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STRUM-LIOUVILLE FORM AND OTHER IMPORTANT PROPERTIES OF MODIFIED ORTHOGONAL BOUBAKER POLYNOMIALS

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Abstract

In this paper, some classical properties of modified orthogonal Boubaker polynomials (MOBPs) are considered, which are: the three-term recurrence relation, Rodriguez formula, characteristic differential equation and the Strum-Liouville form. The only properties of the MOBPs known so far are orthogonality evidence, weight function, orthonormality evidence and zeros. The new properties established in this work will to the applicability of the MOBPs in different areas of science and engineering where the classical non-orthogonal Boubaker polynomials could be applied, and even in cases where these cannot be applied.

Keywords: Recurrence relation, Rodriguez's formula, Orthogonality, Strum-Liouville form.

I. Introduction

Polynomial sequences and the theory of orthogonal polynomials have always been a topic of interest for scientists and engineers because the mathematical models involving singularities can be solved efficiently using orthogonal polynomial-based methods [I], [X]. The classical Boubaker polynomials (BPs) were proposed in [III],[IV] while discussing some heat equation applications, and later several other properties of BPs have been established in the literature. The BPs were not an orthogonal set of polynomials and were defined in [0, 2]. Shaikh and Boubaker [XII] discussed the application of BPs for counting inside the unit disk, the number of complex roots of polynomial equations. A second-order differential equation was obtained for a modification, and higher-order differential equations with physics applications were also addressed using the BPs [VI], [XIV].

The BPs were applied on Lotka–Volterra equation in [VII]. In [V], an asymptotic expression was presented calculating two oppositely charged discs' capacitance. In [IX], the authors presented a Sturm-Liouville shaped characteristic differential equation to the Boubaker polynomials as a supply to further efforts for proposing different analytic expression. Resistance spot welding applications in steel material [XIII], solving optimal control problems indirectly [XI], and numerical solution of boundary-value problems [II] were also acquired using BPs.

Recently, in [VIII], the orthogonality of the classical Boubaker polynomials [III] was discussed by using an arbitrary basis. Besides, orthonormality was proved for the new modified orthogonal Boubaker polynomials (MOBPs), and zeros were proved to be all reals in [0, 2]. So, the MOBPs in [VIII] can be used like other orthogonal polynomials: Chebyshev, Legendre, Hermite, Laguerre, etc. in many areas of application.

In this work, we derive the three-term recurrence relation, Rodriguez formula and characteristic differential equation of the MOBPs to finally obtain the Sturm–Liouville form. Theorems concerning the main results have been proved.

II. On Modified Orthogonal Boubaker Polynomials (MOBPs)

The classical properties of other orthogonal polynomials family are characteristics differential equation, series solution, Rodriguez's formula, the three-term recurrence relation, generating function, orthogonality relation, self-adjoint form or Strum-Liouville form, weight function, eigenvalues and eigenfunction, norm, zeros, etc [VIII]. Not all such were available for the classical BPs [III]. Therefore, in [VIII], by using the Gram-Schmidt orthogonalization process in [0, 2] on the BPs, a new family of orthogonal polynomials was established, being referred to here as MOBPs. The first ten MOBPs are:

$P_0(x) = 1$

$P_1(x) = (x-1)$

$P_2(x) = (x^2-2x+\frac{2}{3})$

$P_3(x) = (x^3-3x^2+\frac{12}{5}x-\frac{2}{5})$

$P_4(x) = (x^4-4x^3+\frac{36}{7}x^2-\frac{16}{7}x+\frac{8}{35})$

$P_5(x) = (x^5-5x^4+\frac{80}{9}x^3-\frac{20}{3}x^2+\frac{40}{21}x-\frac{8}{63})$

$P_6(x) = (x^6-6x^5+\frac{150}{11}x^4-\frac{160}{11}x^3+\frac{80}{11}x^2-\frac{16}{11}x+\frac{16}{231})$

$P_7(x) = (x^7-7x^6+\frac{252}{13}x^5-\frac{350}{13}x^4+\frac{2800}{143}x^3-\frac{1008}{143}x^2+\frac{448}{429}x-\frac{16}{429})$

$P_8(x) = (x^8-8x^7+\frac{392}{15}x^6-\frac{224}{5}x^5+\frac{560}{13}x^4-\frac{896}{39}x^3+\frac{896}{143}x^2-\frac{512}{715}x+\frac{128}{6435})$

$P_9(x) = (x^9-9x^8+\frac{7616}{225}x^7-\frac{15512}{225}x^6+\frac{26768}{325}x^5-\frac{6832}{117}x^4+\frac{30464}{1287}x^3-\frac{17792}{3575}x^2+\frac{42368}{96525}x-\frac{896}{96525})$

The general expression of obtaining the MOBPs from the BPs was obtained in [VIII] as:

$$P_n(x) = B_n(x) - \sum_{i=0}^{n-1}\frac{<B_n(x),P_i(x)>}{<p_i(x),P_i(x)>}P_i(x), \qquad (1)$$

Where $B_n(\mathrm{x}) = \sum_{n=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-4p}{n-p} \binom{n-p}{p} (-1)^p x^{(n-2p)}$ are classical BPs.

The orthogonality was verified for the MOBPs for the case of distinct polynomials, and the following was established:

$$\int_0^2 [P_m(x)P_n(x)]\, dx = 0 \;\; if\; m \neq n \tag{2}$$

Equation (2) demonstrates the orthogonality of MOBPs, and the weight function for the MOBPs from (2) appears as:

$$w\,(x) = 1 \tag{3}$$

with the interval of orthogonality being [0, 2]. Further in [VIII], the orthogonal basis was developed, and it was shown that all zeros of the MOBPs are reals and contained in [0, 2].

III. Present Analysis and Results

We prove theorems concerning some new classical properties of the MOBPs which include three-term recurrence relation, Rodriguez formula, characteristic differential equation and the Sturm-Liouville form.

Theorem 1. *The three-term recurrence relation for the MOBPs is:*

$$P_{n+1}\,(x) = (x\; - 1)P_n(x) - \frac{n^2}{4n^2-1} P_{n-1}(x) \;\;, n = 1,2,3,\ldots \tag{4}$$

with initial conditions $P_0\,(x) = 1$ *and* $P_1(x) = x-1$.

Proof of Theorem 1.

From [I],[X], the general form of the recurrence relation for a sequence of orthogonal polynomials can be expressed as:

$$P_{n+1}\,(\mathrm{x}) = (\mathrm{x} - a_n)\, P_n(\mathrm{x}) - b_n P_{n-1}(\mathrm{x}) \tag{5}$$

Where

$$a_n = \frac{<xP_n(x), P_n(x)>}{<P_n(x), P_n(x)>} \tag{6}$$

and,

$$b_n = \frac{<P_n(x), P_n(x)>}{<P_{n-1}(x), P_{n-1}(x)>} \tag{7}$$

The inner product in (6)-(7) may be defined from [III] as :

$$<f(x), g(x)> = \int_0^2 f(x)g(x)dx \tag{8}$$

We now attempt to identify the general expression of the coefficients a_n and b_n in (6)-(7), respectively, by putting n =1,2,3… in (5) for the MOBPs.

Using n = 1 in (5), we have:

$$P_2(\mathrm{x}) = (\mathrm{x} - a_1)\, P_1(\mathrm{x}) - b_1 P_0(\mathrm{x}) \tag{9}$$

where a_1 and b_1 can be got from (6) and (7) as:

$$a_1 = \frac{<xP_1(\mathrm{x}),P_1(\mathrm{x})>}{<P_0(\mathrm{x}),P_0(\mathrm{x})>} = \frac{<x(x-1),(x-1)>}{<(x-1),(\mathrm{x}-1)>} = \frac{\int_0^2 x(x-1)(x-1)dx}{\int_0^2 (x-1)(x-1)dx} = 1.$$

$$b_1 = \frac{<P_1(\mathrm{x}),P_1(\mathrm{x})>}{<P_0(\mathrm{x}),P_0(\mathrm{x})>} = \frac{<(x-1),(x-1)>}{<1,1>} = \frac{\int_0^2 (x-1)(x-1)dx}{\int_0^2 1dx} = \frac{1}{3}.$$

Using *n* = 2 in (5), and through (6) and (7), we get:

$$a_2 = 1, \quad b_2 = \frac{4}{15}$$

We observe that, we always have:

$$a_n = 1,\ n = 1,2,3\ldots \tag{10}$$

and, b_n values represents the sequence: $\left\{1, \frac{4}{15}, \frac{9}{35}, \frac{16}{63}, \ldots\right\}$. The sequence can easily be generalized as:

$$b_n = \frac{n^2}{4n^2-1},\ n = 1,2,3,\ldots \tag{11}$$

Substituting (10) and (11) in (5), we have the required three-term recurrence relation for the MOBPs as expressed in (4). This completes the proof of Theorem 1.

It is immediate to verify that with the known initial two polynomials P_0 and P_1, all other forthcoming MOBPs can easily be generated using (4).
For example. With *n* =1 and the two initial polynomials in (4), we have:
$P_2(\mathrm{x}) = (\mathrm{x} - 1)(x-1) - \frac{1}{3}(1) = x^2 - 2x + \frac{2}{3}$, which is correct.

Lemma 1. *The general form of the Rodriquez formula for a family of orthogonal polynomials can be described from [I].[X] as:*

$$\alpha_n P_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n} [w(x)\{A(x)\}^n] \tag{12}$$

where w (x) is the weight function,

and $A(x) = ax^2 + bx + c$ (13)

Theorem 2. *The Rodriguez formula for the MOBPs is given as:*

$$P_n(x) = \frac{(n-1)!}{2(2n-1)!} \frac{d^n}{dx^n} (x^2 - 2x)^n \ , Where\ n = 0,1,2,\ldots \tag{14}$$

Proof of Theorem 2.

The weight function for the MOBPs is known from [VIII] and is given in (3). We have to find coefficients *a*, *b* and *c* in *A* (*x*) through Lemma, as described in (13) and also the coefficient α_n depending on *n*.
For *n* = 0 in (12) and using P_0, we have: $\alpha_0 = 1$.
For *n* = 1 in (12) and using P_1, we have: $\alpha_1 P_1(x) = 2ax + b$, which gives:

$$a = \frac{\alpha_1}{2} \text{ and } b = -\alpha_1. \tag{15}$$

For *n* = 1 in (12) and using P_2 we finally get: $\alpha_2 = 3\alpha_1^2$ and *c* = 0.

Thus, from (13) we have for the case of MOBPs:

$$A(x) = \frac{\alpha_1}{2}x^2 - \alpha_1 x \tag{16}$$

Using (16) and (3) in (12) for $n = 1$, we get: $\alpha_1 = 2$, and thus $a = 1$ and $b = -2$. So, for the MOBPs case (16) becomes:

$$A(x) = x^2 - 2x \tag{17}$$

The sequence of values of α_n takes the form: $\{1,2,12,120,1680,\ldots\}$ and can be expressed as:

$$\alpha_n = \frac{2\,(2n-1)!}{(n-1)!}\ , \text{n} = 0,1,2,\ldots \tag{18}$$

Using (18), (3) and (17) in (12), we have for the MOBPs the Rodriguez formula as:

$$\frac{2\,(2n-1)!}{(n-1)!}\,P_n(x) = \ \frac{d^n}{dx^n}(x^2 - 2x)^n \tag{19}$$

Simplifying, we have: $P_n(x) = \frac{(n-1)!}{2(2n-1)!}\,\frac{d^n}{dx^n}(x^2 - 2x)^n$, which is same as (14). This completes the proof of Theorem 2.

Lemma 2. *The characteristic differential equation for polynomial sequence y_n=Q_n can be described as:*

$$a(x)y''(x) + b(x).y'(x) + c(x) = \lambda y \tag{20}$$

where a(x) =A(x) from Rodriguez formula, b(x) = c(x) + $a'(x)$and

c(x) = [a(x)log (w(x))]'. Also, $\lambda_n = \ p.n(n-1) + q\ .n$, where p is coefficient of x^2 in a(x) and q is coefficient of x in b(x).

Theorem 3. *The characteristic differential equation for the MOBPs is given as:*

$$(x^2 - 2x)\ \frac{d^2y}{dx^2} + \ 2(x-1)\ \frac{dy}{dx} - \quad n\,(n+1)y\ = 0 \tag{22}$$

Proof of Theorem 3.

From Rodriguez formula (14) of the MOBPs y_n = P_n, and using Lemma 2, we have that: $a(x) = A(x) = x^2 - 2x$. From (3), we have that *w(x)* = 1.

Using Lemma 2, and the known values, we have c(x) = 0, and therefore: b(x) = 2*x–2*. The coefficient of x^2 term in a(x) is 1, i.e. p = 1 and the coefficient of *x* in b(x) is 2, i.e. q = 2, so: $\lambda_n = \ 1.n(n-1) + 2\,.n$ or $\lambda_n = \ n(n+1)$. Using a(x), b(x), c(x) and λ_n in (20), we have the desired characteristic differential equation (22) for the MOBPs. The proof of Theorem 3 competes here.

Theorem 4. *The Sturm-Liouville form of the MOBPs is:*

$$\frac{d}{dx}[(x^2 - 2x)\ \frac{dy}{dx}] + \lambda\,y\ = 0 \tag{23}$$

Proof of Theorem 4.

The characteristic differential equation for the MOBPs is given in (22), and comparing it with the general form (20), we have:
$a(x) = x^2 - 2x$, b(x) = 2x–2 and c(x) = 0.
By using the functions defined in Lemma 1 of [VIII], we have:

$$p(x) = exp\left[\int_0^x \frac{b(x)}{a(x)} dx\right] => p(x) = (x^2 - 2x)\,, \tag{24}$$

$$r(x) = \frac{1}{a(x)} \exp\left[\int_0^x \frac{b(x)}{a(x)}\, dx\right] => r(x) = 1\,, \tag{25}$$

$$and,\ q(x) = \frac{c(x)}{a(x)}\, exp\left[\int_0^x \frac{b(x)}{a(x)} dx\right] => q(x) = 0 \tag{26}$$

By using (24)-(26) in the general form of Strum-Liouville form from Lemma 1 in [VIII], which is:
$\frac{1}{r(x)}\frac{d}{dx}\left[A(x)\,\frac{dy}{dx}\right] + q(x)y = -\lambda\, y$, we have the Strum-Liouville form of the MOBPs in the form:

$$\frac{d}{dx}\left[(x^2 - 2x)\,\frac{dy}{dx}\right] = -\lambda\, y$$

Or, $\frac{d}{dx}\left[(x^2 - 2x)\ \frac{dy}{dx}\right] + \lambda\, y = 0$. This completes the proof of Theorem 4.

IV. Conclusion

The modified orthogonal Boubaker polynomials were considered in this study, and an attempt was made to discuss the Rodriguez formula, formulation of modified Boubaker differential equation and Strum-Liouville modified Boubaker differential equation form so that these can also be used in future for applications. All theorems concerning the main results were proved.

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Conflict of Interest:

No conflict of interest regarding this article

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