Let Boole's rule h rule for n=4, such as

$$\int_{x_0}^{x} f(x)dx = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$$

Taking derivative for solving integration,

$$\int_{x_0}^{x} f`(x)dx = \frac{2h}{45}[7f`(x_0) + 32f`(x_1) + 12f`(x_2) + 32f`(x_3) + 7f`(x_4)]$$

The placement of certain integration, we get the following statement

$$f(x) = f(x_0) + \frac{2h}{45}[7f`(x_0) + 32f`(x_1) + 12f`(x_2) + 32f`(x_3) + 7f`(x_4)] \quad (1)$$

Where n=4 then h become,

$$h = \frac{x - x_0}{4}$$

h substitute in Eq. (1), we get

$$f(x) = f(x_0) + \frac{(x-x_0)}{90}[7f`(x_0) + 32f`(x_1) + 12f`(x_2) + 32f`(x_3) + 7f`(x_4)] \quad (2)$$

Using Eq. (1), then solving Eq. (2), we obtain

$$x = x_0 - \frac{90f(x_0)}{7f`(x_0)+32f`(x_1)+12f`(x_2)+32f`(x_3)+7f`(x_4)} \quad (3)$$

From basic numerical technique $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ and $x_3 = x_0 + 3h$ using in Eq. (3), we get

$$x = x_0 - \frac{90f(x_0)}{7f`(x_0)+32f`(x_0+h)+12f`(x_0+2h)+32f`(x_0+3h)+7f`(x_0+4h)} \quad (4)$$

To using $h = f(x_0)$ from reference [X] in Eq. (4), such as

$$x = x_0 - \frac{90f(x_0)}{7f`(x_0)+32f`(x_0+f(x_0))+12f`(x_0+2f(x_0))+32f`(x_0+3f(x_0))+7f`(x_0+4f(x_0))} \quad (5)$$

Where x_0 is an initial guess, and which is near to root `x`, so finally in general Eq. (6), we get

$$x_{n+1} = x_n - \frac{90f(x_n)}{7f`(x_n)+32f`(x_n+f(x_n))+12f`(x_n+2f(x_n))+32f`(x_n+3f(x_n))+7f(x_n+4f(x_n))} \quad (6)$$

Hence, Eq. (6) is Boole's iterated method for Solving Nonlinear Problems.

b) Weddle's rule iterated method

This segment developed a Weddle`s iterated method for solving nonlinear equations with the help of Weddle`s rule and Numerical Technique. Let Boole's rule [h] rule for n=4, such as

$$\int_{x_0}^{x} f(x)dx = \frac{3h}{10}[f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + f(x_4) + 5f(x_5) + f(x_6)]$$

Taking derivative for solving integration,

$$\int_{x_0}^{x} f`(x)dx = \frac{3h}{10}[f`(x_0) + 5f`(x_1) + f`(x_2) + 6f`(x_3) + f`(x_4) + 5f`(x_5) + f`(x_6)]$$

The placement of certain integration, we get the following statement,

$$f(x) = f(x_0) + \frac{3h}{10}[f`(x_0) + 5f`(x_1) + f`(x_2) + 6f`(x_3) + f`(x_4) + 5f`(x_5) + f`(x_6)] \quad (7)$$

Where n=6 then h become,

$$h = \frac{x-x_0}{6}$$

h substitute in Eq. (7), we get

$$f(x) = f(x_0) + \frac{x-x_0}{20}[f`(x_0) + 5f`(x_1) + f`(x_2) + 6f`(x_3) + f`(x_4) + 5f`(x_5) + f`(x_6)] \quad (8)$$

Using Eq. (1.1), then solving Eq. (8), we obtain

$$x = x_0 - \frac{20f(x_0)}{f`(x_0)+5f`(x_1)+f`(x_2)+6f`(x_3)+f`(x_4)+5f`(x_5)+f`(x_6)} \quad (9)$$

From basic numerical technique $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$, $x_4 = x_0 + 4h$, $x_4 = x_0 + 5h$ and $x_6 = x_0 + 6h$ using in Eq. (9), we get

$$x = x_0 - \frac{20f(x_0)}{f`(x_0)+5f`(x_0+h)+f`(x_0+h)+6f`(x_0+h)+f`(x_0+h)+5f`(x_5)+f`(x_6)} \quad (10)$$

To using $h = f(x_0)$ from reference [IX] in Eq. (10), such a

$$x = x_0 - 20f(x_0)/[f`(x_0) + 5f`(x_0 + f(x_0)) + f`(x_0 + 2f(x_0)) + 6f`(x_0 + 3f(x_0)) + f`(x_0 + 4f(x_0)) + 5f`(x_0 + 5f(x_0)) + f`(x_0 + 6f(x_0)) \quad (11)$$

Where x_0 is an initial guess, and which is near to root \`x\`, so finally in general Eq. (11), we get

$$x_{n+1} = x_n - 20f(x_n)/[f`(x_n) + 5f`(x_n + f(x_n)) + f`(x_n + 2f(x_n)) + 6f`(x_n + 3f(x_n)) + f`(x_n + 4f(x_n)) + 5f`(x_n + 5f(x_n)) + f`(x_n + 6f(x_n))] \quad (12)$$

Hence, Eq. (12) is Weddle\`s iterated method for Solving Nonlinear Problems.

III. Convergence Analysis

a) The convergence of Boole's rule iterated method

This section is giving the key results of this paper. We have given here the Mathematical proof that the proposed Method is converged quadratically.

Proof:

By using the Taylor series, we are expanding

$f(x_n), f`(x_n), f`(x_n + f(x_n))$, $f`(x_n + 2f(x_n))$, $f`(x_n + 3f(x_n))$ and $f`(x_n + 4f(x_n))$ an only second-order term about \`a\`, such as

$$f(x_n) = e_n f`(a)(1 + ce_n) \quad (13)$$

$$f`(x_n) = f`(a)(1 + 2ce_n) \quad (14)$$

and

$$f(x_n + f(x_n)) = f`(a)\left[(e_n + f(x_n)) + (e_n + f(x_n))^2 c\right]$$

Or

$$f'(x_n + f(x_n)) = f`(a)[(1 + f'(x_n)) + 2(e_n + f(x_n))(1 + f'(x_n))c]$$

Eq. (13) and Eq. (14) substitute in above Equation, we get

$$f`(x_n + f(x_n)) = f`(a)[1 + f`(a) + 2ce_n + 6ce_n f`(a) + 2ce_n f`^2(a)] \quad (15)$$

Now,

$$f(x_n + 2f(x_n)) = f`(a)[(e_n + 2f(x_n)) + c(e_n + 2f(x_n))^2]$$

Taking derivative,

$$f`(x_n + 2f(x_n)) = f`(a)[(1 + 2f`(x_n)) + 2c(e_n + 2f(x_n))(1 + 2f`(x_n))]$$

Using Eq. (13) and Eq. (14) in above, we get

$$f`(x_n + 2f(x_n)) = f`(a)[1 + 2f`(a) + 2ce_n + 12ce_n f`(a)] \tag{16}$$

and

$$f(x_n + 3f(x_n)) = f`(a)[(e_n + 3f(x_n)) + c(e_n + 3f(x_n))^2]$$

Taking derivative,

$$f`(x_n + 3f(x_n)) = f`(a)[(1 + 3f`(x_n)) + 2c(e_n + 3f(x_n))(1 + 3f`(x_n))]$$

Using Eq. (13) and Eq. (14) in above, we get

$$f`(x_n + 3f(x_n)) = f`(a)[(1 + 3f`(a)(1 + 2ce_n)) + 2c(e_n + 3e_n f`(a)(1 + ce_n))(1 + 3f`(a)(1 + 2ce_n))]$$

or

$$f`(x_n + 3f(x_n)) = f`(a)[1 + 3f`(a) + 6ce_n f`(a)) + 2ce_n + 6ce_n f`(a)(1 + 3f`(a) + 6ce_n f`(a) + ce_n + 3ce_n f`(a) + 6ce_n ce_n f`(a)]$$

or

$$f`(x_n + 3f(x_n)) = f`(a)[1 + 3f`(a) + 6ce_n f`(a)) + 2ce_n + 6ce_n f`(a)(1 + 3f`(a) + ce_n(1 + 9f`(a))]$$

or

$$f`(x_n + 3f(x_n)) = f`(a)[1 + 3f`(a) + 2ce_n(1 + (6f`(a) + 9f`(a)f`(a) + 3ce_n f`(a)(1 + 9f`(a))]$$

or

$$f`(x_n + 3f(x_n)) = f`(a)[1 + 3f`(a) + 2ce_n(1 + (6f`(a) + 3ce_n f`(a)(1 + 9f`(a))]$$

or

$$f`(x_n + 3f(x_n)) = f`(a)[1 + 3f`(a) + 2ce_n + 12ce_n f`(a)] \tag{17}$$

Finally,

$$f(x_n + 4f(x_n)) = f`(a)[(e_n + 4f(x_n)) + c(e_n + 4f(x_n))^2]$$

Taking derivative,

$$f`(x_n + 4f(x_n)) = f`(a)[(1 + 4f`(x_n)) + 2c(e_n + 4f(x_n))(1 + 4f`(x_n))]$$

Using Eq. (13) and Eq. (14) in above, we get

$$f`(x_n + 4f(x_n)) = f`(a)[(1 + 4f`(a)(1 + 2ce_n)) + 2c(e_n + 4e_n f`(a)(1 + ce_n))(1 + 4f`(a)(1 + 2ce_n))]$$

or

$$f`(x_n + 4f(x_n)) = f`(a)[1 + 4f`(a) + 8ce_n f`(a) + 2ce_n + 8ce_n f`(a)(1 + 4f`(a) + 8ce_n f`(a) + ce_n + 4ce_n f`(a) + 8ce_n ce_n f`(a)]$$

or

$$f`(x_n + 4f(x_n)) = f`(a)[1 + 4f`(a) + 8ce_n f`(a) + 2ce_n + 8ce_n f`(a)(1 + 4f`(a) + ce_n(1 + 12f`(a))]$$

For solving above eq. and ignoring second-order term, we obtain

$$f`(x_n + 4f(x_n)) = f`(a)[1 + 4f`(a) + 2ce_n + 16ce_n f`(a)] \quad (18)$$

For solving Eq. (14), Eq. (15), Eq. (16), Eq. (17) and Eq. (18), such as

$$\begin{aligned} 7f`(x_n) + 32f`(x_n + f(x_n)) + 12f`(x_n + 2f(x_n)) + 32f`(x_n + 3f(x_n)) \\ + 7f`(x_n + 4f(x_n)) \\ = f`(a)[7 + 14ce_n + 32 + 64f`(a) + 64ce_n + 384ce_n f`(a) \\ + 256ce_n f`^2(a) + 12 + 24f`(a) + 24ce_n + 144ce_n f`(a) + 32 \\ + 96f`(a) + 64ce_n + 384ce_n f`(a) + 7 + 28f`(a) + 14ce_n \\ + 112ce_n f`(a)] \end{aligned}$$

or

$$7f`(x_n) + 32f`(x_n + f(x_n)) + 12f`(x_n + 2f(x_n)) + 32f`(x_n + 3f(x_n)) + 7f`(x_n + 4f(x_n)) = 90f`(a)[1 + 2.4f`(a) + 2ce_n + 14.2ce_n f`(a)] \quad (19)$$

Now using Eq. (13) and Eq. (19) in the developed method, we get

$$e_{n+1} = e_n - \frac{90e_n f`(a)(1+ce_n)}{90f`(a)[1+2.4f`(a)+2ce_n+14.2ce_n f`(a)]}$$

$$e_{n+1} = e_n - \frac{e_n(1+ce_n)}{[1+2.4f`(a)+2ce_n+14.2ce_n f`(a)]}$$

$$e_{n+1} = e_n - e_n(1 + ce_n)[1 + 2.4f`(a) + 2ce_n + 14.2ce_n f`(a)]^{-1}$$

$$e_{n+1} = e_n - e_n(1 + ce_n)[1 - 2.4f`(a) - 2ce_n - 14.2ce_n f`(a)]$$

$$e_{n+1} = e_n - e_n[1 - 2.4f`(a) - ce_n - 16.6ce_n f`(a)]$$

$$e_{n+1} = e_n - e_n + 2.4e_n f`(a) + ce^2{}_n(1 + 16.6f`(a))$$

$$e_{n+1} = 2.4e_n f`(a) + ce^2{}_n(1 + 16.6f`(a)) \quad (20)$$

Finally, $f(x) = 0$ using in Eq. (13) then put in Eq. (20), we get

$$e_{n+1} = -2.4e^2{}_n f``(a) + ce^2{}_n(1 + 16.6f`(a))$$

or

$$e_{n+1} = e^2{}_n[-2.4f``(a) + c(1 + 16.6f`(a)) \quad (21)$$

Hence, Eq. (21) has been proven that the proposed Boole method converges quadratically.

b) The convergence of Weddle's rule iterated method

This section is giving the key results of this paper. We have given here the Mathematical proof that the proposed Method is converged quadratically.

Proof:

By using the Taylor series, we are expanding

$$f(x_n), f`(x_n), f`(x_n + f(x_n)), f`(x_n + 2f(x_n)), f`(x_n + 3f(x_n)),$$
$$f`(x_n + 4f(x_n)), f`(x_n + 5f(x_n))$$

and

$$f`(x_n + 6f(x_n))$$

only second-order term about `a`, such as

$$f(x_n) = e_n f`(a)(1 + ce_n) \tag{22}$$

$$f`(x_n) = f`(a)(1 + 2ce_n) \tag{23}$$

and

$$f(x_n + f(x_n)) = f`(a)\left[(e_n + f(x_n)) + (e_n + f(x_n))^2 c\right]$$

Or

$$f'(x_n + f(x_n)) = f`(a)[(1 + f'(x_n)) + 2(e_n + f(x_n))(1 + f'(x_n))c]$$

Eq. (22) and Eq. (23) substitute in above Equation, we get

$$f`(x_n + f(x_n)) = f`(a)[1 + f`(a) + 2ce_n + 6ce_n f`(a) + 2ce_n f`^2(a)] \tag{24}$$

Now,

$$f(x_n + 2f(x_n)) = f`(a)[(e_n + 2f(x_n)) + c(e_n + 2f(x_n))^2]$$

Taking derivative,

$$f`(x_n + 2f(x_n)) = f`(a)[(1 + 2f`(x_n)) + 2c(e_n + 2f(x_n))(1 + 2f`(x_n))]$$

Using Eq. (22) and Eq. (23) in above, we get

$$f`(x_n + 2f(x_n)) = f`(a)[1 + 2f`(a) + 2ce_n + 12ce_n f`(a)] \tag{25}$$

and

$$f(x_n + 3f(x_n)) = f`(a)[(e_n + 3f(x_n)) + c(e_n + 3f(x_n))^2]$$

Taking derivative,

$$f`(x_n + 3f(x_n)) = f`(a)[(1 + 3f`(x_n)) + 2c(e_n + 3f(x_n))(1 + 3f`(x_n))]$$

Using Eq. (22) and Eq. (23) in above, we get

$$f`(x_n + 3f(x_n)) = f`(a)[(1 + 3f`(a)(1 + 2ce_n)) + 2c(e_n + 3e_n f`(a)(1 + ce_n))(1 + 3f`(a)(1 + 2ce_n))]$$

or

$$f`(x_n + 3f(x_n)) = f`(a)[1 + 3f`(a) + 6ce_n f`(a)) + 2ce_n + 6ce_n f`(a)(1 + 3f`(a) + 6ce_n f`(a) + ce_n + 3ce_n f`(a) + 6ce_n ce_n f`(a)]$$

or

$$f`(x_n + 3f(x_n)) = f`(a)[1 + 3f`(a) + 6ce_n f`(a)) + 2ce_n + 6ce_n f`(a)(1 + 3f`(a) + ce_n(1 + 9f`(a))]$$

or

$$f`(x_n + 3f(x_n)) = f`(a)[1 + 3f`(a) + 2ce_n(1 + (6f`(a) + 9f`(a)f`(a) + 3ce_n f`(a)(1 + 9f`(a))]$$

or

$$f`(x_n+3f(x_n)) = f`(a)[1+3f`(a)+2ce_n(1+(6f`(a)+3ce_nf`(a)(1+9f`(a))]$$

or

$$f`(x_n+3f(x_n)) = f`(a)[1+3f`(a)+2ce_n+12ce_nf`(a)] \quad (26)$$

Further,

$$f(x_n+4f(x_n)) = f`(a)[(e_n+4f(x_n))+c(e_n+4f(x_n))^2]$$

Taking derivative,

$$f`(x_n+4f(x_n)) = f`(a)[(1+4f`(x_n))+2c(e_n+4f(x_n))(1+4f`(x_n))]$$

Using Eq. (22) and Eq. (23) in above, we get

$$f`(x_n+4f(x_n)) = f`(a)[(1+4f`(a)(1+2ce_n)) + 2c(e_n+4e_nf`(a)(1+ce_n))(1+4f`(a)(1+2ce_n))]$$

or

$$f`(x_n+4f(x_n)) = f`(a)[1+4f`(a)+8ce_nf`(a)+2ce_n+8ce_nf`(a)(1+4f`(a) + 8ce_nf`(a)+ce_n+4ce_nf`(a)+8ce_nce_nf`(a)]$$

or

$$f`(x_n+4f(x_n)) = f`(a)[1+4f`(a)+8ce_nf`(a)+2ce_n+8ce_nf`(a)(1+4f`(a) + ce_n(1+12f`(a))]$$

For solving above eq. and ignoring second order term, we obtain

$$f`(x_n+4f(x_n)) = f`(a)[1+4f`(a)+2ce_n+16ce_nf`(a)] \quad (27)$$

Now,

$$f(x_n+5f(x_n)) = f`(a)[(e_n+5f(x_n))+c(e_n+5f(x_n))^2]$$

Taking derivative,

$$f`(x_n+5f(x_n)) = f`(a)[(1+5f`(x_n))+2c(e_n+5f(x_n))(1+5f`(x_n))]$$

Using Eq. (22) and Eq. (23) in above, we get

$$f`(x_n+5f(x_n)) = f`(a)[(1+5f`(a)(1+2ce_n)) + 2c(e_n+5f`(a)(1+ce_n))(1+5f`(a)(1+2ce_n))]$$

or

$$f`(x_n+5f(x_n)) = f`(a)[1+5f`(a)+10ce_nf`(a)+2ce_n+10ce_nf`(a)(1+5f`(a) + 10ce_nf`(a)+ce_n+5ce_nf`(a)+10ce_nce_nf`(a)]$$

or

$$f`(x_n+5f(x_n)) = f`(a)[1+5f`(a)+10ce_nf`(a)+2ce_n + 10ce_nf`(a)(1+5f`(a)+ce_n+25ce_nf`(a))]$$

Or

$$f`(x_n+5f(x_n)) = f`(a)[1+5f`(a)+2ce_n+20ce_nf`(a)] \quad (28)$$

Finally,

$$f(x_n + 6f(x_n)) = f`(a)[(e_n + 6f(x_n)) + c(e_n + 6f(x_n))^2]$$

Taking derivative,

$$f`(x_n + 6f(x_n)) = f`(a)[(1 + 6f`(x_n)) + 2c(e_n + 6f(x_n))(1 + 6f`(x_n))]$$

Using Eq. (21) and Eq. (22) in above, we get

$$f`(x_n + 6f(x_n)) = f`(a)[(1 + 6f`(a)(1 + 2ce_n)) + 2c(e_n + 6f`(a)(1 + ce_n))(1 + 6f`(a)(1 + 2ce_n))]$$

or

$$f`(x_n + 6f(x_n)) = f`(a)[1 + 6f`(a) + 12ce_nf`(a) + 4ce_n + 10cf`(a)(1 + 6f`(a) + 12ce_nf`(a) + ce_n + 6ce_nf`(a) + 12ce_nce_nf`(a)]$$

or

$$f`(x_n + 6f(x_n)) = f`(a)[1 + 6f`(a) + 12ce_nf`(a) + 4ce_n + 10cf`(a)(1 + 6f`(a) + ce_n + 18ce_nf`(a)]$$

For solving above eq. and ignoring second-order term, we obtain

$$f`(x_n + 6f(x_n)) = f`(a)[1 + 6f`(a) + 12ce_nf`(a) + 4ce_n] \quad (29)$$

For solving Eq. (23), Eq. (24), Eq. (25), Eq. (26), Eq. (27), Eq. (28) and Eq. (29), such as

$$f`(x_n) + 5f`(x_n + f(x_n)) + f`(x_n + 2f(x_n)) + 6f`(x_n + 3f(x_n)) + f`(x_n + 4f(x_n)) + 5f`(x_n + 5f(x_n)) + f`(x_n + 6f(x_n)) = f`(a)[1 + 2ce_n + 5 + 5f`(a) + 10ce_n + 30ce_nf`(a) + 1 + 2f`(a) + 2ce_n + 12ce_nf`(a) + 6 + 18f`(a) + 12ce_n + 72ce_nf`(a) + 1 + 4f`(a) + 2ce_n + 16ce_nf`(a) + 5 + 25f`(a) + 10ce_n + 100ce_nf`(a) + 1 + 6f`(a) + 4ce_n + 12ce_nf`(a)$$

Or

$$f`(x_n) + 5f`(x_n + f(x_n)) + f`(x_n + 2f(x_n)) + 6f`(x_n + 3f(x_n)) + f`(x_n + 4f(x_n)) + 5f`(x_n + 5f(x_n)) + f`(x_n + 6f(x_n)) = 20f`(a)[1 + 3f`(a) + 2ce_n + 12ce_nf`(a)] \quad (30)$$

Now using Eq. (22) and Eq. (30) in the developed method, we get

$$e_{n+1} = e_n - \frac{20e_nf`(a)(1 + ce_n)}{20f`(a)[1 + 3f`(a) + 2ce_n + 12ce_nf`(a)]}$$

$$e_{n+1} = e_n - \frac{e_n(1 + ce_n)}{[1 + 3f`(a) + 2ce_n + 12ce_nf`(a)]}$$

$$e_{n+1} = e_n - e_n(1 + ce_n)[1 + 3f`(a) + 2ce_n + 12ce_nf`(a)]^{-1}$$

$$e_{n+1} = e_n - e_n(1 + ce_n)[1 - 3f`(a) - 2ce_n - 12ce_nf`(a)]$$

$$e_{n+1} = e_n - e_n[1 - 3f`(a) - ce_n - 15ce_nf`(a)]$$

$$e_{n+1} = 3e_nf`(a) + ce^2{}_n + 15ce^2{}_nf`(a)$$

$$e_{n+1} = 3e_nf`(a) + ce^2{}_n(1 + 15f`(a))b \quad (31)$$

Finally, $f(x) = 0$ using in Eq. (22) then put in Eq. (31), we get

$$e_{n+1} = -3e^2{}_n f``(a) + ce^2{}_n(1 + 15f`(a))$$

Or

$$e_{n+1} = e^2{}_n\left[-3f``(a) + c\left(1 + 15f`(a)\right)\right] \quad (32)$$

Hence, Eq. (32) has been proven that the proposed Weddle`s iterated method is converged quadratically.

IV. Numerical Results

To justify the numerical results of proposed modified quadrature iterated methods by using C++ and EXCEL. In this section, Modified Quadrature Iterated Methods are applied on few nonlinear application problems i.e.

i. Mass of the jumper ($sin\boldsymbol{x} - x - \mathbf{1} = \mathbf{0}$)
ii. The diameter of the pipe ($\mathbf{2}\boldsymbol{x}^{\mathbf{2}} - \mathbf{5}\boldsymbol{x} - \mathbf{2} = \mathbf{0}$)
iii. The volume of the gas depends on the temperatures ($\boldsymbol{x}^{\mathbf{2}} - \boldsymbol{e}^{\boldsymbol{x}} = \mathbf{0}$)
iv. Anti-symmetric buckling of a beam ($\boldsymbol{e}^{-\boldsymbol{x}} - cosx = \mathbf{0}$)

The proposed Modified Quadrature Iterated Methods compared with Newton Raphson Method, Steffensen method, trapezoidal method and Quadrature Method, such as

Table 1. Results Table

S#	Functions	i	NR Method	S Method	T Method	Q Method	P B Method	P W Method
1	$\boldsymbol{sinx - x + 1 = 0}$	1	1.06086	2.02963	0.751682	1.19666	0.850991	0.137848
	$\boldsymbol{x_0} = 3$	2	1.26018	1.93335	1.18987	1.75841	1.46135	1.09217
		3	2.25674	1.93456	2.21057	1.919	1.78016	1.71803
		4	1.96109	1.93456	4.00889	1.93442	1.90755	1.87457
		5	1.9348	1.93456	2.89742	1.93456	1.93351	1.92726
		6	1.93456		1.96053	1.93456	1.93456	1.93444
		7	1.93456		1.93434	1.93456	1.93456	1.93456
		8	1.93456		1.93456			
		9			1.93456			
		10			1.93456			

Table 2. Error Table

S #	Functions	I	NR Method	S Method	T Method	Q Method	P B Method	P W Method
1	$\boldsymbol{sinx - x + 1 = 0}$ $x_0 = 3$	1	1.93914	5.02963	2.24832	4.19666	3.85099	3.13785
		2	2.32104	0.0962742	0.43819	0.561751	0.610357	0.954321
		3	0.996557	0.0012114	1.0207	0.160589	0.318808	0.625864
		4	0.29565	1.79787e-007	6.21945	0.0154234	0.127393	0.156539
		5	0.0262884	3.9968e-015	1.11146	0.000140799	0.025961	0.0526889
		6	0.000236493		0.936891	1.1695e-008	0.00105184	0.00717435
		7	1.92751e-008		0.0261916	2.22045e-016	1.68841e-006	0.000127988
		8	2.22045e-016		0.000225115			
		9			1.74734e-008			
		10			2.22045e-016			

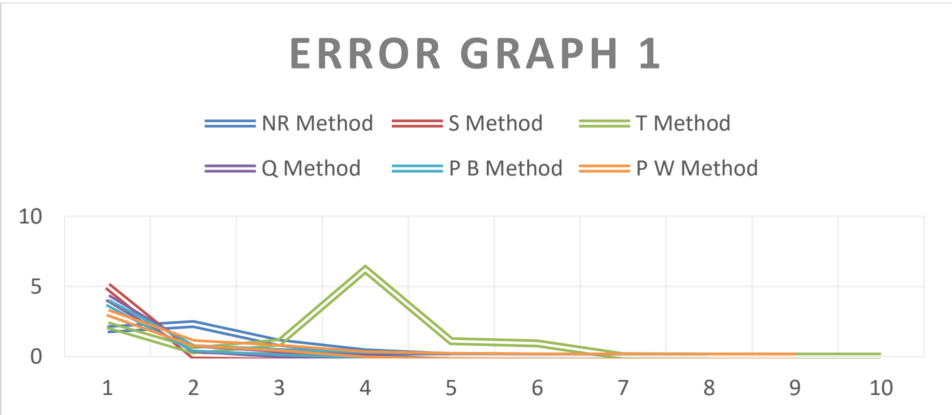


Table 3. Results Table

S#	Functions	I	NR Method	S Method	T Method	Q Method	P B Method	P W Method
2	$\boldsymbol{2x^2 - 5x - 2 = 0}$ $x_0 = 1$	1	4	0.545455	0.761905	0.166667	0.545455	0.6875
		2	1.61905	0.172598	0.356846	0.223958	0.172598	0.401857
		3	0.631101	0.110105	0.112234	0.340784	0.110105	0.146775
		4	0.371668	0.282008	0.330164	0.350713	0.282008	0.0687419
		5	0.350916	0.343655	0.350647	0.350781	0.343655	0.229952
		6	0.350781	0.350696	0.350781	0.350781	0.350696	0.321263
		7	0.350781	0.350781	0.350781		0.350781	0.348614
		8	0.350781	0.350781	0.350781		0.350781	0.350769
		9		0.350781				

Table 4. Error Table

S #	Functions	I	NR Method	S Method	T Method	Q Method	P B Method	P W Method
2	$2x^2 - 5x - 2 = 0$ $x_0 = 1$	1 2 3 4 5 6 7 8 9	5 2.38095 0.987947 0.259433 0.0207519 0.000134499 5.65037e-009 5.55112e-017	0.454545 0.372856 0.282704 0.171903 0.0616468 0.00704128 8.46319e-005 1.20896e-008 2.77556e-016	0.238095 0.405058 0.469081 0.217929 0.0204829 0.000134488 5.65037e-009 5.55112e-017	0.833333 0.390625 0.116825 0.00992928 6.81612e-005 3.19492e-009	0.454545 0.372856 0.282704 0.171903 0.0616468 0.00704128 8.46319e-005 1.20896e-008	0.3125 0.285643 0.255081 0.215517 0.16121 0.0913112 0.0273504 0.00215491

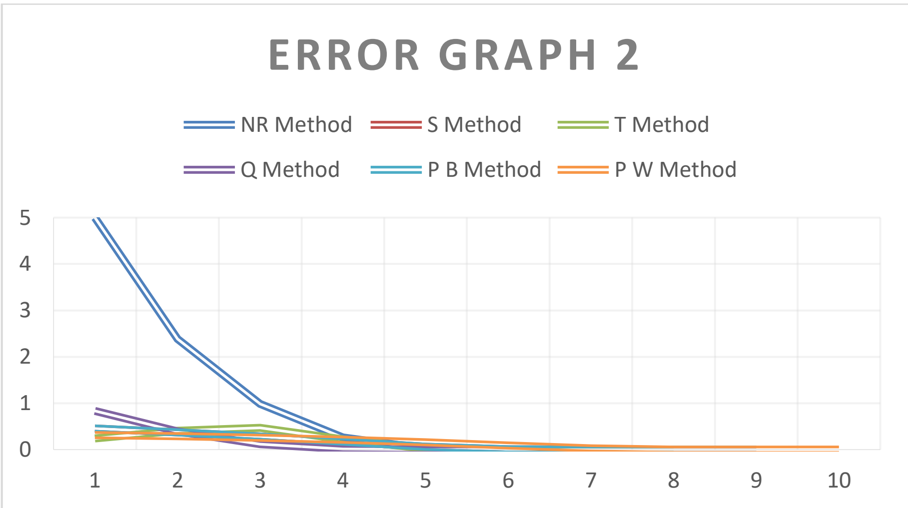


Table 5. Results Table

S#	Functions	I	NR Method	S Method	T Method	Q Method	P B Method	P B Method
3	$x^2 - e^x = 0$ $x_0 = 0$	1 2 3 4 5 6	1 0.733044 0.703808 0.703467 0.703467 0.703467	0.6127 0.700737 0.703465 0.703467 0.703467	0.5 0.686102 0.703347 0.703467 0.703467	0.681304 0.703412 0.703467 0.703467	0.445285 0.658405 0.702141 0.703466 0.703467	0.315816 0.548202 0.673735 0.702418 0.703466

Table 6. Error Table

S #	Functions	I	NR Method	S Method	T Method	Q Method	P B Method	P B Method
3	$x^2 - e^x = 0$ $x_0 = 0$	1 2 3 4 5 6	1 0.266956 0.0292358 0.000340318 4.58334e-008 7.77156e-016	0.6127 0.0880376 0.00272734 2.65374e-006 2.5131e-012	0.5 0.186102 0.017245 0.000120043 5.70312e-009	 0.681304 0.0221084 5.502e-005 3.31265e-010	0.445285 0.21312 0.0437362 0.00132507 1.08329e-006	0.315816 0.232386 0.125533 0.0286834 0.00104804

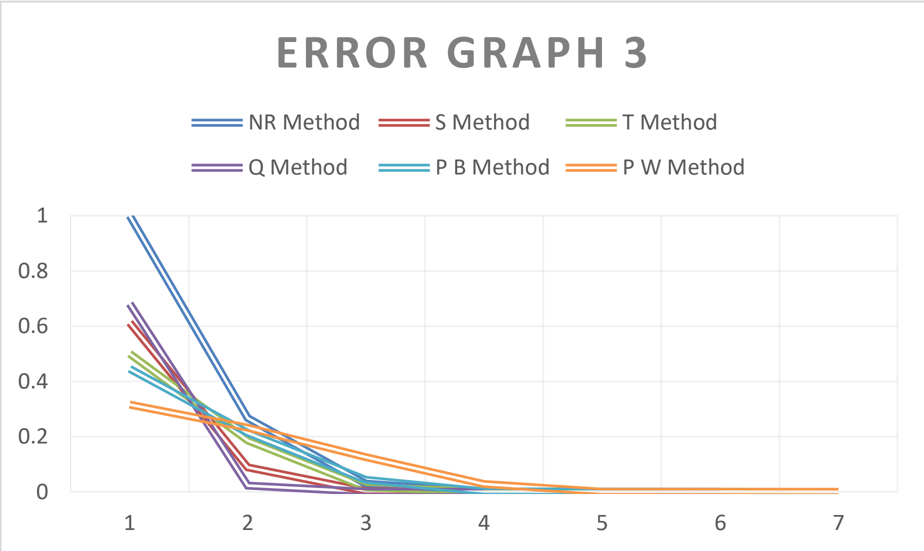
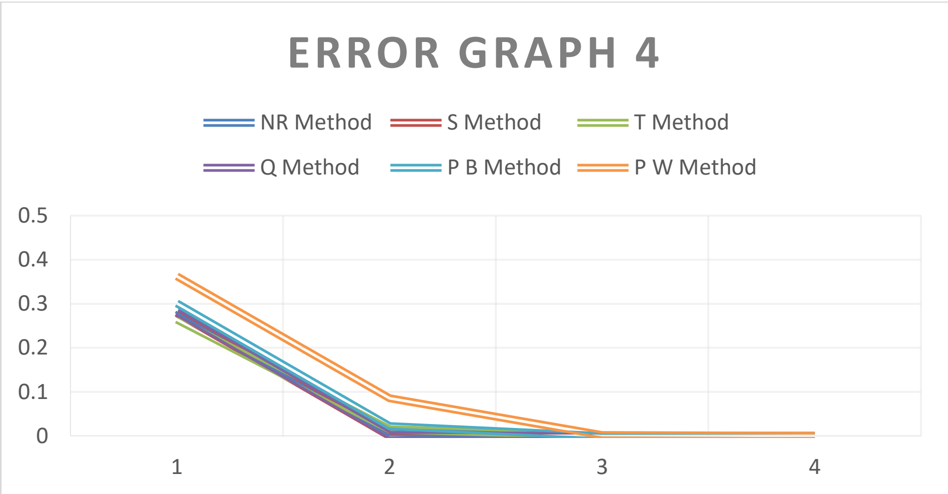


Table 7. Results Table

S#	Functions	I	NR Method	S Method	T Method	Q Method	P B Method	P W Method
4	$e^{-x} - cosx = 0$ $x_0 = 5$	1 2 3 4	4.71323 4.72129 4.72129 4.72129	4.72127 4.72129 4.72129	4.736 4.7213 4.72129 4.72129	4.71917 4.72129 4.72129	4.69908 4.7213 4.72129	4.63764 4.72317 4.72129

Table 8. Error Table

S #	Functions	I	NR Method	S Method	T Method	Q Method	P B Method	P W Method
4	$e^{-x} - cosx = 0$ $x_0 = 5$	1 2 3 4	0.286771 0.00806369 4.02154e-007 8.88178e-016	0.278735 2.75358e-005 5.9508e-014	0.264004 0.0146995 3.99067e-006 1.40332e-013	0.280827 0.00212015 3.90183e-008	0.300917 0.0222161 6.32483e-006	0.362356 0.0855304 0.00188125



V. Conclusion

This article has been proposed modified quadrature iterated methods for solving nonlinear application equations, which are derived from the Boole rule and Weddle rule, and they are converged quadratically. Modified quadrature iterated methods have compared with Newton Raphson Method, Steffensen method, trapezoidal method and quadrature method. From graphical representation and tables, it has been confirmed that the present modified quadrature iterated methods seem to be an imperious improvement and better execution compared with existing methods for solving nonlinear application equations.

VI. Acknowledgement

Thanks to all teachers, friends and family members who supported us in making this article publishable.

Conflict of Interest:

Authors decleared : No conflict of interest regarding this article**.**

References

I. Akram, A. and Q. U. Ann, Newton Raphson Method, *International Journal of Scientific & Engineering Research,* Vol. 6. 456-462, 2015.

II. Khatri, A., A. A. Shaikh and K. A. Abro, Closed Newton Cotes Quadrature Rules with Derivatives, ,, Vol.9, No.5, 2019.

III. Liu, Z. and H. Zhang, Steffensen-Type Method of super third-order convergence for solving nonlinear equations, *Journal of Applied Mathematics and Physics*, vol 2, 581-586, 2014.

IV. Memon Kashif, Muhammad Mujtaba Shaikh, Muhammad Saleem Chandio, Abdul Wasim Shaikh, : A NEW AND EFFICIENT SIMPSON'S 1/3-TYPE QUADRATURE RULE FOR RIEMANN-STIELTJES INTEGRAL, J. Mech. Cont. & Math. Sci., Vol.-15, No.-11, November (2020) pp 132-148

V. Perhiyar, M. A., S. F. Shah and A. A. Shaikh, Modified Trapezoidal Rule Based Different Averages for Numerical Integration, *Mathematical Theory and Modeling*, Vol.9, No.9, 2019.

VI. Qureshi, U. K., A New Accelerated Third-Order Two-Step Iterative Method for Solving Nonlinear Equations, *Mathematical Theory and Modeling*, Vol: 8(5), 2018.

VII. Qureshi, U. K., I. A. Bozdar, A. Pirzada and M. B. Arain, Quadrature Rule Based Iterative Method for the Solution of Non-Linear Equations, *Proceedings of the Pakistan Academy of Sciences*, *Pakistan Academy of Sciences A. Physical and Computational Sciences*, vol: 56 (1): 39–43, 2019.

VIII. Qureshi, U. K., Z. A. Kalhoro, A. A. Shaikh and A. R. Nagraj, Trapezoidal Second Order Iterated Method for Solving Nonlinear Problems, *University of Sindh Journal of Information and Communication Technology* (USJICT), Vol: 2(2), 2018.

IX. Shaikh, M. M., M. S. Chandio and A. S. Soomro, A Modified Four-point Closed Mid-point Derivative Based Quadrature Rule for Numerical Integration, *Sindh University Research Journal*, SURJ (Science Series), 48(2), 2016.

X. Umar Sehrish , Muhammad Mujtaba Shaikh , Abdul Wasim Shaikh, : A NEW QUADRATURE-BASED ITERATIVE METHOD FOR SCALAR NONLINEAR EQUATIONS, J. Mech. Cont.& Math. Sci., Vol.-15, No.-10, October (2020) pp 79-93.

XI. Weerakoon, S. and T. G. I. Fernando, 2000, A Variant of Newton' s Method with Accelerated Third-Order Convergence, *Applied Mathematics Letters,* vol 13 87-93.

XII. Zafar, F., Saira S. and C. O. E. Burg, New Derivative Based Open Newton-Cotes Quadrature Rules, *Hindawi Publishing Corporation Abstract and Applied Analysis* Volume 2014.

XIII. Zhao, W. and L. Hongxing, Midpoint Derivative-Based Closed Newton-Cotes Quadrature, *Hindawi Publishing Corporation Abstract and Applied Analysis* Volume 2013.