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AN EFFICIENT FOUR-POINT QUADRATURE SCHEME FOR RIEMANN-STIELTJES INTEGRAL

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Abstract

In this work, a new four-point quadrature scheme is proposed for efficient approximation of the Riemann-Stieltjes integral (RS-integral). The composite form of the proposed scheme is also derived for the RS-integral from the concept of precision. Theoretically, the theorems related to the basic form, composite form, local and global errors of the new scheme are proved on the RS-integral. The correctness of the new proposed scheme is checked by g(t) = t, which reduces the proposed scheme into the original form of Simpson's 3/8 rule for Riemann integral. The efficiency of the new proposed scheme is demonstrated by experimental work using programming in MATLAB against existing schemes. The order of accuracy and computational cost of the new proposed scheme is computed. The average CPU time is also measured in seconds. The obtained results demonstrate the efficiency of the proposed scheme over the existing schemes.

Keywords: Quadrature rule, Riemann-Stieltjes, Simpson's 3/8 rule, Composite form, Local error, Global error, Cost-effectiveness, Time-efficiency

I. Introduction

Finding the area under the regular and irregular curves when the functions or only data is available has remained an important topic for scientists and engineers. Such a problem can also be dealt with the methods of numerical integration in best as well as worst cases. The worst cases consist of a definite integral whose analytical evaluation is not possible because of mixed transcendental and nonlinear behavior of integrands, for example: – e^{x^2} and $sin\, x^2$. The methods of numerical integration of such worst cases in one dimension are referred to as quadrature schemes. But, in most of the past literature quadrature rules have been quite frequently used on Riemann integrals only in one dimension. A few recent studies have focused on cubature rules

for approximating Riemann integral in higher dimensions [V], [VI]. The derivative-based and quadrature-based schemes have also been used for the nonlinear equations [VII], [XVIII], [XV], [XVI]. The Riemann-Stieltjes (RS) integral, which is an extension of the Riemann integral. For a function $f(x)$ which is bounded in $[a, b]$, and another function $\alpha(x)$ which is monotonically increasing in $[a, b]$, the RS-integral is defined [IV] as:

$$RS(f(x): \alpha; a, b) = \int_a^b f(x) d\alpha(x) \tag{1}$$

where $f(x)$ is integrand and $\alpha(x)$ is integrator.

Numerical evaluation of RS-integral can be applied in several areas of mathematics. For instance, statistics and probability theory, complex analysis, functional analysis, operator theory and others. Quadrature approximations for the Riemann integral are enormously available in literature as in [II], [XIV], extended for integral equations in [XVII], and applied to switched reluctance motors [X]. On the other hand, only some works in the past focused numerical approximation for the RS-integrals. The very first Trapezoid approximation with Hadamard inequality can be found in [XI]. Utilizing the relative convexity concept in [XII], some important inequalities were developed for the RS-integral approximation using the midpoint and Simpson rules. Authors in [XIX] presented a new family of closed Newton-Cotes quadrature schemes with Midpoint derivative for the Riemann integral at first instance, then in [XXI], the midpoint derivative-based trapezoid scheme for the RS-integral was proposed. It was the first time that a derivative-based rule was suggested for the RS-integrals. Later in [XX], the authors presented the composite form of trapezoidal rule for the RS-integral, but the work lacked numerical experiments. Recently, Memon *et al.* [VIII], [IX] presented efficient derivative-based and derivative-free schemes for the RS-integral with experimental verification of the theoretical results.

In this work, a new four-point scheme of Simpson's 3/8-type is proposed for the approximation of the RS-integral. The derivation in basic and composite forms and corresponding error analysis are discussed. The theoretical results have been verified using numerical experiments from literature to demonstrate cost efficiency, time efficiency and rapid convergence of the new scheme.

II. Some Existing Schemes for the RS-Integral

The RS-integral, as defined in (1) can be evaluated numerically by some existing schemes: T [XI], ZT [XXI], MZT [VIII], which are defined in basic form in (2)-(4):

$$T = \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right) f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right) f(b) \tag{2}$$

$$ZT = \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right) f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right) f(b)$$
$$+ \left(\int_a^b \int_a^t g(x)dxdt - \frac{b-a}{2}\int_a^b g(t)dt\right) f''(c_{ZT}) \tag{3}$$

$$where, c_{ZT} = \frac{(-2b^2+a^2-ab)\int_a^b g(t)dt+6b\int_a^b\int_a^t g(x)dxdt-6\int_a^b\int_a^t\int_a^y g(x)dxdydt}{6\int_a^b\int_a^t g(x)dxdt-3(b-a)\int_a^b g(t)dt}.$$

$$MZT = \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right)f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right)f(b)$$
$$+\left(\int_a^b\int_a^t g(x)dxdt - \frac{b-a}{2}\int_a^b g(t)dt\right)f''(c_{MZT}) \quad (4)$$

$$where, c_{MZT} = \frac{(-2b^2+a^2+ab)\int_a^b g(t)dt+6b\int_a^b\int_a^t g(x)dxdt-6\int_a^b\int_a^t\int_a^y g(x)dxdydt}{6\int_a^b\int_a^t g(x)dxdt-3(b-a)\int_a^b g(t)dt}.$$

The composite forms of the CT, ZCT and MZCT schemes are defined in (5)-(7) as:

$$CT = \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) \quad (5)$$

$$ZCT = \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) + \sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{h}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right]f''(c_{ZT,k})$$

$$\int_a^b f(t)dg \approx MZCT = \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right]f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right]f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right]f(b) + \sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{h}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right]f''(c_k) \quad (6)$$

Where,

$$\boldsymbol{c_{ZT,k}} = \frac{(-2x_k^2+x_{k-1}^2-x_{k-1}x_k)\int_{x_{k-1}}^{x_k} g(t)dt+6b\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt}{6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-\frac{3(x_k-x_{k-1})}{n}\int_{x_{k-1}}^{x_k} g(t)dt}$$

$$\boldsymbol{c_{MZT,k}} = \frac{(-2x_k^2+x_{k-1}^2+x_{k-1}x_k)\int_{x_{k-1}}^{x_k} g(t)dt+6b\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt}{6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-\frac{3(x_k-x_{k-1})}{n}\int_{x_{k-1}}^{x_k} g(t)dt}.$$

III. Proposed Four-point Scheme for the Riemann-Stieltjes Integral

We base the proposed four-point scheme for the RS-integral approximation on the well-known closed Newton-Cotes four-point quadrature rule – the Simpson's 3/8 (S38) rule – which is defined in basic form in (7).

$$S38 = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] \quad (7)$$

The Simpson's 3/8 rule for the Riemann integral as in (7) provides an exact answer for all polynomials whose degree is three or less, so its precision is 3 and global order of accuracy is 4.

On the basis of (7), the proposed four-point scheme, i.e. S38 for the RS-integral (RSI), in basic form is derived in Theorem 1.

Theorem 1. Let f(t) and g(t) be continuous on [a, b] and g(t) be increasing there. Simpson's 3/8 scheme for the RS-integral can be described as

$$\int_a^b f(x)dg \approx S38 = \left(\frac{1}{b-a}\int_a^b g(t)dt - \frac{9}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt + \frac{27}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt - g(a)\right)f(a) + \left(\frac{36}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{9}{2(b-a)}\int_a^b g(t)dt - \frac{81}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt\right)f\left(\frac{2a+b}{3}\right) + \left(\frac{9}{(b-a)}\int_a^b g(t)dt - \frac{45}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt + \frac{81}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt\right)f\left(\frac{a+2b}{3}\right) + \left(g(b) - \frac{11}{2(b-a)}\int_a^b g(t)dt + \frac{18}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{27}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt\right)f(b) \quad (8)$$

Proof of Theorem 1.

To derive the proposed four-point, i.e. Simpson's 3/8 type scheme for the RSI, we search numbers: a_0, b_0, c_0, d_0 such that:

$$\int_a^b f(t)dg \approx a_0 f(a) + b_0 f\left(\frac{2a+b}{3}\right) + c_0 f\left(\frac{a+2b}{3}\right) + d_0 f(b) \quad (9)$$

is exact for $f(t) = 1, t, t^2, t^3$. That is,

$$\int_a^b 1dg = a_0 + b_0 + c_0 + d_0$$

$$\int_a^b tdg = a_0 a + b_0\left(\frac{2a+b}{3}\right) + c_0\left(\frac{a+2b}{3}\right) + d_0 b$$

$$\int_a^b t^2 dg = a_0 a^2 + b_0\left(\frac{2a+b}{3}\right)^2 + c_0\left(\frac{a+2b}{3}\right)^2 + d_0 b^2$$

$$\int_a^b t^3 dg = a_0 a^3 + b_0\left(\frac{2a+b}{2}\right)^3 + c_0\left(\frac{a+2b}{2}\right)^3 + d_0 b^3$$

By using integration by parts of the RS integral, as in [XXI], we have the following system of equations (*11*)-(*14*).

$$a_0 + b_0 + c_0 + d_0 = g(b) - g(a) \quad (10)$$

$$a_0 a + b_0\left(\frac{2a+b}{3}\right) + c_0\left(\frac{a+2b}{3}\right) + d_0 b = bg(b) - ag(a) - \int_a^b g(t)dt \quad (11)$$

$$a_0a^2 + b_0\left(\frac{2a+b}{3}\right)^2 + c_0\left(\frac{a+2b}{3}\right)^2 + d_0b^2 = b^2g(b) - a^2g(a) - 2b\int_a^b g(t)dt + 2\int_a^b\int_a^t g(x)dx\,dt \quad (12)$$

$$a_0a^3 + b_0\left(\frac{2a+b}{3}\right)^3 + c_0\left(\frac{a+2b}{3}\right)^3 + d_0b^3 = b^3g(b) - a^3g(a) - 3b^2\int_a^b g(t)dt + 6b\int_a^b\int_a^t g(x)dxdt - 6\int_a^b\int_a^t\int_a^y g(x)dxdt \quad (13)$$

The system of linear equations (*10*)-(*13*) can be written in the coefficient matrix as:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & \frac{2a+b}{3} & \frac{a+2b}{3} & b \\ a^2 & \left(\frac{2a+b}{3}\right)^2 & \left(\frac{a+2b}{3}\right)^2 & b^2 \\ a^3 & \left(\frac{2a+b}{3}\right)^3 & \left(\frac{a+2b}{3}\right)^3 & b^3 \end{bmatrix}$$

The reduced row echelon form of *M* is:

$$M \overset{R}{\approx} M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As in M_R, all four rows are nonzero rows, so *M* has four linearly independent rows, and rank(*M*) = 4. To find the coefficients a_0, b_0, c_0 and d_0, we solve equations (*10*)-(13) simultaneously, to have:

$$a_0 = \frac{1}{b-a}\int_a^b g(t)dt - \frac{9}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt + \frac{27}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt - g(a)$$

$$b_0 = \frac{36}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{81}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt - \frac{9}{2(b-a)}\int_a^b g(t)dt$$

$$c_0 = \frac{9}{b-a}\int_a^b g(t)dt - \frac{45}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt + \frac{81}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt$$

$$d_0 = g(b) - \frac{11}{2(b-a)}\int_a^b g(t)dt$$

$$+\frac{18}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{27}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt$$

Putting the values of coefficients a_0, b_0, c_0 and d_0 in (9), we get the required scheme (8) as stated in the statement of Theorem 1 with the precision equal to 3.

In Theorem 2, we derive the local error term in the proposed four-point S38 schemes for the RS-integral as stated in (8). The successful reduction of the proposed scheme

for RS integral (8) to the corresponding Riemann integral scheme (7) is then established in Theorem 3.

Theorem 2. Let f(t) and g(t) be continuous on [a, b] and g(t) be increasing there. The proposed four-point scheme S38 for the RS-integral with local error term is:

$$\int_a^b f(x)\,dg = S38 = \left(\frac{1}{b-a}\int_a^b g(t)dt - \frac{9}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt + \frac{27}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt - g(a)\right) f(a) \tag{14}$$
$$+\left(\frac{36}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{9}{2(b-a)}\int_a^b g(t)dt - \frac{81}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt\right) f\left(\frac{2a+b}{3}\right)$$
$$+\left(\frac{9}{(b-a)}\int_a^b g(t)dt - \frac{45}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt + \frac{81}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt\right) f\left(\frac{a+2b}{3}\right)$$
$$+\left(g(b) - \frac{11}{2(b-a)}\int_a^b g(t)dt + \frac{18}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{27}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt\right) f(b)$$
$$+\left(\frac{(a-b)^3}{108}\int_a^b g(t)dt + \frac{11(a-b)^2}{108}\int_a^b\int_a^t g(x)dxdt + \frac{a-b}{2}\int_a^b\int_a^t\int_a^y g(x)dxdydt\right.$$
$$\left.+\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt\right) f^{(4)}(\mu)\,g'(\eta),$$

$where \mu, \eta \in (a, b).$

Proof of Theorem 2.

The precision of the proposed scheme (8) is 3, so taking $f(t) = \frac{t^4}{4!}$.

For the basic form scheme of the proposed S38 rule for RSI, the local error term thus takes the form:

$$R_{S38}[f] = \frac{1}{4!}\int_a^b t^4 dg - S38(t^4, g; a, b) \tag{15}$$

From [XXI], we know that

$$\frac{1}{4!}\int_a^b t^4 dg = \frac{1}{24}\left(b^4 g(b) - a^4 g(a)\right) - \frac{b^3}{6}\int_a^b g(t)dt + \frac{b^2}{2}\int_a^b\int_a^t g(x)dxdt \tag{16}$$
$$-b\int_a^b\int_a^t\int_a^y g(x)dxdydt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt$$

By Theorem 1 and scheme (8), we have:

$$S38\left(t^4;g;a,b\right)=\left(\frac{1}{b-a}\int_a^b g(t)dt-\frac{9}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt+\frac{27}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt-g(a)\right)\frac{a^4}{4!} \quad (17)$$
$$+\left(\frac{36}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt-\frac{9}{2(b-a)}\int_a^b g(t)dt-\frac{81}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt\right)\frac{(2a+b)^4}{3^4.4!}$$
$$+\left(\frac{9}{(b-a)}\int_a^b g(t)dt-\frac{45}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt+\frac{81}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt\right)\frac{(a+2b)^4}{3^4.4!}$$
$$+\left(g(b)-\frac{11}{2(b-a)}\int_a^b g(t)dt+\frac{18}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt-\frac{27}{(b-a)^3}\int_a^b\int_a^t\int_a^y g(x)dxdydt\right)\frac{b^4}{4!}$$

Using (16) and (17) in (15), we have:

$$R_{S38}[f]=\left(\frac{(a-b)^3}{108}\int_a^b g(t)dt+\frac{11(a-b)^2}{108}\int_a^b\int_a^t g(x)dxdt+\frac{a-b}{2}\int_a^b\int_a^t\int_a^y g(x)dxdydt\right. \quad (18)$$
$$\left.+\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt\right)f^{(4)}(\mu)g'(\eta),$$

Which is the required local error term of proposed Simpson's 3/8 scheme for RS-integral.

Theorem 3. With g(t) = t, the proposed S38 scheme with the error term (14) for the RS-integral reduces to the corresponding S38 scheme i.e. (7) for the classical Riemann integral.

Proof of Theorem 3.

By Theorem 2, we get

$$\int_a^b f(t)dg=\int_a^b f(t)dt=\left(\frac{1}{b-a}\int_a^b tdt-\frac{9}{(b-a)^2}\int_a^b\int_a^t xdxdt+\right.$$
$$\frac{27}{(b-a)^3}\int_a^b\int_a^t\int_a^y xdxdydt-g(a)\Big)f(a)+\left(\frac{36}{(b-a)^2}\int_a^b\int_a^t xdxdt-\frac{9}{2(b-a)}\int_a^b tdt-\right.$$
$$\left.\frac{81}{(b-a)^3}\int_a^b\int_a^t\int_a^y xdxdydt\right)f\left(\frac{2a+b}{3}\right)+\left(\frac{9}{(b-a)}\int_a^b tdt-\frac{45}{(b-a)^2}\int_a^b\int_a^t xdxdt+\right.$$
$$\left.\frac{81}{(b-a)^3}\int_a^b\int_a^t\int_a^y xdxdydt\right)f\left(\frac{a+2b}{3}\right)+\left(g(b)-\frac{11}{2(b-a)}\int_a^b tdt+\frac{18}{(b-a)^2}\int_a^b\int_a^t xdxdt-\right.$$
$$\left.\frac{27}{(b-a)^3}\int_a^b\int_a^t\int_a^y xdxdydt\right)f(b)+\left(\frac{(a-b)^3}{108}\int_a^b tdt+\frac{11(a-b)^2}{108}\int_a^b\int_a^t xdx\,dt+\right.$$
$$\left.\frac{a-b}{2}\int_a^b\int_a^t\int_a^y xdx\,dy\,dt+\int_a^b\int_a^t\int_a^z\int_a^y xdx\,dy\,dz\,dt\right)f^{(4)}(\mu)g'(\eta) \quad (19)$$

It is direct to observe that:

$$\int_a^b tdt=\frac{b^2-a^2}{2},$$
$$\int_a^b\int_a^t xdxdt=\frac{b^3}{6}-\frac{a^2b}{2}+\frac{a^3}{3},$$
$$\int_a^b\int_a^t\int_a^y xdxdy\,dt=\frac{b^4}{24}-\frac{a^2b^2}{4}+\frac{a^3b}{3}-\frac{a^4}{8},$$
$$\int_a^b\int_a^t\int_a^z\int_a^y xdxdy\,dz\,dt=\frac{b^5}{120}-\frac{a^2b^3}{12}+\frac{a^3b^2}{6}-\frac{a^4b}{8}+\frac{a^5}{30},$$

and, using these in (19), we finally have:

$$\int_a^b f(x)dx = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480}f^{(4)}(\xi), \quad (20)$$

where $\xi \in (a, b)$, which shows the reducibility of the proposed S38 scheme to the classical Riemann integral form (7).

Now, the proposed composite Simpson's 3/8 scheme for the RSI integral is derived by dividing the interval into small subintervals and applying integration rule to each subinterval, and the results are showcased in Theorem 4.

Theorem 4. Let f(t) and g(t) be continuous on [a, b] and g(t) be increasing there. Let the interval [a, b] be subdivided into 3n subintervals $[x_k, x_{k+1}]$ with width $h = \frac{b-a}{n}$ by using the equally spaced nodes $x_k = a+kh$, where k = 0, 1, …, n. The composite Simpson's 3/8 scheme to 3n subintervals for the RSI can be described as

$$\int_a^b f(t)dg \approx CS38 = \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - \frac{9n^2}{(b-a)^2}\int_a^{x_1}\int_a^t g(x)dxdt + \frac{27n^3}{(b-a)^3}\int_a^{x_1}\int_a^t\int_a^y g(x)dxdyd\,t - g(a)\right]f(a) + \frac{9n}{b-a}\sum_{k=1}^{n}\left[\frac{4n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt - \frac{1}{2}\int_{x_{k-1}}^{x_k} g(t)dt - \frac{9n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdydt\right]f\left(\frac{2x_{k-1}+x_k}{3}\right) + \frac{9n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt - \frac{5n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt + \frac{9n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdydt\right]f\left(\frac{x_{k-1}+2x_k}{3}\right) + \sum_{k=1}^{n-1}\left[\begin{array}{l}\frac{n}{b-a}\left\{\int_{x_k}^{x_{k+1}} g(t)dt - \frac{11}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right\} \\ \quad + \frac{9n^2}{(b-a)^2}\left\{2\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt - \int_{x_k}^{x_{k+1}}\int_{x_k}^t g(x)dxdt\right\} \\ + \frac{27n^3}{(b-a)^3}\left\{\int_{x_k}^{x_{k+1}}\int_{x_k}^t\int_{x_k}^y g(x)dxdydt - \int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdydt\right\}\end{array}\right]f(x_k) + \left[g(b) - \frac{11n}{b-a}\int_{x_{n-1}}^b g(t)dt + \frac{18n^2}{(b-a)^2}\int_{x_{n-1}}^b\int_{x_{n-1}}^t g(x)dxdt - \frac{27n^3}{(b-a)^3}\int_{x_{n-1}}^b\int_{x_{n-1}}^t\int_{x_{n-1}}^y g(x)dxdydt\right]f(b)$$

Proof of Theorem 4.

The proposed basic form S38 scheme for the RSI is given in (8). Applying proposed S38 rule over each subinterval, we have:

$$\int_a^b f(t)dg \approx \left[\frac{1}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt - \frac{9}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt + \frac{27}{\left(\frac{b-a}{n}\right)^3}\int_a^{x_1}\int_a^t\int_a^y g(x)dxdydt - g(a)\right]f(a) + \left[\frac{36}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt -\right.$$

$$\frac{81}{\left(\frac{b-a}{n}\right)^3}\int_a^{x_1}\int_a^t\int_a^y g(x)dxdydt-\frac{9}{2\left(\frac{b-a}{n}\right)}\int_a^{x_1} g(t)dt\Bigg]f\left(\frac{2a+x_1}{3}\right)+\Bigg[\frac{9}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt-$$

$$\frac{45}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt+\frac{81}{\left(\frac{b-a}{n}\right)^3}\int_a^{x_1}\int_a^t\int_a^y g(x)dxdydt\Bigg]f\left(\frac{a+2x_1}{3}\right)+\Bigg[g(x_1)-$$

$$\frac{11}{2\left(\frac{b-a}{n}\right)}\int_a^{x_1} g(t)dt+\frac{18}{\left(\frac{b-a}{n}\right)^2}\int_a^{x_1}\int_a^t g(x)dxdt-\frac{27}{\left(\frac{b-a}{n}\right)^3}\int_a^{x_1}\int_a^t\int_a^y g(x)dxdydt\Bigg]f(x_1)$$

$$+\Bigg[\frac{1}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt-\frac{9}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt+\frac{27}{\left(\frac{b-a}{n}\right)^3}\int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^y g(x)dxdydt\Bigg]f(x_1)$$

$$+\Bigg[\frac{36}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt-\frac{81}{\left(\frac{b-a}{n}\right)^3}\int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^y g(x)dxdydt-$$

$$\frac{9}{2\left(\frac{b-a}{n}\right)}\int_{x_1}^{x_2} g(t)dt\Bigg]f\left(\frac{2x_1+x_2}{3}\right)+\Bigg[\frac{9}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt-\frac{45}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt+$$

$$\frac{81}{\left(\frac{b-a}{n}\right)^3}\int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^y g(x)dxdydt\Bigg]f\left(\frac{x_1+2x_2}{3}\right)+\Bigg[g(x_2)-\frac{11}{2\left(\frac{b-a}{n}\right)}\int_{x_1}^{x_2} g(t)dt+$$

$$\frac{18}{\left(\frac{b-a}{n}\right)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt-\frac{27}{\left(\frac{b-a}{n}\right)^3}\int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^y g(x)dxdydt\Bigg]f(x_2)+\ldots$$

$$+\Bigg[\frac{1}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k} g(t)dt-\frac{9}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt+$$

$$\frac{27}{\left(\frac{b-a}{n}\right)^3}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdydt-g(x_{k-1})\Bigg]f(x_{k-1})$$

$$+\Bigg[\frac{36}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt-\frac{81}{\left(\frac{b-a}{n}\right)^3}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdydt-$$

$$\frac{9}{2\left(\frac{b-a}{n}\right)}\int_{x_{k-1}}^{x_k} g(t)dt\Bigg]f\left(\frac{2x_{k-1}+x_k}{3}\right)+\Bigg[\frac{9}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k} g(t)dt-$$

$$\frac{45}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt+\frac{81}{\left(\frac{b-a}{n}\right)^3}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdydt\Bigg]f\left(\frac{x_{k-1}+2x_k}{3}\right)$$

$$+\Bigg[g(x_k)-\frac{11}{2\left(\frac{b-a}{n}\right)}\int_{x_{k-1}}^{x_k} g(t)dt+\frac{81}{\left(\frac{b-a}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt-$$

$$\frac{27}{\left(\frac{b-a}{n}\right)^3}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdydt\Bigg]f(x_k)+\ldots$$

$$+\Bigg[\frac{1}{\frac{b-a}{n}}\int_{x_{n-1}}^{b} g(t)dt-\frac{9}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^t g(x)dxdt+$$

$$\frac{27}{\left(\frac{b-a}{n}\right)^3}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdydt\Bigg] f(x_{n-1}) +$$
$$\Bigg[\frac{36}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt - \frac{81}{\left(\frac{b-a}{n}\right)^3}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdydt -$$
$$\frac{9}{2\left(\frac{b-a}{n}\right)}\int_{x_{n-1}}^{b} g(t)dt\Bigg] f\left(\frac{2x_{n-1}+b}{3}\right) + \Bigg[\frac{9}{\frac{b-a}{n}}\int_{x_{n-1}}^{b} g(t)dt -$$
$$\frac{45}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt +$$
$$\frac{81}{\left(\frac{b-a}{n}\right)^3}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdydt\Bigg] f\left(\frac{x_{n-1}+2b}{3}\right) + \Bigg[g(b) -$$
$$\frac{11}{2\left(\frac{b-a}{n}\right)}\int_{x_{n-1}}^{b} g(t)dt + \frac{81}{\left(\frac{b-a}{n}\right)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt -$$
$$\frac{27}{\left(\frac{b-a}{n}\right)^3}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdydt\Bigg] f(b)$$

$$= \Big[\frac{n}{b-a}\int_a^{x_1} g(t)dt - \frac{9n^2}{(b-a)^2}\int_a^{x_1}\int_a^t g(x)dxdt +$$
$$\frac{27n^3}{(b-a)^3}\int_a^{x_1}\int_a^t\int_a^y g(x)dxdydt - g(a)\Big] f(a) + \Big[\frac{36n^2}{(b-a)^2}\int_a^{x_1}\int_a^t g(x)dxdt -$$
$$\frac{81n^3}{(b-a)^3}\int_a^{x_1}\int_a^t\int_a^y g(x)dxdydt - \frac{9n}{2(b-a)}\int_a^{x_1} g(t)dt\Big] f\left(\frac{2a+x_1}{3}\right)$$
$$+ \Big[\frac{36n^2}{(b-a)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt - \frac{81n^3}{(b-a)^3}\int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^y g(x)dxdydt -$$
$$\frac{9n}{2(b-a)}\int_{x_1}^{x_2} g(t)dt\Big] f\left(\frac{2x_1+x_2}{3}\right) + \ldots + \Big[\frac{36n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt -$$
$$\frac{81n^3}{(b-a)^3}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdydt - \frac{9n}{2(b-a)}\int_{x_{k-1}}^{x_k} g(t)dt\Big] f\left(\frac{2x_{k-1}+x_k}{3}\right)$$
$$+ \ldots + \Big[\frac{36n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^t g(x)dxdt - \frac{81n^3}{(b-a)^3}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^t\int_{x_{n-1}}^y g(x)dxdydt -$$
$$\frac{9n}{2(b-a)}\int_{x_{n-1}}^{b} g(t)dt\Big] f\left(\frac{2x_{n-1}+b}{3}\right) + \Big[\frac{9n}{b-a}\int_a^{x_1} g(t)dt - \frac{45n^2}{(b-a)^2}\int_a^{x_1}\int_a^t g(x)dxdt +$$
$$\frac{81n^3}{(b-a)^3}\int_a^{x_1}\int_a^t\int_a^y g(x)dxdydt\Big] f\left(\frac{a+2x_1}{3}\right) + \Big[\frac{9n}{b-a}\int_{x_1}^{x_2} g(t)dt -$$
$$\frac{45n^2}{(b-a)^2}\int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt + \frac{81n^3}{(b-a)^3}\int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^y g(x)dxdydt\Big] f\left(\frac{x_1+2x_2}{3}\right) + \ldots$$
$$+ \Big[\frac{9n}{b-a}\int_{x_{k-1}}^{x_k} g(t)dt - \frac{45n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t g(x)dxdt +$$
$$\frac{81n^3}{(b-a)^3}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^t\int_{x_{k-1}}^y g(x)dxdydt\Big] f\left(\frac{x_{k-1}+2x_k}{3}\right) + \ldots + \Big[\frac{9n}{b-a}\int_{x_{n-1}}^{b} g(t)dt -$$
$$\frac{45n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^t g(x)dxdt + \frac{81n^3}{(b-a)^3}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^t\int_{x_{n-1}}^y g(x)dxdydt\Big] f\left(\frac{x_{n-1}+2b}{3}\right)$$
$$+ \left[\begin{array}{l}\frac{n}{b-a}\left(\int_{x_1}^{x_2} g(t)dt - \frac{11}{2}\int_a^{x_1} g(t)dt\right) + \frac{9n^2}{(b-a)^2}\left(2\int_a^{x_1}\int_a^t g(x)dxdt - \int_{x_1}^{x_2}\int_{x_1}^t g(x)dxdt\right) \\ + \frac{27n^3}{(b-a)^3}\left(\int_{x_1}^{x_2}\int_{x_1}^t\int_{x_1}^y g(x)dxdydt - \int_a^{x_1}\int_a^t\int_a^y g(x)dxdydt\right)\end{array}\right] f(x_1)$$

$$+\left[\begin{matrix}\frac{n}{b-a}\left(\int_{x_2}^{x_3} g(t)dt-\frac{11}{2}\int_{x_1}^{x_2} g(t)dt\right)+\frac{9n^2}{(b-a)^2}\left(2\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt-\int_{x_2}^{x_3}\int_{x_2}^{t} g(x)dxdt\right)\\ +\frac{27n^3}{(b-a)^3}\left(\int_{x_2}^{x_3}\int_{x_2}^{t}\int_{x_2}^{y} g(x)dxdydt-\int_{x_1}^{x_2}\int_{x_1}^{t}\int_{x_1}^{y} g(x)dxdydt\right)\end{matrix}\right] f(x_2)$$

$$+\ldots+\left[\begin{matrix}\frac{n}{b-a}\left(\int_{x_{n-1}}^{x_n} g(t)dt-\frac{11}{2}\int_{x_{n-2}}^{x_{n-1}} g(t)dt\right)\\ +\frac{9n^2}{(b-a)^2}\left(2\int_{x_{n-2}}^{x_{n-1}}\int_{x_{n-2}}^{t} g(x)dxdt-\int_{x_{n-1}}^{x_n}\int_{x_{n-1}}^{t} g(x)dxdt\right)\\ +\frac{27n^3}{(b-a)^3}\left(\int_{x_{n-1}}^{x_n}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdydt-\int_{x_{n-2}}^{x_{n-1}}\int_{x_{n-2}}^{t}\int_{x_{n-2}}^{y} g(x)dxdydt\right)\end{matrix}\right] f(x_{n-1})$$

$$+\left[g(b)-\frac{11n}{2(b-a)}\int_{x_{n-1}}^{b} g(t)dt+\frac{18n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt-\right.$$
$$\left.\frac{27n^3}{(b-a)^3}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdydt\right] f(b)$$

$$=\left[\frac{n}{b-a}\int_{a}^{x_1} g(t)dt-\frac{9n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t} g(x)dxdt+\frac{27n^3}{(b-a)^3}\int_{a}^{x_1}\int_{a}^{t}\int_{a}^{y} g(x)dxdydt-\right.$$
$$\left.g(a)\right] f(a)+\frac{9n}{b-a}\sum_{k=1}^{n}\left[\frac{4n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-\right.$$
$$\left.\frac{9n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt-\frac{1}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right] f\left(\frac{2x_{k-1}+x_k}{3}\right)$$
$$+\frac{9n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt-\frac{5n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt+\right.$$
$$\left.\frac{9n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt\right] f\left(\frac{x_{k-1}+2x_k}{3}\right)$$

$$+\sum_{k=1}^{n-1}\left[\begin{matrix}\frac{n}{b-a}\left(\int_{x_k}^{x_{k+1}} g(t)dt-\frac{11}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right)\\ +\frac{9n^2}{(b-a)^2}\left(2\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-\int_{x_k}^{x_{k+1}}\int_{x_k}^{t} g(x)dxdt\right)\\ +\frac{27n^3}{(b-a)^3}\left(\int_{x_k}^{x_{k+1}}\int_{x_k}^{t}\int_{x_k}^{y} g(x)dxdydt-\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt\right)\end{matrix}\right] f(x_k)$$

$$+\left[g(b)-\frac{11n}{2(b-a)}\int_{x_{n-1}}^{b} g(t)dt+\frac{18n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt-\right.$$
$$\left.\frac{27n^3}{(b-a)^3}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdydt\right] f(b) \qquad (22)$$

The proposed composite scheme for the RS-integral integral is proved.

The global error term of proposed Simpson's 3/8 scheme for RS-integral is determined in Theorem 5.

Theorem 5. Let f(t) and g(t) be continuous on [a, b] and g(t) be increasing there. Let the interval [a, b] be subdivided into 3n subintervals $[x_k, x_{k+1}]$ with width $h = \frac{b-a}{n}$ by using the equally spaced nodes $x_k = a+kh$, where k = 0, 1, …, n. The composite Simpson's 3/8 scheme with global error term $R_{CS38}[f]$ for the RS-integral can be described as

$$\int_a^b f(t)dg = CS38 + R_{CS38}[f] = \left[\frac{n}{b-a}\int_{a}^{x_1} g(t)dt-\frac{9n^2}{(b-a)^2}\int_{a}^{x_1}\int_{a}^{t} g(x)dxdt+\right.$$
$$\left.\frac{27n^3}{(b-a)^3}\int_{a}^{x_1}\int_{a}^{t}\int_{a}^{y} g(x)dxdyd\,t-g(a)\right] f(a)+\frac{9n}{b-a}\sum_{k=1}^{n}\left[\frac{4n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-\right.$$
$$\left.\frac{1}{2}\int_{x_{k-1}}^{x_k} g(t)dt-\frac{9n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt\right] f\left(\frac{2x_{k-1}+x_k}{3}\right)$$
$$+\frac{9n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt-\frac{5n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt+\right.$$

$$\frac{9n^2}{(b-a)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt\Big] f\left(\frac{x_{k-1}+2x_k}{3}\right)$$
$$+\sum_{k=1}^{n-1}\begin{bmatrix}\frac{n}{b-a}\left\{\int_{x_k}^{x_{k+1}} g(t)dt-\frac{11}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right\}\\ +\frac{9n^2}{(b-a)^2}\left\{2\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-\int_{x_k}^{x_{k+1}}\int_{x_k}^{t} g(x)dxdt\right\}\\ +\frac{27n^3}{(b-a)^3}\left\{\int_{x_k}^{x_{k+1}}\int_{x_k}^{t}\int_{x_k}^{y} g(x)dxdydt-\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt\right\}\end{bmatrix} f(x_k)$$
$$+\Big[g(b)-\frac{11n}{b-a}\int_{x_{n-1}}^{b} g(t)dt+\frac{18n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)dxdt-$$
$$\frac{27n^3}{(b-a)^3}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y} g(x)dxdydt\Big] f(b)$$
$$+\begin{bmatrix}\frac{(a-b)^3}{108}\int_a^b g(t)dt+\frac{11(a-b)^2}{108}\int_a^b\int_a^t g(x)dx\,dt+\frac{a-b}{2}\int_a^b\int_a^t\int_a^y g(x)dx\,dy\,dt\\ +\int_a^b\int_a^t\int_a^z\int_a^y g(x)dx\,dy\,dz\,dt\end{bmatrix} f^{(4)}(\mu)g'(\eta),$$

$where \mu,\eta\in(a,b).$

Proof of Theorem 5.

We know from (18) that the local error in a sub-interval $[x_{p-1}, x_p]$, can be written as

$$\left(\frac{(x_{p-1}-x_p)^3}{108}\int_{x_{p-1}}^{x_p} g(t)dt+\frac{11(x_{p-1}-x_p)^2}{108}\int_{x_{p-1}}^{x_p}\int_{x_{p-1}}^{t} g(x)dx\,dt+\right.$$
$$\left.\frac{x_{p-1}-x_p}{2}\int_{x_{p-1}}^{x_p}\int_{x_{p-1}}^{t}\int_{x_{p-1}}^{y} g(x)dx\,dy\,dt+\int_{x_{p-1}}^{x_p}\int_{x_{p-1}}^{t}\int_{x_{p-1}}^{z}\int_{x_{p-1}}^{y} g(x)dx\,dy\,dz\,dt\right) f^{(4)}(\mu)g'(\eta),$$

$where\ \mu_p,\eta\in\left(x_{p-1},x_p\right).$

Hence, the global error is obtained by summing over *n* such terms:

$$\sum_{k=1}^{n}\begin{bmatrix}\frac{(x_{k-1}-x_k)^3}{108}\int_{x_{k-1}}^{x_k} g(t)dt+\frac{11(x_{k-1}-x_k)^2}{108}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dx\,dt\\ +\frac{x_{k-1}-x_k}{2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dx\,dy\,dt\\ +\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y} g(x)dx\,dy\,dz\,dt\end{bmatrix} f^{(4)}(\mu)g'(\eta),$$
$$=n\begin{bmatrix}\frac{(a-b)^3}{108}\int_a^b g(t)dt+\frac{11(a-b)^2}{108}\int_a^b\int_a^t g(x)dx\,dt\\ +\frac{a-b}{2}\int_a^b\int_a^t\int_a^y g(x)dx\,dy\,dt\\ +\int_a^b\int_a^t\int_a^z\int_a^y g(x)dx\,dy\,dz\,dt\end{bmatrix}\left[\frac{1}{n}\sum_{k=1}^{n} f^{(4)}(\mu_k)g'(\eta)\right]$$

Let $M=\frac{1}{n}\sum_{k=1}^{n} f^{(4)}(\mu_k)g'(\eta)$, so $\min_{x\in[a,b]}\left\{f^{(4)}(x)g'(x)\right\}\le M\le\max_{x\in[a,b]}\left\{f^{(4)}(x)g'(x)\right\}$. Since $f^{(4)}(t)$ and g'(*t*) are continuous in [*a*, *b*], then there exist two points μ and η such that $M=f^{(4)}(\mu)g'(\eta)$. This implies that the error term $R_{S38}[f]$ is

$$=n\begin{bmatrix}\frac{(a-b)^3}{108}\int_a^b g(t)dt+\frac{11(a-b)^2}{108}\int_a^b\int_a^t g(x)dx\,dt\\ +\frac{a-b}{2}\int_a^b\int_a^t\int_a^y g(x)dx\,dy\,dt+\int_a^b\int_a^t\int_a^z\int_a^y g(x)dx\,dy\,dz\,dt\end{bmatrix} f^{(4)}(\mu)g'(\eta),$$

$where\mu,\eta\in(a,b)$ and $h=\frac{b-a}{n}$.

IV. Results and Discussion

It is noted in previous studies [XI], [XII], [XXI] that the experimental works were not conducted to confirm the theoretical works on quadrature schemes for RS-integral. In this study, experimental works have been conducted on quadrature schemes for RS-integral which confirm the validity of theoretical results. Three numerical problems have been solved for each scheme taken from [VIII], [IX], [III], [XIII], [I] etc, which were determined using MATLAB software. All the results are noted in Intel (R) Core (TM) Laptop with RAM 8.00GB and a processing speed of 1.00GHz-1.61GHz. Double-precision arithmetic is used for numerical results. Similarly, the absolute error and computational order of accuracy (COC) formulae are also taken from [IX].

In Table 1, the absolute error drops have been compared for the proposed four-point CS38 and other schemes: CT, ZCT and MZCT under similar conditions, and it is observed that the proposed CS38 rule results in the smallest error for all examples. When the number of strips is increased, it is obvious to conclude from Figs. 1-3 through the line plots of decreasing error distributions that errors in the proposed scheme reduce rapidly as compared to other accompanying schemes.

Using the COC formula, the observed COC have been computed for the used methods, and are listed in Tables 2-4 for Examples 1-3, respectively versus several strips. The numbers in Tables 2-4 confirm the theoretical accuracy of the discussed methods, including the proposed CS38 scheme for the RS-integral. The proposed CS38's order of accuracy is 4 which is the same as the MZCT scheme, but the error reduction is rapid for the former. The CT scheme exhibits an order of accuracy of 2, whereas for the ZCT scheme, due to the issues and mistakes highlighted in [VIII], the order oscillates and doesn't converge to 4.

In Table 5, the total evaluations required per strip are summarized for the discussed methods, which are necessary to compute the computational costs. In Table 6, we list the total computational cost and the average CPU usage in seconds for the three integrals mentioned in Examples 1-3 using CT, ZCT, MZCT and CS38 schemes. It is observed from numerical results that the proposed scheme took less cost to achieve the error 10^{-5} as compared to existing schemes for all test problems, and the similar performance is obvious from Table 6 regarding the smaller average CPU time to achieve the error 10^{-5} for the proposed method against others for Examples 1-3.

Table 1: Absolute error comparison by CS38 and other schemes for Examples 1-3.

Quadrature variants	Example 1 (m=20)	Example 2 (m=100)	Example 3 (m=20)
CT	1.1862E-03	4.9713E-04	3.9042E-02
ZCT	1.6698 E-03	5.5959 E-04	3.9042E-02
MZCT	1.8552 E-06	1,2428E-09	2.4399E-06
CS38	1.6946E-09	1.1351E-12	2.2317E-09

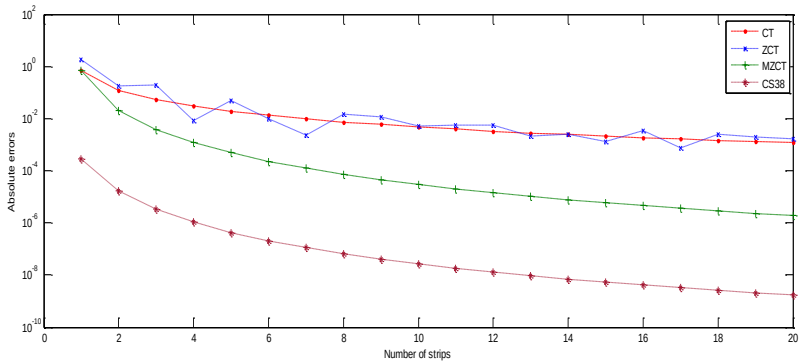

Fig 1. Comparison of error drops by all methods for Example 1

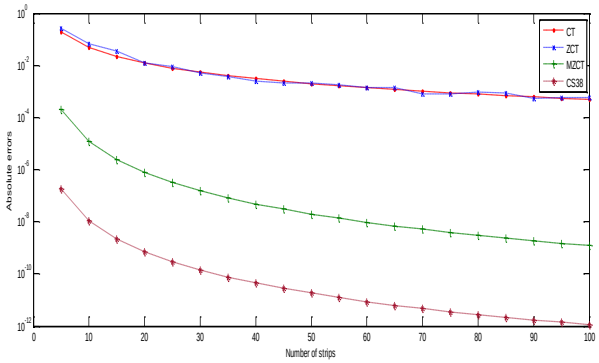

Fig 2. Comparison of error drops by all methods for Example 2

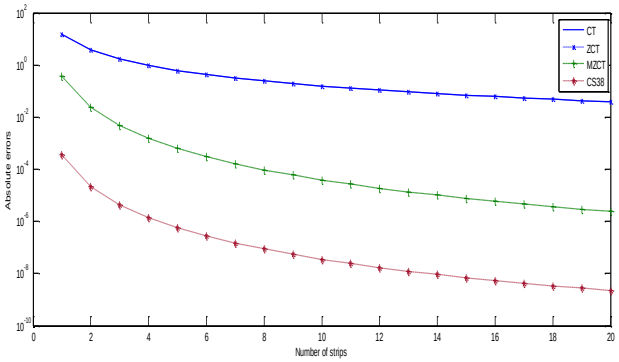

Fig 3. Comparison of error drops by all methods for Example 3

Table 2: Comparison of COC in all methods for Example 1

Number of strips (m)	CT	ZCT	MZCT	CS38
1	NA	NA	NA	NA
2	2.5728	3.3788	5.0724	4.0495
4	2.0420	4.3106	4.1474	4.0121
8	2.0091	-0.8154	4.0348	4.0030
16	2.0022	2.1713	4.0086	4.0008
32	2.0006	3.2118	4.0021	4.0002
64	2.0001	1.1185	4.0006	4.0001

Table 3: Comparison of COC in all methods for Example 2

Number of strips (m)	CT	ZCT	MZCT	CS38
5	NA	NA	NA	NA
10	2.0018	1.9243	4.0025	4.0003
20	2.0004	2.4487	4.0007	4.0000
40	2.0001	2.3985	4.0001	exact
80	2.0001	2.3985	4.0000	exact
160	2.0000	2.4557	3.9998	exact

Table 4: Comparison of COC in all methods for Example 3

Number of strips (m)	CT	ZCT	MZCT	CS38
1	NA	NA	NA	NA
2	1.9319	1.9319	3.8984	3.9862
4	1.9844	1.9844	3.9761	3.9965
8	1.9962	1.9962	3.9941	3.9992
16	1.9990	1.9990	3.9985	3.9998
32	1.9998	1.9998	3.9985	3.9999

Table 5: Computational cost in quadrature variants for m strips.

Quadrature Variants	Total evaluations
CT	2m+3 [XI]
ZCT	5m+3 [XI]
MZCT	5m+3 [XI]
Proposed CS38	6m+3

Table 6: Computational cost and CPU time comparison to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

Quadrature Variants	Computational cost			CPU time (in seconds)		
	Example 1	Example 2	Example 3	Example 1	Example 2	Example 3
CT	439	1415	2503	68.04	12.82	432.30
ZCT	1043	3503	6253	552.60	153.98	6469.38
MZCT	73	78	78	28.01	5.26	27.35
CS38	21	15	21	8.25	2.84	7.61

V Conclusion

A new four-point quadrature scheme of Simpson's 3/8 type was developed for efficient approximations of the RS-integral and extended for higher strips in a composite sense. The theorems concerning the local and global error terms were proved. Three numerical problems were considered from the literature to test the performance of the proposed scheme versus a few other existing schemes. The error drops, observed orders of accuracy and computations performance in terms of evaluations and CPU usage demonstrate the ascendance of the proposed scheme over other discussed schemes for the evaluation for the RS-integral numerically.

Conflict of Interest:

There is no conflict of interest regarding this article.

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