



A COMPARATIVE EXPLORATION ON DIFFERENT NUMERICAL METHODS FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

In this paper, the initial value problem of Ordinary Differential Equations has been solved by using different Numerical Methods namely Euler's method, Modified Euler method, and Runge-Kutta method. Here all of the three proposed methods have to be analyzed to determine the accuracy level of each method. By using MATLAB Programming language first we find out the approximate numerical solution of some ordinary differential equations and then to determine the accuracy level of the proposed methods we compare all these solutions with the exact solution. It is observed that numerical solutions are in good agreement with the exact solutions and numerical solutions become more accurate when taken step sizes are very much small. Lastly, the error of each proposed method is determined and represents them graphically which reveals the superiority among all the three methods. We find that, among the proposed methods Runge-Kutta 4th order method gives the accurate result and minimum amount of error.

Keywords: Initial Value Problems (IVP), Euler's Method, Modified Euler Method, Fourth-order Runge-Kutta Method, and Error Estimation.

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I. Introduction

Differential Equations are of great help for solving complex mathematical problems in almost every section of Engineering, Science, and Mathematics. In mathematics, many real problems arise in the form of differential equations. These differential equations are either in the form of an ordinary differential equation or a partial differential equation. Numerical approximation methods are frequently used for solving mathematical problems where it is so much difficult or even impossible to obtain exact results. Though there are many analytical methods for solving ordinary differential equations, a large number of ordinary differential equations cannot be solved by using analytical methods. In that situation, numerical methods help us for finding the approximate solution of an ODE.

From the literature review analysis, we can understand that many authors have worked on numerical solutions of the ordinary differential equation using numerous numerical methods such as Euler's Method, Modified Euler's method, and Runge-Kutta 4th order method, etc. Many authors have taken initiative, to maintain the high correctness for the solution of initial value problems (IVP). In [I] the author discusses the error of Euler's method. In [VII] the author represents the accuracy analysis of initial value problems (IVP) for ordinary differential equations (ODE) using the Euler method, and also in [VIII], the author tries to find out accurate solutions by using the fourth-order Runge Kutta method of initial value problems for ordinary differential equations. Various numerical methods are discussed in [XIV], for solving initial value problems of ordinary differential equations. A detailed explanation can be found in [II], [III], [IV], [V], [VI], [IX], [X], [XI], [XII], [XIII], [XV], [XVI] about the solution of the initial value problems of the ordinary differential equation where numerous numerical methods are discussed.

In this paper, the Runge-Kutta method, Modified Euler's method, and Euler's Method are used to solve ordinary differential equations without any alteration, discretization, or limiting assumptions. Compare to Euler and Modified Euler method, Runge- Kutta method is the best fitted numerical method since it gives consistent initial values and is also very effective when the calculation of higher derivatives is very intricate. In the Runge-Kutta method, it is not required to find the derivatives of the superior order and the results found in the Runge-Kutta method converges closer to the analytical solution, compared to other methods.

II. Formulation of the Problem

For obtaining the approximate solutions of the initial value problem of an ordinary differential equation, we consider three numerical methods consisting of the form

$$y' = f(x, y(x)), \quad x \in (x_0, x_n), \quad y(x_0) = y_0 \quad (1)$$

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Here, $y' = \frac{dy}{dx}$ and $f(x, y(x))$ is a function that is given and the solution of the equation (1) is $y(x)$. A continuous approximation to the solution $y(x)$ will not be found; instead, approximation result to y will be generated at various values, in the interval (x_0, x_n) . Numerical methods employ the equation (1) to obtain the approximations to the values of the solution corresponding to various selected values of $x = x_n = x_0 + nh$, $n = 1, 2, 3, \dots$ and “h” denotes the step size. The solution of equation (1) is given by a set of points $\{(x_n, y_n): n = 0, 1, 2, 3, \dots, n\}$ and each point (x_n, y_n) is an approximation to the corresponding point $(x_n, y(x_n))$ on the curve of the solution.

Euler’s Method

Euler Method is the most elementary approximation procedure for obtaining the result of the initial value problem. In 1768, Euler introduced this method for solving the initial value problem. It is the simplest one-step method. Error analysis can be easily understood by studying this method. Euler Approximation is generally denoted by: $y_{n+1}(x) = y_n(x) + hf(x_n, y_n)$, where $n = 0, 1, 2, \dots$

Modified Euler Method

In this method the curve in the interval (x_0, x_1) where $x_1 = x_0 + h$ is approximated by the line through (x_0, y_0) with the slope $f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))$ which is the slope as the middle point whose abscissa is average of x_0 and x_1 . The generalized Modified Euler method is

$$y_{n+1}(x) = y_n(x) + h f(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n))$$

Runge-Kutta Method

Runge-Kutta Method is the most well-known method because it is stable, the accuracy level is high and so much easy to coding. In 1894, it was first introduced by Runge and then extended by Kutta. Both of them were German mathematicians. The general solution for Runge-Kutta fourth-order method is

$$y_{n+1}(x) = y_n(x) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$
$$n = 0, 1, 2, \dots$$

Where,

$$k_1 = hf(x, y)$$
$$k_2 = hf(x + \frac{h}{2}, y + \frac{k_1}{2})$$
$$k_3 = hf(x + \frac{h}{2}, y + \frac{k_2}{2})$$
$$k_4 = hf(x + h, y + k_3)$$

III. Error Estimation Process

There exist two kinds of Numerical errors namely Truncation errors and Round-off errors. Truncation errors encounter when initial value problems are solved numerically. Round-off error begins from the actuality that computers can only illustrate the numbers by using stable and confined numbers of momentous figures. As a result, these numbers cannot be represented precisely in the memory of the computer. The disproportion initiated by this limitation is called Round-off errors. Truncation errors generate when approximations are used to determine some quantity. Here h represents the step size, and accuracy depends on how small the size of h , has to be taken. Mathematically, a numerical method is said to be convergent if $\log_{h \rightarrow 0} \max_{1 \leq n \leq N} |y(x_n) - y_n| = 0$. where, $y(x_n)$ stands for the approximate solution and y_n stands for the exact solution.

To verify the proposed methods, two Initial Value Problems are considered. The approximate solutions are evaluated for the proposed three methods by using the MATLAB Programming language.

Here the highest amount of error is obtained by $E_r = \log_{h \rightarrow 0} \max_{1 \leq n \leq N} |y(x_n) - y_n|$

IV. Numerical Examples Solved by the Proposed Methods

Here, we consider two numerical examples and solve them to determine which proposed method converges quickly to the exact solution. We also calculate the error for each method and represent the approximate result and error graphically. All of the calculations and graphs are conducted by MATLAB software. Lastly, we use a bar diagram for the easy visualization of the error estimation.

Example 1: Taking the initial value problem $y' = -5y + 5x^2 + 2x$, $y(0) = \frac{1}{3}$ on the interval $0 \leq x \leq 1$. The exact solution is given by $y(x) = x^2 + \frac{1}{3}e^{-5x}$.

The approximate results of the proposed three methods with the exact solution are given in **Table 1**, and the error estimation of the proposed three methods is represented in **Table 2**.

Table 1: Approximate solutions of the proposed three methods

| x_n | Approximate Solutions | | | Exact Solution |
|-------|-----------------------|-------------------------|--|----------------|
| | Euler's Method | Modified Euler's Method | Runge-Kutta 4 th order Method | |
| 0 | 0.333333333333 | 0.333333333333 | 0.333333333333 | 0.333333333333 |
| 0.1 | 0.166666666667 | 0.220833333333 | 0.212282986111 | 0.212176886571 |
| 0.2 | 0.108333333333 | 0.174270833333 | 0.162765457718 | 0.162626480390 |
| 0.3 | 0.114166666667 | 0.176419270833 | 0.164516540751 | 0.164376720049 |
| 0.4 | 0.162083333333 | 0.216512044271 | 0.205240505195 | 0.205111761079 |
| 0.5 | 0.241041666667 | 0.287820027669 | 0.277476660704 | 0.277361666208 |
| 0.6 | 0.345520833333 | 0.386137517293 | 0.376698077980 | 0.376595689456 |
| 0.7 | 0.472760416667 | 0.508835948308 | 0.500157948357 | 0.500065794474 |
| 0.8 | 0.621380208333 | 0.654272467693 | 0.646189588456 | 0.646105212963 |
| 0.9 | 0.790690104167 | 0.821420292308 | 0.813781703412 | 0.813702998846 |
| 1 | 0.980345052083 | 1.009637682692 | 1.002320668998 | 1.002245982333 |

Table 2: Error estimation of the proposed three methods compare to exact solution

| x_n | Approximate Error | | | Exact Solution |
|-------|-------------------|-------------------------|--|----------------|
| | Euler's Method | Modified Euler's Method | Runge-Kutta 4 th order Method | |
| 0 | 0.000000000000 | 0.000000000000 | 0.000000000000 | 0.333333333333 |
| 0.1 | 0.045510219904 | 0.008656446762 | 0.000106099540 | 0.212176886571 |
| 0.2 | 0.054293147057 | 0.011644352943 | 0.000138977328 | 0.162626480390 |
| 0.3 | 0.050210053383 | 0.012042550784 | 0.000139820702 | 0.164376720049 |
| 0.4 | 0.043028427746 | 0.011400283192 | 0.000128744116 | 0.205111761079 |
| 0.5 | 0.036319999541 | 0.010458361461 | 0.000114994496 | 0.277361666208 |
| 0.6 | 0.031074856123 | 0.009541827837 | 0.000102388524 | 0.376595689456 |
| 0.7 | 0.027305377807 | 0.008770153834 | 0.000092153883 | 0.500065794474 |
| 0.8 | 0.024725004630 | 0.008167254730 | 0.000084375494 | 0.646105212963 |
| 0.9 | 0.023012894679 | 0.007717293462 | 0.000078704566 | 0.813702998846 |
| 1 | 0.021900930250 | 0.007391700359 | 0.000074686665 | 1.002245982333 |

Figure 1, represents the approximate solutions of the proposed three methods with the exact solution, and **Figure 2 & Figure 3**, represents the error estimation and comparison of the proposed methods.

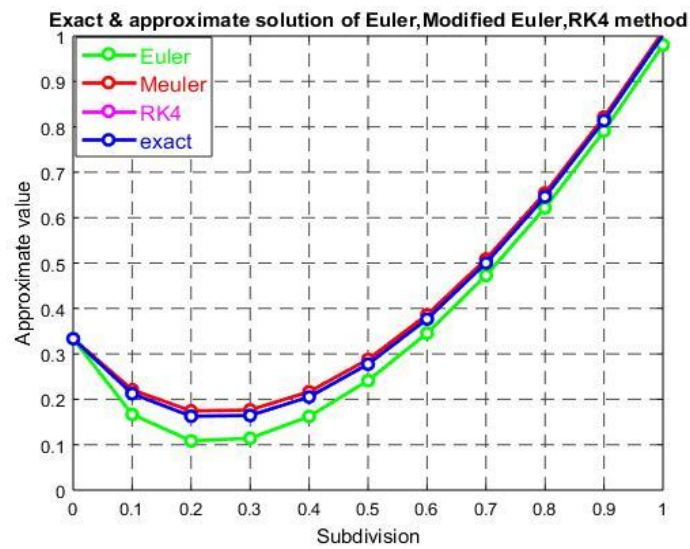


Figure 1: Approximate solution for the proposed three methods with the exact solution

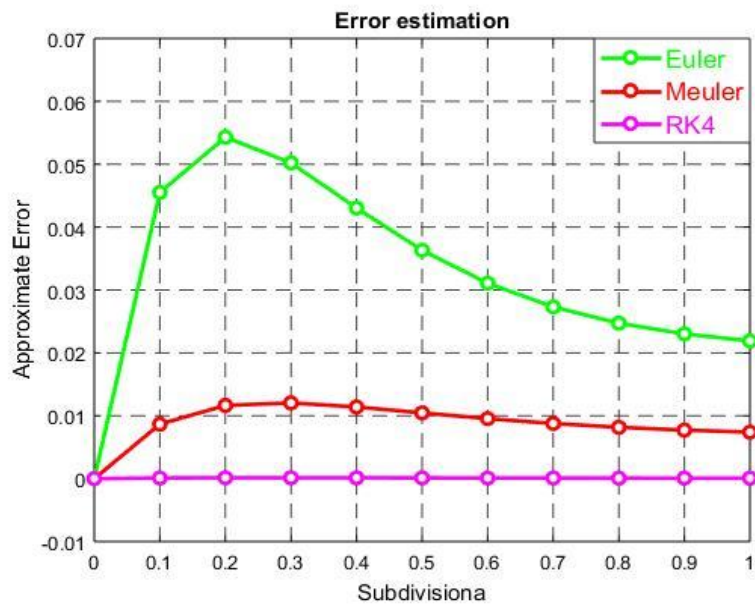


Figure 2: Error graph for the proposed three methods using MATLAB

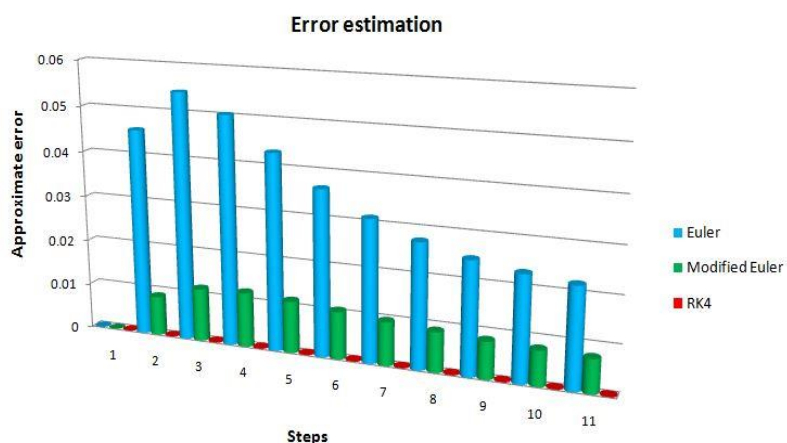


Figure 3: Graphical Error Comparison of the proposed three methods

Example 2: Taking the initial value problem $y' = \frac{2-2xy}{x^2+1}$, $y(0) = 1$ on the interval $0 \leq x \leq 1$. The exact solution is given by $y(x) = \frac{2x+1}{x^2+1}$

The approximate results of the proposed three methods with the exact solution are given in **Table 3**, and the error estimation of the proposed three methods is represented in **Table 4**.

Table 3: Approximate solutions of the proposed three methods

| x_n | Approximate Solutions | | | Exact Solution |
|-------|-----------------------|-------------------------|--|----------------|
| | Euler's Method | Modified Euler's Method | Runge-Kutta 4 th order Method | |
| 0.00 | 1.000000000000 | 1.000000000000 | 1.000000000000 | 1.000000000000 |
| 0.10 | 1.200000000000 | 1.187128712871 | 1.188118764675 | 1.188118811881 |
| 0.20 | 1.374257425743 | 1.344353306237 | 1.346153608558 | 1.346153846154 |
| 0.30 | 1.513709063214 | 1.465526999548 | 1.467889340630 | 1.467889908257 |
| 0.40 | 1.613871867074 | 1.549060650318 | 1.551723173323 | 1.551724137931 |
| 0.50 | 1.674984152103 | 1.597265955399 | 1.599998664190 | 1.600000000000 |
| 0.60 | 1.700985419935 | 1.615015698993 | 1.617645444883 | 1.617647058824 |
| 0.70 | 1.697957294647 | 1.608321183516 | 1.610736481379 | 1.610738255034 |
| 0.80 | 1.672645870988 | 1.583221347337 | 1.585364030205 | 1.585365853659 |
| 0.90 | 1.631412127477 | 1.545108042338 | 1.546959536955 | 1.546961325967 |
| 1.00 | 1.579669484966 | 1.498430678105 | 1.499998300645 | 1.500000000000 |

Table 4: Error estimation of the proposed three methods compare to exact solution

| x_n | Approximate Error | | | Exact Solution |
|-------|-------------------|-------------------------|--|----------------|
| | Euler's Method | Modified Euler's Method | Runge-Kutta 4 th order Method | |
| 0.00 | 0.000000000000 | 0.000000000000 | 0.000000000000 | 1.000000000000 |
| 0.10 | 0.011881188119 | 0.000990099010 | 0.000000047206 | 1.188118811881 |
| 0.20 | 0.028103579589 | 0.001800539917 | 0.000000237596 | 1.346153846154 |
| 0.30 | 0.045819154957 | 0.002362908709 | 0.000000567627 | 1.467889908257 |
| 0.40 | 0.062147729143 | 0.002663487613 | 0.000000964608 | 1.551724137931 |
| 0.50 | 0.074984152103 | 0.002734044601 | 0.000001335810 | 1.600000000000 |
| 0.60 | 0.083338361111 | 0.002631359830 | 0.000001613941 | 1.617647058824 |
| 0.70 | 0.087219039613 | 0.002417071517 | 0.000001773654 | 1.610738255034 |
| 0.80 | 0.087280017330 | 0.002144506322 | 0.000001823453 | 1.585365853659 |
| 0.90 | 0.084450801511 | 0.001853283629 | 0.000001789012 | 1.546961325967 |
| 1.00 | 0.079669484966 | 0.001569321895 | 0.000001699355 | 1.500000000000 |

Figure 4, represents the approximate solutions of the proposed three methods with the exact solution, and **Figure 5** & **Figure 6**, represent the error estimation and comparison of the proposed methods.

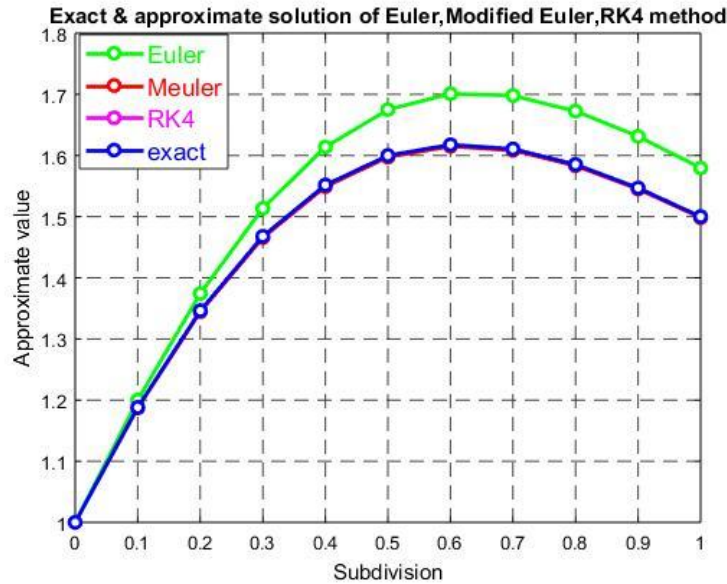


Figure 4: Approximate solution for the proposed three methods with the exact solution

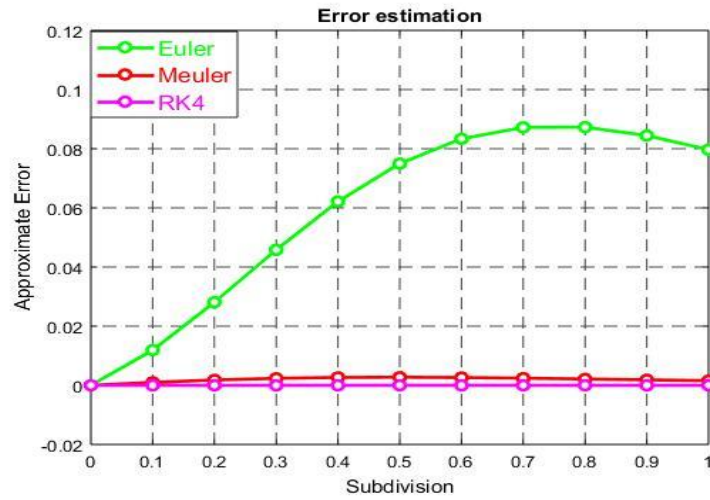


Figure 5: Error graph for the proposed three methods using MATLAB

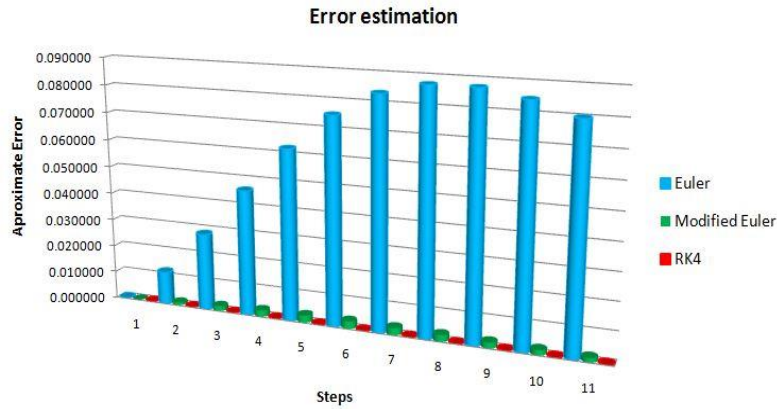


Figure 6: Graphical Error Comparison of the proposed three methods

V. Result Discussion

Table 1 & Table 3 represents the obtained approximate results of the two examples, where Table 2 & Table 4 represents the error analysis of the three methods compare to the exact solution. The approximate results of the two examples are demonstrated in Figure 1 & Figure 4. Besides Figure 2 & Figure 5 represents the error estimation of the proposed three methods graphically for two examples using MATLAB. Again Figure 3 & Figure 6 reveals the comparison of the error estimation of the proposed three methods for two examples using MS Excel. From the result table, we see that the converge rate with the exact solution is more for the Runge-Kutta fourth-order method compares to the Euler method and Modified Euler method. Error graph and Error Bar-Diagram reveals that the amount of error is maximum for the Euler

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method and minimum for Runge-Kutta fourth-order method. In a nutshell, it has been observed that the Runge-Kutta fourth-order method converges accurately and swiftly compare to Modified Euler and Euler method, and it is the most fruitful way for solving the IVPs of ordinary differential equations.

VI. Conclusion

In this study, the Initial Value Problems of ordinary differential equations are solved by the Runge-Kutta method, Modified Euler method, and Euler's method at the same time accuracy level of these proposed methods are also determined. The error table and graph of error which is found from MATLAB Programmed, reveal a clear scenario about the superiority of each method. Though numerical solutions of all the methods are in good agreement with the exact solution, the fourth-order Runge-Kutta method was found to be more reliable and also converged faster to the exact solution compared to other methods. It is presumed that the Runge-Kutta method is quite stable, consistent, convergent, and more accurate than all other proposed methods and it is extensively used in solving IVPs of ordinary differential equations.

Future plan

In future research, we will try to find out the accurate result of the ordinary differential equations (Initial Value Problems) with these proposed methods by taking the different step size which is very much small.

Conflict of Interest

Authors guarantee that in this article, none of the authors have any contest of interest.

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