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CLOSED NEWTON-COTES CUBATURE SCHEMES FOR TRIPLE INTEGRALS WITH ERROR ANALYSIS

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Abstract

Most of the problems in applied sciences in engineering contain integrals, not only in one dimension but also in higher dimensions. The complexity of integrands of functions in one variable or higher variables motivates the quadrature and cubature approximations. Much of the work is focused on the literature on single integral quadrature approximations and double integral cubature schemes. On the other hand, the work on triple integral schemes has been quite rarely focused. In this work, we propose the closed Newton-Cotes-type cubature schemes for triple integrals and discuss consequent error analysis of these schemes in terms of the degree of precision and local error terms for the basic form approximations. The results obtained for the proposed triple integral schemes are in line with the patterns observed in single and double integral schemes. The theorems proved in this work on the local error analysis will be a great aid in extending the work towards global error analysis of the schemes in the future.

Keywords: Cubature, Triple integrals, closed Newton-Cotes, Precision, Order of accuracy, Local error, Global error.

I. Introduction

Numerical methods are utilized to deal with the highly complicated and nonlinear equations arising from the modeling of physical systems in science and engineering [III]. The traditional methods mostly cannot be applied to such systems because of nonlinearity and complex nature terms (derivatives, integrals, algebraic, rational) involved in equations [III],[VIII], and numerical methods efficiently deal with such problems [XIV],[XVII]. The complicated integrals in this way are approximated by using methods of numerical integration, also called the quadrature rules in one dimension [I]. Isaac Newton and his colleague Roger Cotes first contributed quadrature rules for the approximate evaluation of such integrals in one dimension [III] for single-variable integration [I]. The basic quadrature rules of Newton-Cotes were modified quite frequently, as in [VI], [VII] without derivatives and in [II], [IV], [V], [XIX], [XVIII], [XVI] with derivatives. The basic and modified rules were used in other applications as well, for instance, to approximate solutions of Fredholm integral equations of the second kind [XV], reliable numerical simulation of switched reluctance machines [XIII], approximating efficiently the Riemann-Stieltjes integral [XI],[XII], proposing cubature rules with derivative-based schemes and error analysis [IX], [X], etc. However, for the integration in higher dimensions the basic closed Newton-Cotes schemes are although available in the literature [XIV] for double integrals only and their error analysis [X], but for higher are not available to the best of our knowledge.

This work focuses on the extension of closed Newton-Cotes integration schemes for the approximation of triple integrals. In this regard, three cubature schemes for the triple integrals have been proposed, and the local error analysis has been conducted for the proposed schemes. The theorems concerning precision and error terms in basic form have been proved. It is demonstrated through exhaustive theoretical comparison of the obtained results that the results of this study match with the previous works in the literature on single and double integral closed Newton-Cotes schemes. The degrees of precision, local orders of accuracy and local error terms have been derived successfully for triple integral cases.

II. Existing Closed Newton-Cotes Quadrature and Cubature Scheme

For single integrals approximation in closed Newton-Cotes sense, the starting three schemes, also referred to as quadrature rules, in the basic form are described in (1)-(3) [III]:

$$\int_a^b f(x)\,dx = CNCT_{1D} = \frac{b-a}{2}[f(a)+f(b)] - \frac{(b-a)^3}{12} f''(\xi) \tag{1}$$

$$\int_a^b f(x)\,dx = CNCS13_{1D} = \frac{b-a}{6}\begin{bmatrix} f(a) + 4f\left(\frac{a+b}{2}\right) \\ +f(b) \end{bmatrix} - \frac{(b-a)^5}{2880} f^{(4)}(\xi) \tag{2}$$

$$\int_a^b f(x)\,dx = CNCS38_{1D} = \frac{b-a}{8}\begin{bmatrix} f(a) + 3f\left(\frac{2a+b}{3}\right) \\ +3f\left(\frac{a+2b}{3}\right) + f(b) \end{bmatrix} - \frac{(b-a)^5}{6480} f^{(4)}(\xi) \tag{3}$$

where $\xi \in (a,b)$, and are known as trapezoidal, Simpson's 1/3 and Simpson's 3/8 rules. For the double integrals, the corresponding schemes to (1)-(3) with error terms from [X] are expressed in (4)-(6):

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = CNCT_{2D} = \frac{(b-a)(d-c)}{4}\begin{bmatrix} f(a,c) + f(a,d) \\ +f(b,c) + f(b,d) \end{bmatrix}$$
$$- \frac{(b-a)^3(d-c)}{12} f_{xx}(\xi,\eta) - \frac{(b-a)(d-c)^3}{12} f_{yy}(\xi,\eta) \tag{4}$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = CNCS13_{2D}$$

$$= \frac{(b-a)(d-c)}{36}\begin{bmatrix} f(a,c)+f(a,d)+f(b,c)+f(b,d) \\ +4\begin{Bmatrix} f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right) \\ +f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\end{Bmatrix}+16f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\end{bmatrix}$$

$$-\frac{(b-a)^5(d-c)}{2880}f_{xxxx}(\xi,\eta)-\frac{(b-a)(d-c)^5}{2880}f_{yyyy}(\xi,\eta) \qquad (5)$$

$$\int_c^d\int_a^b f(x,y)\,dx\,dy = CNCS38_{2D}$$

$$a = \frac{(b-a)(d-c)}{64}\begin{bmatrix} f(a,c)+f(a,d)+f(b,c)+f(b,d) \\ +3\begin{Bmatrix} f\left(a,\frac{2c+d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{c+2d}{3}\right) \\ +f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\end{Bmatrix} \\ +9\begin{Bmatrix} f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right) \\ +f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\end{Bmatrix}\end{bmatrix}$$

$$-\frac{(b-a)^5(d-c)}{6480}f_{xxxx}(\xi,\eta)-\frac{(b-a)(d-c)^5}{6480}f_{yyyy}(\xi,\eta) \qquad (6)$$

where $\xi\in(a,b)$ *and* $\eta\in(c,d)$.

III. Proposed Closed Newton-Cotes Triple Integration Schemes

The proposed triple integral cubature schemes in the closed Newton-Cotes sense denoted as $CNCT_{3D}$, $CNCS13_{3D}$ and $CNCS38_{3D}$ are described here as (7)-(9). Following the extension mechanism of single integral quadrature schemes (1)-(3) to double integral cubature schemes (4)-(6), we propose the following three corresponding closed Newton-Cotes triple integral cubature schemes:

$$\int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dydz \approx \frac{(b-a)(d-c)(s-r)}{8}\begin{bmatrix} f(a,c,r)+f(a,c,s) \\ +f(a,d,r)+f(a,d,s) \\ +f(b,c,r)+f(b,c,s) \\ +f(b,d,r)+f(b,d,s)\end{bmatrix} \qquad (7)$$

$$\int_r^s\int_c^d\int_a^b f(x,y,z)dxdydz \approx \frac{(b-a)(d-c)(s-r)}{216}\Bigg[\begin{Bmatrix} f(a,c,r)+f(a,c,s)+f(a,d,r)+f(a,d,s) \\ +f(b,c,r)+f(b,c,s)+f(b,d,r)+f(b,d,s)\end{Bmatrix}$$

$$+4\begin{Bmatrix} f\left(a,c,\frac{r+s}{2}\right)+f\left(a,\frac{c+d}{2},r\right)+f\left(a,\frac{c+d}{2},s\right)+f\left(a,d,\frac{r+s}{2}\right)+f\left(\frac{a+b}{2},c,r\right)+f\left(\frac{a+b}{2},c,s\right) \\ +f\left(\frac{a+b}{2},d,r\right)+f\left(\frac{a+b}{2},d,s\right)+f\left(b,c,\frac{r+s}{2}\right)+f\left(b,\frac{c+d}{2},r\right)+f\left(b,\frac{c+d}{2},s\right)+f\left(b,d,\frac{r+s}{2}\right)\end{Bmatrix}$$

$$+16\begin{Bmatrix} f\left(a,\frac{c+d}{2},\frac{r+s}{2}\right)+f\left(\frac{a+b}{2},c,\frac{r+s}{2}\right)+f\left(\frac{a+b}{2},\frac{c+d}{2},r\right) \\ +f\left(\frac{a+b}{2},\frac{c+d}{2},s\right)+f\left(\frac{a+b}{2},d,\frac{r+s}{2}\right)+f\left(b,\frac{c+d}{2},\frac{r+s}{2}\right)\end{Bmatrix}+64f\left(\frac{a+b}{2},\frac{c+d}{2},\frac{r+s}{2}\right)\Bigg] \qquad (8)$$

$$
\begin{aligned}
\int_r^s\int_c^d\int_a^b f(x,y,z)dxdydz \approx \frac{(b-a)(d-c)(s-r)}{512}\Bigg[&\left\{\begin{array}{l} f(a,c,r)+f(a,c,s)+f(a,d,r)+f(a,d,s)\\ +f(b,c,r)+f(b,c,s)+f(b,d,r)+f(b,d,s)\end{array}\right\}\\
+3&\left\{\begin{array}{l}
f\left(a,c,\frac{2r+s}{3}\right)+f\left(a,c,\frac{r+2s}{3}\right)+f\left(a,\frac{2c+d}{3},r\right)+f\left(a,\frac{2c+d}{3},s\right)+f\left(a,\frac{c+2d}{3},r\right)+f\left(a,\frac{c+2d}{3},s\right)\\
+f\left(a,d,\frac{2r+s}{3}\right)+f\left(a,d,\frac{r+2s}{3}\right)+f\left(\frac{2a+b}{3},c,r\right)+f\left(\frac{2a+b}{3},c,s\right)+f\left(\frac{2a+b}{3},d,r\right)+f\left(\frac{2a+b}{3},d,s\right)\\
+f\left(\frac{a+2b}{3},c,r\right)+f\left(\frac{a+2b}{3},c,s\right)++f\left(\frac{a+2b}{3},d,r\right)+f\left(\frac{a+2b}{3},d,s\right)+f\left(b,c,\frac{2r+s}{3}\right)+f\left(b,c,\frac{r+2s}{3}\right)\\
+f\left(b,\frac{2c+d}{3},r\right)+f\left(b,\frac{2c+d}{3},s\right)+f\left(b,\frac{c+2d}{3},r\right)+f\left(b,\frac{c+2d}{3},s\right)+f\left(b,d,\frac{2r+s}{3}\right)+f\left(b,d,\frac{r+2s}{3}\right)
\end{array}\right\}\\
+9&\left\{\begin{array}{l}
f\left(a,\frac{2c+d}{3},\frac{2r+s}{3}\right)+f\left(a,\frac{2c+d}{3},\frac{r+2s}{3}\right)+f\left(a,\frac{c+2d}{3},\frac{2r+s}{3}\right)+f\left(a,\frac{c+2d}{3},\frac{r+2s}{3}\right)\\
+f\left(\frac{2a+b}{3},c,\frac{2r+s}{3}\right)+f\left(\frac{2a+b}{3},c,\frac{r+2s}{3}\right)+f\left(\frac{2a+b}{3},\frac{2c+d}{3},r\right)+f\left(\frac{2a+b}{3},\frac{2c+d}{3},s\right)\\
+f\left(\frac{2a+b}{3},\frac{c+2d}{3},r\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3},s\right)+f\left(\frac{2a+b}{3},d,\frac{2r+s}{3}\right)+f\left(\frac{2a+b}{3},d,\frac{r+2s}{3}\right)\\
+f\left(\frac{a+2b}{3},c,\frac{2r+s}{3}\right)+f\left(\frac{a+2b}{3},c,\frac{r+2s}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3},r\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3},s\right)\\
+f\left(\frac{a+2b}{3},\frac{c+2d}{3},r\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3},s\right)+f\left(\frac{a+2b}{3},d,\frac{2r+s}{3}\right)+f\left(\frac{a+2b}{3},d,\frac{r+2s}{3}\right)\\
+f\left(b,\frac{2c+d}{3},\frac{2r+s}{3}\right)+f\left(b,\frac{2c+d}{3},\frac{r+2s}{3}\right)+f\left(b,\frac{c+2d}{3},\frac{2r+s}{3}\right)+f\left(b,\frac{c+2d}{3},\frac{r+2s}{3}\right)
\end{array}\right\}\\
+27&\left\{\begin{array}{l}
f\left(\frac{2a+b}{3},\frac{2c+d}{3},\frac{2r+s}{3}\right)+f\left(\frac{2a+b}{3},\frac{2c+d}{3},\frac{r+2s}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3},\frac{2r+s}{3}\right)\\
+f\left(\frac{2a+b}{3},\frac{c+2d}{3},\frac{r+2s}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3},\frac{2r+s}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3},\frac{r+2s}{3}\right)\\
+f\left(\frac{a+2b}{3},\frac{c+2d}{3},\frac{2r+s}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3},\frac{r+2s}{3}\right)
\end{array}\right\}\Bigg]
\end{aligned} \tag{9}
$$

IV. Error Analysis, Main Results and Discussion

Now, the theorems concerning their degree of precision and local error term are also proved in this section, which are the main contributions from this study. Particularly, in Theorems, 1-3 refer to our main contributions for the proposed closed Newton-Cotes triple integral cubature schemes: CNCT, CNCS13 and CNCS38, respectively.

Theorem 1. Let a, b, c, d, r, s be finite real numbers, and f (x, y, z) along with its second-order partial derivatives exist and are continuous in [a, b]×[c, d]× [r, s], then the $CNCT_{3D}$ scheme has precision one, and in basic form with the local error term is defined as:

$$
\int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dydz = CNCT + R_{CNCT}[f]
$$

$$
= \frac{(b-a)(d-c)(s-r)}{8}[f(a,c,r)+f(a,c,s)+f(a,d,r)+f(a,d,s)+f(b,c,r)+f(b,c,s)+f(b,d,r)+f(b,d,s)] - \frac{(b-a)^3(d-c)(s-r)}{12}f_{xx}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)^3(s-r)}{12}f_{yy}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)(s-r)^3}{12}f_{zz}(\xi,\eta,\gamma)
$$

$$
where\ \ \xi\in(a,b)\,,\ \eta\in(c,d)\ \ and\ \ \gamma\in(r,s)
$$

Proof of Theorem 1.

Following the definition of degree of precision as mentioned in [IX],[X], the exact results of the triple integral for $f(x, y, z) = (x\,y\,z)^n$ with $n = 0,1,2$ are:

$$\int_r^s \int_c^d \int_a^b (x^0 y^0 z^0)\, dx\, dydz = (b-a)(d-c)(s-r) \quad (10)$$

$$\int_r^s \int_c^d \int_a^b (x^1 y^1 z^1)\, dx\, dydz = \frac{(b^2-a^2)(d^2-c^2)(s^2-r^2)}{8} \quad (11)$$

$$\int_r^s \int_c^d \int_a^b (x^2 y^2 z^2)\, dx\, dydz = \frac{(b^3-a^3)(d^3-c^3)(s^3-r^3)}{27} \quad (12)$$

The approximate results using (7) are:

$$CNCT(x^0 y^0 z^0) = (b-a)(d-c)(s-r) \quad (13)$$

$$CNCT(x^1 y^1 z^1) = \frac{(b^2-a^2)(d^2-c^2)(s^2-r^2)}{8} \quad (14)$$

$$CNCT(x^2 y^2 z^2) = \frac{(b-a)(b^2+a^2)(d-c)(d^2+c^2)(s-r)(s^2+r^2)}{8} \quad (15)$$

Comparison of (10)-(12) with (13)-(15) gives,

For $f(x,y,z) = (xyz)^n,\ n \le 1$,

$$\int_r^s \left(\int_c^d \left(\int_a^b (xyz)^n\, dx\right) dy\right) dz - CNCT(xyz)^n = 0 \quad (16)$$

But if $f(x,y,z) = x^2y^2z^2$, then

$$\int_r^s \left(\int_c^d \left(\int_a^b x^2y^2z^2\, dx\right) dy\right) dz - CNCT(x^2y^2z^2) \neq 0 \quad (17)$$

So, from (16)-(17), the derived precision of CNCT triple integral scheme is 1.

From [IX], [X], the 2nd order partial derivative term in Taylor series of $f(x,y,z)$ about (x_0, y_0, z_0) is:

$$\frac{1}{2!}\left[\begin{array}{c}(x-x_0)^2\frac{\partial^2 f}{\partial x^2}(x_0,y_0,z_0) + (y-y_0)^2\frac{\partial^2 f}{\partial y^2}(x_0,y_0,z_0) + (z-z_0)^2\frac{\partial^2 f}{\partial y^2}(x_0,y_0,z_0)\\ +2(x-x_0)(y-y_0)\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0,z_0) + 2(y-y_0)(z-z_0)\frac{\partial^2 f}{\partial y \partial z}(x_0,y_0,z_0)\\ +2(z-z_0)(x-x_0)\frac{\partial^2 f}{\partial z \partial x}(x_0,y_0,z_0)\end{array}\right] \quad (18)$$

Focusing (18), the local error term of CNCT triple integral scheme is expressed as:

$$Error\ term\ (CNCT) = \frac{1}{2!}\left[\int_r^s \int_c^d \int_a^b x^2\, dxdydz - CNCT(x^2)\right] f_{xx}(\xi,\eta,\gamma) + \frac{1}{2!}\left[\int_r^s \int_c^d \int_a^b y^2\, dxdydz - CNCT(y^2)\right] f_{yy}(\xi,\eta,\gamma) + \frac{1}{2!}\left[\int_r^s \int_c^d \int_a^b z^2\, dxdydz - CNCT(z^2)\right] f_{zz}(\xi,\eta,\gamma) + \left[\int_r^s \int_c^d \int_a^b xy\, dxdydz - CNCT(xy)\right] f_{yx}(\xi,\eta,\gamma) + \left[\int_r^s \int_c^d \int_a^b yz\, dxdydz - CNCT(yz)\right] f_{zy}(\xi,\eta,\gamma) + \left[\int_r^s \int_c^d \int_a^b zx\, dxdydz - CNCT(zx)\right] f_{xz}(\xi,\eta,\gamma) \quad (19)$$

The exact results in (19) are:

$\int_r^s \int_c^d \int_a^b x^2 dxd\, ydz = \frac{(b^3-a^3)(d-c)(s-r)}{3}$, $\int_r^s \int_c^d \int_a^b y^2 dxd\, ydz = \frac{(b-a)(d^3-c^3)(s-r)}{3}$,

$\int_r^s \int_c^d \int_a^b z^2 dxd\, ydz = \frac{(b-a)(d-c)(s^3-r^3)}{3}$, $\int_r^s \int_c^d \int_a^b xydxdydz = \frac{(b^2-a^2)(d^2-c^2)(s-r)}{4}$,

$\int_r^s \int_c^d \int_a^b yzdxdydz = \frac{(b-a)(d^2-c^2)(s^2-r^2)}{4}$, $\int_r^s \int_c^d \int_a^b zxdxdydz = \frac{(b^2-a^2)(d-c)(s^2-r^2)}{4}$

The approximate results in (19) using (7) are:

$CNCT(x^2) = \frac{(b-a)(b^2+a^2)(d-c)(s-r)}{2}$, $CNCT(y^2) = \frac{(b-a)(d-c)(d^2+c^2)(s-r)}{2}$,

$CNCT(z^2) = \frac{(b-a)(d-c)(s-r)(s^2+r^2)}{3}$, $CNCT(xy) = \frac{(b^2-a^2)(d^2-c^2)(s-r)}{4}$,

$CNCT(yz) = \frac{(b-a)(d^2-c^2)(s^2-r^2)}{4}$, $CNCT(zx) = \frac{(b^2-a^2)(d-c)(s^2-r^2)}{4}$.

Using the exact and approximate results in (19), and simplifying we finally get the local error term for CNCT triple integral cubature scheme (7) in basic form as:

$$Error\ term = R_{CNCT}[f] = -\frac{(b-a)^3(d-c)(s-r)}{12} f_{xx}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)^3(s-r)}{12} f_{yy}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)(s-r)^3}{12} f_{zz}(\xi,\eta,\gamma) \qquad (20)$$

$where\ \ \xi \in (a,b)\ ,\eta \in (c,d)\ and\ \gamma \in (r,s).$

Theorem 2. Let a, b, c, d, r, s be finite real numbers, and f (x, y, z) along with its fourth order partial derivatives exist and are continuous in [a, b]×[c, d]× [r, s], then the CNCS13 scheme has precision degree three, and in basic form with the local error term is defined as:

$$CNCS13 + R_{CNCS13} = \frac{(b-a)(d-c)(s-r)}{216} \left[\begin{array}{l} \left\{ \begin{array}{l} f(a,c,r) + f(a,c,s) + f(a,d,r) + f(a,d,s) \\ +f(b,c,r) + f(b,c,s) + f(b,d,r) + f(b,d,s) \end{array} \right\} \\ +4 \left\{ \begin{array}{l} f\left(a,c,\frac{r+s}{2}\right) + f\left(a,\frac{c+d}{2},r\right) + f\left(a,\frac{c+d}{2},s\right) + f\left(a,d,\frac{r+s}{2}\right) \\ +f\left(\frac{a+b}{2},c,r\right) + f\left(\frac{a+b}{2},c,s\right) + f\left(\frac{a+b}{2},d,r\right) + f\left(\frac{a+b}{2},d,s\right) \\ +f\left(b,c,\frac{r+s}{2}\right) + f\left(b,\frac{c+d}{2},r\right) + f\left(b,\frac{c+d}{2},s\right) + f\left(b,d,\frac{r+s}{2}\right) \end{array} \right\} \\ +16 \left\{ \begin{array}{l} f\left(a,\frac{c+d}{2},\frac{r+s}{2}\right) + f\left(\frac{a+b}{2},c,\frac{r+s}{2}\right) + f\left(\frac{a+b}{2},\frac{c+d}{2},r\right) \\ +f\left(\frac{a+b}{2},\frac{c+d}{2},s\right) + f\left(\frac{a+b}{2},d,\frac{r+s}{2}\right) + f\left(b,\frac{c+d}{2},\frac{r+s}{2}\right) \end{array} \right\} \\ +64 f\left(\frac{a+b}{2},\frac{c+d}{2},\frac{r+s}{2}\right) \end{array} \right]$$

$$-\frac{(b-a)^5(d-c)(s-r)}{2880} f_{xxxx}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)^5(s-r)}{2880} f_{yyyy}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)(s-r)^5}{2880} f_{zzzz}(\xi,\eta,\gamma)\ ,where\ \ \xi \in (a,b)\ ,\ \eta \in (c,d)\ \ and\ \ \gamma \in (r,s)\ \ .$$

Proof of Theorem 2.

It is immediate to note for $f(x,y,z) = (xyz)^n$, and $n \le 3$ through (8) that:

$$\int_r^s \left(\int_c^d \left(\int_a^b (xyz)^n \, dx\right) dy\right) dz - CNCS13(xyz)^n = 0 \tag{21}$$

But if $f(x,y,z) = x^4y^4z^4$, then:

$$\int_r^s \left(\int_c^d \left(\int_a^b x^4y^4z^4 \, dx\right) dy\right) dz - CNCS13(x^4y^4z^4) \neq 0 \tag{22}$$

So, from (21)-(22)), it appears that the derived precision of the CNCS13 triple integral scheme is 3.

From [IX],[X], using the coefficients of 4th order partial derivative terms in the Taylor series the local error term of the CNCS13 triple integral scheme can be defined as:

$$\begin{aligned}
&Error\ term\ (CNCS13\ triple) = \frac{1}{24}\left[\int_r^s \int_c^d \int_a^b x^4 \, dxdydz - CNCS13(x^4)\right] f_{xxxx}(\xi,\eta,\gamma) + \frac{1}{24}\left[\int_r^s \int_c^d \int_a^b y^4 \, dxdydz - CNCS13(y^4)\right] f_{yyyy}(\xi,\eta,\gamma) + \frac{1}{24}\left[\int_r^s \int_c^d \int_a^b z^4 \, dxdydz - CNCS13(z^4)\right] f_{zzzz}(\xi,\eta,\gamma) \\
&+ \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b x^3y \, dxdydz - CNCS13(x^3y)\right] f_{yxxx}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b y^3z \, dxdydz - CNCS13(y^3z)\right] f_{zyyy}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b z^3x \, dxdydz - CNCS13(z^3x)\right] f_{xzzz}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b x^2y^2 \, dxdydz - CNCS13(x^2y^2)\right] f_{yyxx}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b y^2z^2 \, dxdydz - CNCS13(y^2z^2)\right] f_{zzyy}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b z^2x^2 \, dxdydz - CNCS13(z^2x^2)\right] f_{xxzz}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b xy^3 \, dxdydz - CNCS13(xy^3)\right] f_{yyyx}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b yz^3 \, dxdydz - CNCS13(yz^3)\right] f_{zzzy}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b zx^3 \, dxdydz - CNCS13(zx^3)\right] f_{xxxz}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b xyz^2 \, dxdydz - CNCS13(xyz^2)\right] f_{zzyx}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b xy^2z \, dxdydz - CNCS13(xy^2z)\right] f_{zyyx}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b x^2yz \, dxdydz - CNCS13(x^2yz)\right] f_{zyxx}(\xi,\eta,\gamma)
\end{aligned} \tag{23}$$

Using the exact and approximate evaluations of the integrals in (23), the local error term for the CNCS13 triple integral cubature scheme is

$$R_{CNCS13} = -\frac{(b-a)^5(d-c)(s-r)}{2880} f_{xxxx}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)^5(s-r)}{2880} f_{yyyy}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)(s-r)^5}{2880} f_{zzzz}(\xi,\eta,\gamma) \tag{24}$$

$where\ \xi \in (a,b), \eta \in (c,d)\ and\ \gamma \in (r,s)$

Theorem 3. Let a, b, c, d, r, s be finite real numbers, and f (x, y, z) along with its fourth order partial derivatives exist and are continuous in [a, b]×[c, d]× [r, s], then the CNCS38 scheme has precision degree of three, and in basic form with the local error term is defined as:

$$\int_r^s\int_c^d\int_a^b f\left(x,y,z\right)dxdydz=\frac{(b-a)(d-c)(s-r)}{512}\left[\begin{Bmatrix} f(a,c,r)+f(a,c,s)+f(a,d,r)+f(a,d,s)\\ +f(b,c,r)+f(\text{b},c,s)+f(\text{b},d,r)+f(\text{b},d,s)\end{Bmatrix}\right.$$

$$+3\begin{Bmatrix} f\left(a,c,\frac{2r+s}{3}\right)+f\left(a,c,\frac{r+2s}{3}\right)+f\left(a,\frac{2c+d}{3},\text{r}\right)+f\left(a,\frac{2c+d}{3},\text{s}\right)+f\left(a,\frac{c+2d}{3},\text{r}\right)+f\left(a,\frac{c+2d}{3},\text{s}\right)\\ +f\left(a,d,\frac{2r+s}{3}\right)+f\left(a,d,\frac{r+2s}{3}\right)+f\left(\frac{2a+b}{3},c,r\right)+f\left(\frac{2a+b}{3},\text{c,s}\right)+f\left(\frac{2a+b}{3},d,r\right)+f\left(\frac{2a+b}{3},d,s\right)\\ +f\left(\frac{a+2b}{3},\text{c},r\right)+f\left(\frac{a+2b}{3},\text{c},s\right)++f\left(\frac{a+2b}{3},\text{d},r\right)+f\left(\frac{a+2b}{3},\text{d},s\right)+f\left(\text{b,c},\frac{2r+s}{3}\right)+f\left(\text{b,c},\frac{r+2s}{3}\right)\\ +f\left(\text{b},\frac{2c+d}{3},\text{r}\right)+f\left(\text{b},\frac{2c+d}{3},\text{s}\right)+f\left(\text{b},\frac{c+2d}{3},\text{r}\right)+f\left(\text{b},\frac{c+2d}{3},\text{s}\right)+f\left(\text{b},d,\frac{2r+s}{3}\right)+f\left(\text{b},d,\frac{r+2s}{3}\right)\end{Bmatrix}$$

$$+9\begin{Bmatrix} f\left(a,\frac{2c+d}{3},\frac{2r+s}{3}\right)+f\left(\text{a},\frac{2c+d}{3},\frac{r+2s}{3}\right)+f\left(a,\frac{c+2d}{3},\frac{2r+s}{3}\right)+f\left(a,\frac{c+2d}{3},\frac{r+2s}{3}\right)\\ +f\left(\frac{2a+b}{3},\text{c},\frac{2r+s}{3}\right)+f\left(\frac{2a+b}{3},\text{c},\frac{r+2s}{3}\right)+f\left(\frac{2a+b}{3},\frac{2c+d}{3},\text{r}\right)+f\left(\frac{2a+b}{3},\frac{2c+d}{3},\text{s}\right)\\ +f\left(\frac{2a+b}{3},\frac{c+2d}{3},\text{r}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3},\text{s}\right)+f\left(\frac{2a+b}{3},\text{d},\frac{2r+s}{3}\right)+f\left(\frac{2a+b}{3},\text{d},\frac{r+2s}{3}\right)\\ +f\left(\frac{a+2b}{3},\text{c},\frac{2r+s}{3}\right)+f\left(\frac{a+2b}{3},\text{c},\frac{r+2s}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3},\text{r}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3},\text{s}\right)\\ +f\left(\frac{a+2b}{3},\frac{c+2d}{3},\text{r}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3},\text{s}\right)+f\left(\frac{a+2b}{3},\text{d},\frac{2r+s}{3}\right)+f\left(\frac{a+2b}{3},\text{d},\frac{r+2s}{3}\right)\\ +f\left(\text{b},\frac{2c+d}{3},\frac{2r+s}{3}\right)+f\left(\text{b},\frac{2c+d}{3},\frac{r+2s}{3}\right)+f\left(\text{b},\frac{c+2d}{3},\frac{2r+s}{3}\right)+f\left(\text{b},\frac{c+2d}{3},\frac{r+2s}{3}\right)\end{Bmatrix}$$

$$\left.+27\begin{Bmatrix} f\left(\frac{2a+b}{3},\frac{2c+d}{3},\frac{2r+s}{3}\right)+f\left(\frac{2a+b}{3},\frac{2c+d}{3},\frac{r+2s}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3},\frac{2r+s}{3}\right)\\ +f\left(\frac{2a+b}{3},\frac{c+2d}{3},\frac{r+2s}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3},\frac{2r+s}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3},\frac{r+2s}{3}\right)\\ +f\left(\frac{a+2b}{3},\frac{c+2d}{3},\frac{2r+s}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3},\frac{r+2s}{3}\right)\end{Bmatrix}\right]$$

$$-\frac{(b-a)^5(d-c)(s-r)}{6480}f_{xxxx}(\xi,\eta,\gamma)-\frac{(b-a)(d-c)^5(s-r)}{6480}f_{yyyy}(\xi,\eta,\gamma)$$

$$-\frac{(b-a)(d-c)(s-r)^5}{6480}f_{zzzz}(\xi,\eta,\gamma)$$

where $\xi\in(a,b)\ ,\eta\in(c,d)\ \&\ \gamma\in(r,s)$

Proof of Theorem 3.

Using exact and approximate integrals with the help of (9) for $\mathrm{f(x,y,z)=(xyz)^n}$, we observe that:

$$\int_r^s\left(\int_c^d\left(\int_a^b(xyz)^n\,dx\right)dy\right)dz-CNCS38(xyz)^n=0,\ \text{for n}\le 3 \quad (25)$$

But, if $f(x, y, z) = x^4y^4z^4$, then:

$$\int_r^s \left(\int_c^d \left(\int_a^b x^4y^4z^4\, dx\right) dy\right) dz - CNCS38(x^4y^4z^4) \neq 0 \qquad (26)$$

So, from (25)-(26), the derived precision of CNCS38 triple integral scheme is 3.

Likewise in Theorem 2 for the CNCS13 triple integral scheme, using the coefficients of fourth-order partial derivative terms in the Taylor's series expansion of *f(x, y, z)* about (x_0, y_0, z_0), the local error term of CNCS38 triple integral scheme is sated as:

$$R_{CNCS38}[f] = \frac{1}{24}\left[\int_r^s \int_c^d \int_a^b x^4\, dxdydz - CNCS38(x^4)\right] f_{xxxx}(\xi,\eta,\gamma) + \frac{1}{24}\left[\int_r^s \int_c^d \int_a^b y^4\, dxdydz - CNCS38(y^4)\right] f_{yyyy}(\xi,\eta,\gamma) + \frac{1}{24}\left[\int_r^s \int_c^d \int_a^b z^4\, dxdydz - CNCS38(z^4)\right] f_{zzzz}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b x^3y\, dxdydz - CNCS38(x^3y)\right] f_{yxxx}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b y^3z\, dxdydz - CNCS38(y^3z)\right] f_{zyyy}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b z^3x\, dxdydz - CNCS38(z^3x)\right] f_{xzzz}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b x^2y^2\, dxdydz - CNCS38(x^2y^2)\right] f_{yyxx}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b y^2z^2\, dxdydz - CNCS38(y^2z^2)\right] f_{zzyy}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b z^2x^2\, dxdydz - CNCS38(z^2x^2)\right] f_{xxzz}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b xy^3\, dxdydz - CNCS38(xy^3)\right] f_{yyyx}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b yz^3\, dxdydz - CNCS38(yz^3)\right] f_{zzzy}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b zx^3\, dxdydz - CNCS38(zx^3)\right] f_{xxxz}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b xyz^2\, dxdydz - CNCS38(xyz^2)\right] f_{zzyx}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b xy^2z\, dxdydz - CNCS38(xy^2z)\right] f_{zyyx}(\xi,\eta,\gamma) + \frac{1}{6}\left[\int_r^s \int_c^d \int_a^b x^2yz\, dxdydz - CNCS38(x^2yz)\right] f_{zyxx}(\xi,\eta,\gamma) \qquad (27)$$

Using the exact and approximate evaluations (using (9) of the integrals in (27), the local error term for the CNCS38 triple integral scheme is

$$R\_CNCS38\,[f] = -\frac{(b-a)^5(d-c)(s-r)}{6480} f_{xxxx}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)^5(s-r)}{6480} f_{yyyy}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)(s-r)^5}{6480} f_{zzzz}(\xi,\eta,\gamma) \qquad (28)$$

$where\ \xi \in (a,b)\ ,\eta \in (c,d)\ and\ \gamma \in (r,s)$

The degrees of precision verification is carried out in Table 1 for the proposed CNCT, CNCS13 and CNCS38 cubature schemes for the triple integrals, by applying the rules to $f\,(x, y, z) = (xyz)^n$ to observe the same and different results concerning the exact evaluations of the integrals for n = 0,1,2,3,4. It is observed from Table 1 that the theoretical degrees of precision as also proved in Theorems 1-3 have been verified for triple integral cases proposed in this work, and results are in line with the expressions for the single and double integral closed Newton-Cotes schemes in literature [IX,[X],[XIV].

On the other hand, in Table 2, the local orders of accuracy have been summarized for the closed Newton-Cotes schemes for single (1D), double (2D) and triple (3D) integrals, which are also the same. Moreover, the coefficients of local error terms and the degrees of interval lengths in the coefficients in Table 2 are also the same. Tables 1-2 lead to the verification and summary of the results proved in Theorems 1-3.

The presented work focuses on the derivation and error analysis of the closed Newton-Cotes triple integral cubature schemes in the basic form. The contributions from this work are pioneering work on such extension and may be extended in the future towards the composite forms, global error analysis and consequent verification through numerical examples.

Table 1: Degrees of precision by comparing exact and approximate results

$f(x,y,z) = (xyz)^n$ / schemes	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	precision
CNC T	same	same	different	different	different	**1**
CNCS13	same	same	same	Same	different	**3**
CNCS38	Same	same	same	same	different	**3**

Table 2: Local order of accuracy and error term of cubature schemes up to 3D

Scheme	Local order of accuracy	Local error term		
		1D [III]	2D [IX],[X]	3D (Present work)
CNCT	3	$-\frac{(b-a)^3}{12}f^{(2)}(\xi)$	$-\frac{(b-a)^3(d-c)}{12}f_{xx}(\xi,\eta) - \frac{(b-a)(d-c)^3}{12}f_{yy}(\xi,\eta)$	$-\frac{(b-a)^3(d-c)(s-r)}{12}f_{xx}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)^3(s-r)}{12}f_{yy}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)(s-r)^3}{12}f_{zz}(\xi,\eta,\gamma)$
CNCS13	5	$-\frac{(b-a)^5}{2880}f^{(4)}(\xi)$	$-\frac{(b-a)^5(d-c)}{2880}f_{xxxx}(\xi,\eta) - \frac{(b-a)(d-c)^5}{2880}f_{yyyy}(\xi,\eta)$	$-\frac{(b-a)^5(d-c)(s-r)}{2880}f_{xxxx}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)^5(s-r)}{2880}f_{yyyy}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)(s-r)^5}{2880}f_{zzzz}(\xi,\eta,\gamma)$
CNCS38	5	$-\frac{(b-a)^5}{6480}f^{(4)}(\xi)$	$-\frac{(b-a)^5(d-c)}{6480}f_{xxxx}(\xi,\eta) - \frac{(b-a)(d-c)^5}{6480}f_{yyyy}(\xi,\eta)$	$-\frac{(b-a)^5(d-c)(s-r)}{6480}f_{xxxx}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)^5(s-r)}{6480}f_{yyyy}(\xi,\eta,\gamma) - \frac{(b-a)(d-c)(s-r)^5}{6480}f_{zzzz}(\xi,\eta,\gamma)$

V. Conclusion

In this paper, a pioneering investigation of the development of triple integral cubature schemes of the closed Newton-Cotes was carried out. The proposed triple integral schemes were discussed in basic form, and the theorems on the degrees of precision and local error terms were proved for the CNCT, CNCS13 and CNCS38 schemes for triple integrals. The comparison of the present theoretical results for proposed triple integration schemes was in line with the existing single and double integral schemes. The extension to the composite forms and global error analysis can be focused on in future works.

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Conflict of Interest:

There is no conflict of interest regarding this article.

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