



A NEW QUADRATURE-BASED ITERATIVE METHOD FOR SCALAR NONLINEAR EQUATIONS

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Abstract

Nonlinear equations and their efficient numerical solution is a fundamental issue in the field of research in mathematics because nature is full of nonlinear models demanding careful solution and consideration. In this work, a new two-step iterative method for solving nonlinear equations has been developed by using a quadrature formula so that the cost of evaluations is considerably reduced. The proposed strategy successfully removes the use of an additional derivative in an existing method in literature so that there is no compromise at all on the cubic convergence rate. The developed scheme is cubically convergent and uses a functional and three derivative evaluations only as compared to some other methods in the literature using much higher evaluations. The theorems concerning the derivation of the proposed method and its third order of convergence have been discussed with proof. Performance evaluation of the new proposed scheme has been discussed with some methods from literature including well-known traditional methods. An exhaustive numerical verification has been done under the same numerical conditions on ten examples from the literature. The efficiency index is found to be higher for the new proposed scheme than some schemes with orders more than three, and comparable with some methods. The comparison using observed absolute errors, number of iterations, functional and derivative evaluations, and observed convergence reveals that the proposed method finds the solutions quickly and with lesser computational cost as compared to most of the other methods used in the comparison. The results show the encouraging performance of the proposed method.

Sehrish Umar et al

Keywords: Cost-efficient, Quadrature, Nonlinear equations, Order of convergence, Efficiency index.

I. Introduction

Computational mathematics deals with the proposition of numerical schemes that can efficiently solve complex problems arising from physical case studies in different areas of science and engineering [VIII], [X], [XIV] including the Colebrook's equation [XIV], [XXIII], [XXIV]. When the equation describing the relationship between the variables becomes nonlinear with respect to the unknown variable, then the solution of such nonlinear equation gives rise to the motivation of nonlinear solvers for finding the roots [XIII], [XIII], [V]. Although, there can be more than one variable in such implicit relationships, with the objective to compute one of these when others are specified. But a simple case of such problems can be considered with one variable in the nonlinear relationship. Methods for solving such one variable nonlinear equations form the basis to propose extended schemes for solving multivariable nonlinear equations. The general form of a nonlinear equation in one variable, say x , is:

$$f(x) = 0 \quad (1)$$

The polynomials equations of degree at most 2 can easily be solved by analytical methods. For higher order polynomial equations, besides the Cordano and Ferrari methods for cubic and bi-quadratic equations respectively, if one root can be guessed by hit and trial process then others can be attained using synthetic division method. On the other hand, when the above conditions are not followed then, particularly for polynomials of degree higher than two, numerical methods are used. Similarly, for transcendental equations that cannot be solved analytically, there is a need for efficient numerical methods. But the numerical methods need to be convergent and stable for wider use and applicability [VIII],[X],[XIV],[XXIII],[XXIV].

Finding the root of a nonlinear equation is a very important aspect for researchers in mathematics and practitioner engineers and scientists working in different domains of knowledge [XXVII]. In recent years, many modifications in iterative methods like Bisection, Regula-Falsi and Newton's method have been proposed in the literature, which has either equal or better performance than these methods. In [XXVI], [IX] authors demonstrated the effectiveness of the secant method for nonlinear equations in comparison with other methods. However, the simplest and oldest root-finding algorithm is the Bisection method.

Modifications of the Newton-Raphson [IV],[XXI],[III],[XVII] method have mostly been derived using numerical integration approaches and also by truncating Taylor's series. The error analysis is usually conducted using Taylor's series and fixed point theory. A second-order Newton-like method was proposed in [XXII], whereas [VI],[VII] discussed some biquadratic order methods for nonlinear equations. In [XIX], Noor et al. proposed two multi-step methods of third and fourth-order convergence, respectively; but with a lot of computational costs in form of functional and derivative evaluations. Meanwhile, in [XV] a higher order method was suggested. Similarly, [XI] and [II], respectively, considered and promoted the use of

Shrish Umar et al

derivative-free and time efficient methods for nonlinear equations and systems. Kang et al. in [XXV] proposed a third order method better than the corresponding third order method in [XIX]. A second order method by Noor et al. in [XVII] was recommended because it did not involve higher order derivatives. The method of Hafiz and Bhagat [XV] was also a cubically convergent method better than the similar method in [XIX].

In this work, we construct and propose a new nonlinear solver based on the quadrature rule for solving scalar nonlinear equations, which appears to be an enhancement of the method of Noor et al. [XIX]. Besides being simpler and easy-to-use, the proposed method takes lesser evaluations to achieve the same accuracy as by the Noor et al. method [XIX]. The cost-effectiveness of the proposed method is the main advantage. The proposed method has been compared with many old methods and also some recent methods from literature. We demonstrate using all possible computational test procedures that the proposed method is a cost-efficient modification of the method of Noor et al. [XIX]. For the sake of conformity of the theoretical results about the convergence behaviour and lesser cost of the proposed method with high efficiency, a set of nonlinear equations from literature has been solved by proposed and other methods. The results have been obtained in high precision arithmetic using MATLAB software. The comparison of absolute error distributions, observed orders of convergence, the number of iterations and computational cost to achieve some specified error tolerance have been examined for all methods, and the results confirm the suitability of the proposed method.

II. Material and methods

The general form into which a scalar nonlinear equation in the unknown variable x can best be described is: $f(x) = 0$. The left hand side function can be a polynomial of degree at least 3, purely a transcendental relation or a mixture of both. Nonlinear solvers try to find the approximate solution in a series of steps depending on each other, called iterations so that the iterative solutions by numerical methods approach to the exact solution as iterations advance. Here, a few existing nonlinear solvers in this section along with the derivation and convergence proof of the proposed method.

Some Existing Nonlinear Solvers

The very first method appearing in the literature of iterative nature is the Bisection method (BM) [XIII],[XXVI]. It finds the approximations to root of $f(x) = 0$ using the mid-point of the interval $[a, b]$ containing the root. The BM is always convergent, but it inclines to slow, for discovering the root of an equation.

For an equation, $f(x) = 0$, where $f(x)$ is continuous in the interval $[a, b]$ containing its root such that $f(a).f(b) < 0$ then the approximate root by BM in $[a, b]$ is $(a+b)/2$. The process is continued until the required accuracy is achieved. The order of convergence of BM is one. For an interval containing more than one root of the equation, the BM converges to only one root at a time. However, to find all roots BM must be applied in lower length intervals containing one root at a time.

A faster method than the BM is the Regula-Falsi method (RFM) [XXVI],[IX],[XX]. Most nonlinear equation solving techniques commonly converge faster than BM, and the RFM is one of those. RFM works the same as BM, but generally faster than it, and also sometimes slower than BM when $f(x)$ is a flatter curve around X-axis. Under the assumptions of BM, the RFM approximates the root at the root of the secant line joining the ordinates of the ending points of $[a, b]$. The order of convergence of RFM is also one, but the speed of convergence is higher than BM. Both the BM and RFM are modest in a way that these do not contain derivatives.

The Newton-Raphson method (NRM) [IV],[XXI],[III] is a classical method because of its quick convergence for approximating the root. This method finds successive estimation to the root of the nonlinear equation by estimating and using the first order derivative at several points. The NRM method uses one functional and derivative evaluation and converges quadratically. The NRM scheme is given in equation (1) to find $(n + 1)^{\text{th}}$ approximation of the root through n^{th} approximation. The NRM method is an open method that can be initiated with a starting point x_0 , called an initial guess.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \dots \quad (2)$$

The derivative-term in the denominator of (1) must be nonzero, i.e. the tangent drawn at any iterate x_n must not be horizontal; otherwise the NRM fails.

In 2010, Noor et al. [XVIII] developed a method, say Noor1, which was cubically convergent. Noor1 requires two functional and three derivative evaluations in an iteration. The two steps of Noor1 scheme are described as equations (3) and (4).

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (3)$$

$$x_{n+1} = y_n - \frac{4f(y_n)}{f'(x_n) + 2f'\left(\frac{x_n + y_n}{2}\right) + f'(y_n)}, n = 0, 1, 2, \dots \quad (4)$$

Proposed Method

Using the background of a recently proposed nonlinear solver in [XIX] by Noor et al., we propose an improved method using the quadrature rule and fundamental theorem of integral calculus to solve the scalar nonlinear equations with the lesser computational cost. The first step of the proposed method is the usual NRM. The second step of the proposed cubically convergent method (PM) is given as in (5).

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 2f'\left(\frac{x_n + y_n}{2}\right) + f'(y_n)}, n = 0, 1, 2, \dots \quad (5)$$

The PM is an enhancement of Noor1 method in [XVIII]. The PM uses one functional and three derivative evaluations, i.e. 4 in total instead of 5 as in the method Noor1 [XIX].

The derivation of PM is discussed in the following theorem – Theorem 1.

Theorem 1. *If $f(x) = 0$ is a nonlinear equation to be solved, then the proposed quadrature-based method is defined in equation (5).*

Sehrish Umar et al

Proof of Theorem 1.

The scalar nonlinear equation is:

$$f(x) = 0 \quad (6)$$

Using the combination of midpoint and Trapezoidal rules, equation (6) can be written as

$$f(x) = f(r) + \frac{h}{2} \left[f'(r) + 4f' \left(\frac{(x+r)}{2} \right) + f'(x) \right] \quad (7)$$

If $f(x) = 0$, then (7) becomes

$$f(r) + \frac{(x-r)}{4} \left[f'(r) + 2f' \left(\frac{(x+r)}{2} \right) + f'(x) \right] = 0 \quad (8)$$

Equation (8) can also be written as,

$$x = r - \frac{4f(r)}{f'(r) + 2f' \left(\frac{(x+r)}{2} \right) + f'(x)} \quad (9)$$

Using (9), and by taking $r = x_n$ and $x = x_{n+1}$ the following scheme based on quadrature (7) can be used.

$$x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 2f' \left(\frac{(x_n + x_{n+1})}{2} \right) + f'(x_{n+1})}$$

Where we consider $x_{n+1} = y_n = \text{NRM}$ on the right hand side to remove implicitness, and thus

$$x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 2f' \left(\frac{(x_n + y_n)}{2} \right) + f'(y_n)} \quad \text{for } n = 0, 1, 2, \dots$$

which is the same as the proposed method in equation (5).

The PM method is cubically convergent, and the proof is included below as in Theorem 2.

Theorem 2. Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ for an open interval I . If x_0 is sufficiently close to α , then the two-step iterative methods, i.e. equation (5) has third order of convergence.

Proof of Theorem 2.

Let α be a simple zero of f . Expanding $f(x_n)$ and $f'(x_n)$ in Taylor's Series about α , gives:

$$f(x_n) = f'(\alpha)(e_n + c_2 e_n^2 + c_3 e_n^3 + \dots) \quad (10)$$

$$f'(x_n) = f'(\alpha)(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \dots) \quad (11)$$

Where

$$c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}, \quad k = 2, 3, 4, \dots \text{ and } e_n = x_n - \alpha$$

Shrisha Umar et al

From(10) and(11), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3) e_n^3 + \dots \quad (12)$$

From (12), we get

$$y_n = c_2 e_n^2 - 2(c_2^2 - c_3) e_n^3 + \dots \quad (13a)$$

Expanding $f(y_n)$, $f'(y_n)$ and $f'\left(\frac{x_n+y_n}{2}\right)$ as in (10)-(11) about α , and thus using (8), gives:

$$f(y_n) = f'(a)[c_2 e_n^2 + 2(c_3 - c_2^2) e_n^3 + \dots] \quad (13B)$$

$$f'(y_n) = f'(a)[1 + 2c_2^2 e_n^2 + 4(c_2 c_3 - c_3^2) e_n^3 + \dots] \quad (14)$$

$$f'\left(\frac{x_n+y_n}{2}\right) = f'(a)\left[1 + c_2 e_n + \left(c_2^2 + \frac{3}{4}c_3\right) e_n^2 + \left(\frac{7}{2}c_2 c_3 - 2c_3^2\right) e_n^3 + \dots\right] \quad (15)$$

From(11), (14), and(15), we have

$$f'(y_n) + 2f'\left(\frac{x_n+y_n}{2}\right) + f'(x_n) = f'(a) \left[\frac{4 + 4c_2 e_n + \left(4c_2^2 + \frac{9}{2}c_3\right) e_n^2}{+(11c_2 c_3 + 5c_4 - 8c_3^2) e_n^3 + \dots} \right] \quad (16)$$

From (10) and (16), we obtain

$$\begin{aligned} & \frac{4f(x_n)}{f'(x_n) + 2f'\left(\frac{x_n+y_n}{2}\right) + f'(y_n)} \\ &= \frac{4f'(a)[e_n + c_2 e_n^2 + c_3 e_n^3 + \dots]}{4f'(a) \left[1 + c_2 e_n + \left(c_2^2 + \frac{9}{8}c_3\right) e_n^2 + \left(\frac{11}{4}c_2 c_3 + \frac{5}{4}c_4 - 2c_3^2\right) e_n^3 + \dots \right]} \end{aligned} \quad (17)$$

$$\begin{aligned} & \frac{4f(x_n)}{f'(x_n) + 2f'\left(\frac{x_n+y_n}{2}\right) + f'(y_n)} \\ &= e_n \left[1 - \left(2c_2^2 + \frac{17}{8}c_3\right) e_n^2 + \left(\frac{23}{8}c_2 c_3 + \frac{5}{4}c_4 - 3c_3^2\right) e_n^3 + \dots \right] \end{aligned} \quad (18)$$

Substituting (18) in (5), we obtain

$$e_{n+1} = e_n - e_n \left[1 - \left(2c_2^2 + \frac{17}{8}c_3\right) e_n^2 + \left(\frac{23}{8}c_2 c_3 + \frac{5}{4}c_4 - 3c_3^2\right) e_n^3 + \dots \right]$$

$$e_{n+1} = e_n - e_n + \left(2c_2^2 + \frac{17}{8}c_3\right) e_n^3 - \left(\frac{23}{8}c_2 c_3 + \frac{5}{4}c_4 - 3c_3^2\right) e_n^4 + \dots$$

$$e_{n+1} = \left(2c_2^2 + \frac{17}{8}c_3\right) e_n^3 - \left(\frac{23}{8}c_2 c_3 + \frac{5}{4}c_4 - 3c_3^2\right) e_n^4 + \dots$$

$$e_{n+1} = \left(2c_2^2 + \frac{17}{8}c_3\right) e_n^3 + O(e_n^4)$$

Hence, the third order convergence of PM has been established

Sehrish Umar et al

The formula for the efficiency index is $E = p^{1/m}$, for a p^{th} order convergent method using m functional and derivative evaluations in all in an iteration [II]. In this context, the efficiency index of the PM for scalar nonlinear equations is 1.3160740129. The efficiency indices of PM and some other methods are given in Table 1.

The PM method takes fewer evaluations than the method of Noor1 without compromising upon the third order convergence. Similarly, the evaluations are smaller than the fourth order convergent method, Noor2. The efficiency indices from Table 1 show that the proposed method is efficient than other lower and higher order methods. While the efficiency index of NRM is higher than PM, the former is a lower order method as compared to PM. Moreover, the exhaustive comparison of the computational performance of PM and other methods is carried out in the forthcoming sections, where the ascendancy of PM over other accompanying methods will be established.

Table 1: Efficiency indices for non-linear equation and systems of the discussed methods.

Methods	Order of convergence	Total evaluations in an iteration	Efficiency index
BM	1	1	1
RFM	1	1	1
NRM	2	2	1.4142135623
Noor1 [XVIII]	3	5	1.2457309396
Noor2 [XVIII]	4	8	1.1892071150
Noor3 [XVII]	2	3	1.2599210498
PM (present)	3	4	1.3160740129

III. Numerical Experiments

For performance evaluation of PM and some existing methods, some scalar nonlinear equations have been taken from the literature [XVII], [XXI], [VI], [VII], [XIX], [XV], [XI], [II], [XXV] which are listed in Table 2 along with test conditions. We have used high precision arithmetic of 1000 digits to compare the results and numerical properties of different methods for sufficiently reduced errors, i.e. upto 10^{-50} or lower. In Table 2, the test functions of the considered nonlinear equations, corresponding intervals containing a real root, the used initial guesses and the exact root to 20 decimal places' have been mentioned for reference to conduct and replicate numerical experiments.

For comparison, we use the following formulae of errors and observed order of convergence from [XVII], [XXII], [VI], [VII], [XIX], [XV], [XI], [II].

$$\text{Absolute Error (AE)} = |\text{Approximate solution} - \text{Exact solution}| \quad (19)$$

If exact solution is not known, then:

$$\text{Absolute Error (A.E.)}_i = |x_i - x_{i-1}|, i=1,2,\dots \quad (20)$$

The observed order of convergence (P) [XXII], [VI], [VII], [XIX], [XV], [XI], [II],[XXV],[XVIII] can be obtained as:

$$P = \frac{\ln(\frac{e_{n+1}}{e_n})}{\ln(\frac{e_n}{e_{n-1}})} \quad (21)$$

Table 2: Numerical examples 1-10 and test conditions

Test functions	f(x)	Interval containing a real root	Initial guess used (x ₀)	Exact root upto 20 decimal places'
$f_1(x)$	$3x + \sin(x) - \exp(x)$	[0,1]	0.5	0.36042170296032440136
$f_2(x)$	$\cos(x) - x \exp(x)$	[0,1]	0.5	0.51775736368245829832
$f_3(x)$	$3x + \cos(x) - 1$	[0,1]	0.5	0.60710164810312263122
$f_4(x)$	$\ln x + \sqrt{x} - 5$	[8,9]	8.5	8.30943269423157179534
$f_5(x)$	$\cos(x) - x$	[0,1]	0.5	0.73908513321516064165
$f_6(x)$	$\sin^2(x) - x^2 + 1$	[-2,-1]	-1.5	-1.4044916482153412260
$f_7(x)$	$x^2 - e^x - 3x + 2$	[0,1]	0.5	0.25753028543986076045
$f_8(x)$	$\sqrt{x} - \cos x$	[0,1]	0.5	0.64171437087288265839
$f_9(x)$	$2x - \ln(x) - 7$	[3,5]	4	4.21990648378038144162
$f_{10}(x)$	$\exp(x) - 5(x)$	[0,0.5]	0.25	0.25917110181907374505

IV. Results and Discussion

The PM has been compared with some existing methods: BM, RFM and NRM In Fig. 1, the absolute error distributions for the first few iterations using (20) are shown for all methods for Examples 1-10. The absolute errors are represented by the vertical axis and have been represented in the logarithmic scale with the reversed direction in a way that higher marks in the graph intend to show smaller errors. From Fig. 1, it is clear that the PM at the same iterations exhibits smaller errors than the BM, RFM and NRM methods. Table 3 describes the number of iteration (I), computational order (P) of convergence, absolute errors (AE) and the number of evaluations (EVAL) to attain an absolute error of atmost 10^{-50} for all methods. It is

Sehrish Umar et al

clear from Table 3 that all methods including the PM converge to the theoretical orders of convergence as mentioned in Table 1, and that the number of iterations taken by the PM is lower than BM, RFM and NRM for all examples, as also shown collectively in Fig. 2. It should be noted that the number of evaluations for the PM are sufficiently smaller than those of BM and RFM as shown in Fig. 3 and Table 3 example-wise, whereas in comparison to the NRM the evaluations of the PM are either comparable or a bit more, nevertheless the PM method is better than NRM because of the theoretical order of convergence, which is 3 for PM and 2 for NRM.

The main advantage of PM lies on the fact that it uses only first-order derivative instead of higher ones to approximate the solution of nonlinear equations. From Table 1, we observe that the efficiency indices of BM, RFM, Noor1, Noor2 and Noor3 were higher than PM. The rapid decrease in errors in Fig. 1, lower iterations in Fig. 2, and lesser evaluations as in Fig. 3 than most of the methods are the main characteristics of the PM justifying its cost effectiveness over other methods with only first derivative information instead of higher order derivatives usually used in other methods in the literature. Also, for higher order nonlinear systems the PM method will not contribute much cost than most of the other methods which can be tested in the future.

V. Conclusion

This work has been devoted to the proposition of a cubically convergent quadrature-based iterative method to solve scalar nonlinear equations. The proposed method is a two-step scheme. The proposed method was derived using the quadrature approach, and its third order of convergence was proved. An exhaustive comparison of the new method with several traditional methods from the literature on ten numerical examples under similar conditions was made. The superiority of the proposed method was demonstrated over others, especially the main feature being its cost efficiency as it used lesser evaluations to achieve similar accuracy as compared to other methods. The proposed method can be used as a replacement of some methods contributing a lot of computational burdens to yield the required accuracy.

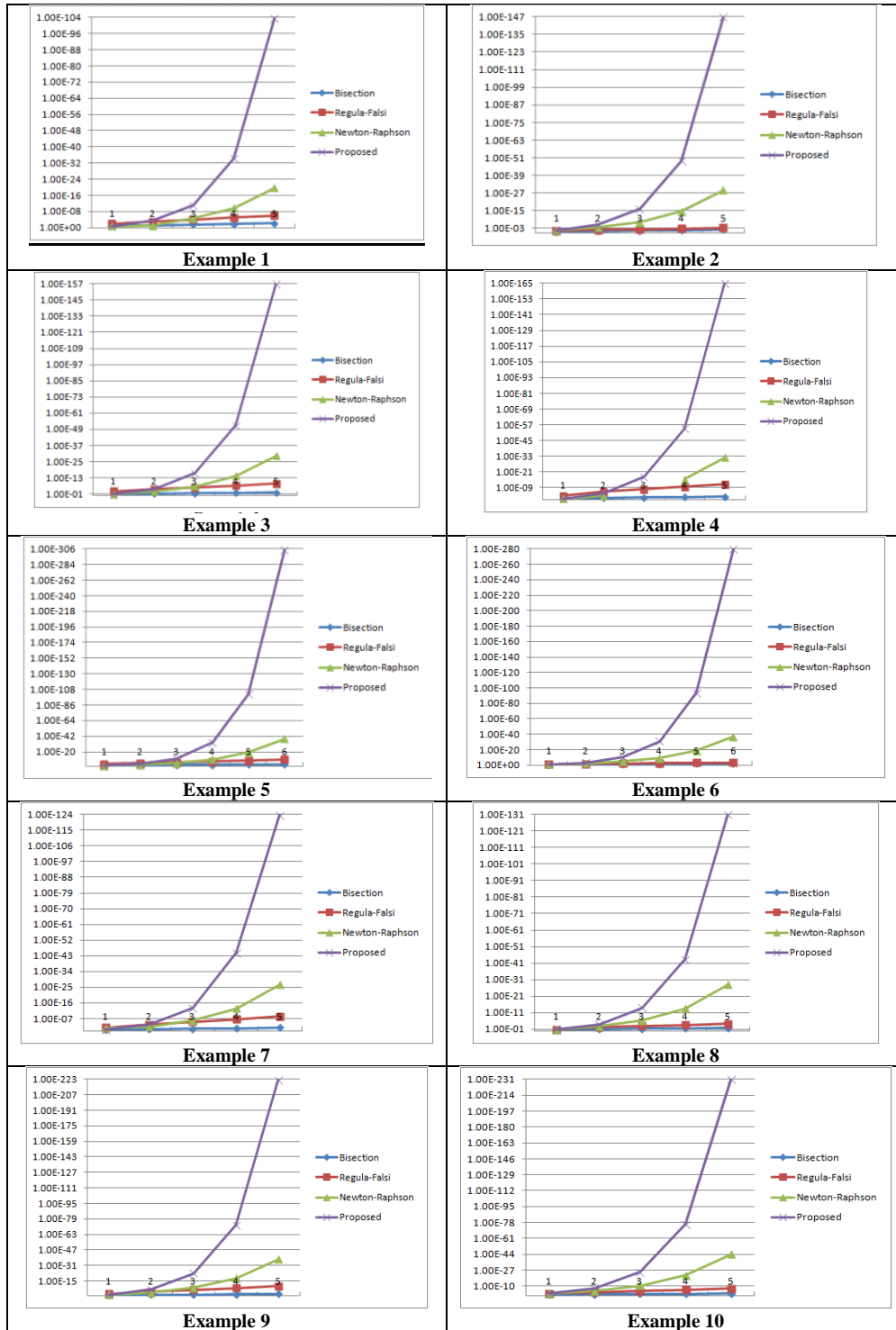


Fig 1. Absolute errors in Examples 1-10 for first few iterations (Along X-axis are number of iterations and along Y-axis are Absolute errors in reversed logarithm scale)

Sehrish Umar et al

Table 3. Comparison of the computational performance in Examples 1-10 for $AE \leq 10^{-50}$.

$f_i, (x_0)$		BM	RFM	NRM	PM
$f1$	I	165	50	7	5
	P	1.04965	0.9999	2.00214	3.10020
	A.E	2.6E-50	1.9E-50	1.79E-80	2.14E-104
	EVAL	166	51	14	20
$f2$	I	161	102	6	4
	P	1.03991	0.92758	1.97621	3.01755
	A.E	2.1E-50	6.1E-50	2.19E-117	1.70E-147
	EVAL	162	103	12	16
$f3$	I	164	36	6	4
	P	1.04965	1.03078	2.04134	2.94945
	A.E	3.0E-50	1.3E-50	2.13E-60	2.21E-52
	EVAL	165	37	12	16
$f4$	I	160	27	6	4
	P	0.90080	0.99490	2.05720	3.09198
	A.E	7.9E-50	1.3E-50	2.16E-66	7.9E-55
	EVAL	161	28	12	16
$f5$	I	164	38	7	5
	P	0.90080	0.99490	2.05720	3.09198
	A.E	5.4E-50	7.1E-50	1.5E-78	6.1E-99
	EVAL	165	39	14	20
$f6$	I	163	99	7	5
	P	0.9999	1.01476	2.00003	3.09198
	A.E	6.2E-50	1.3E-49	1.8E-74	7.8E-94
	EVAL	164	100	14	20
$f7$	I	164	33	6	5
	P	0.9999	0.9999	1.92852	3.00144
	A.E	1.0E-50	1.0E50	8.4E-55	2.9E-124
	EVAL	165	34	12	20
$f8$	I	163	73	6	5
	P	1.00148	0.993452	1.99986	3.000023
	A.E	4.7E-50	4.8E-50	2.14E-57	1.57E-131
	EVAL	164	74	12	20
$f9$	I	162	26	6	4
	P	0.948469	1.01630	2.00033	2.95437
	A.E	2.8E-50	2.0E-50	5.7E-77	2.1E-73
	EVAL	163	27	12	16
$f10$	I	163	37	6	4
	P	0.948469	0.98587	1.99994	3.04071
	A.E	4.2E-50	1.0E-51	1.8E-89	5.0E-77
	EVAL	164	38	12	16

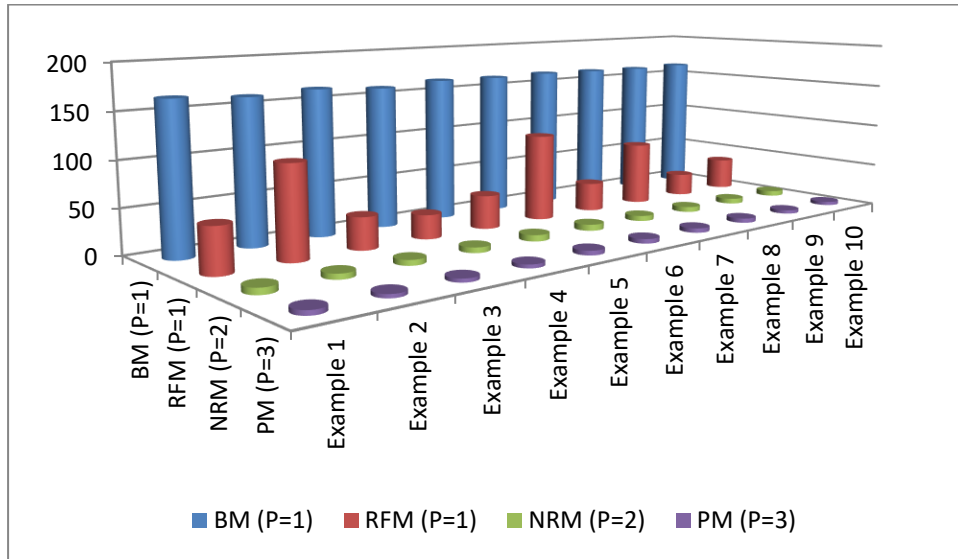


Fig. 2. Number of iterations for Examples 1-10.

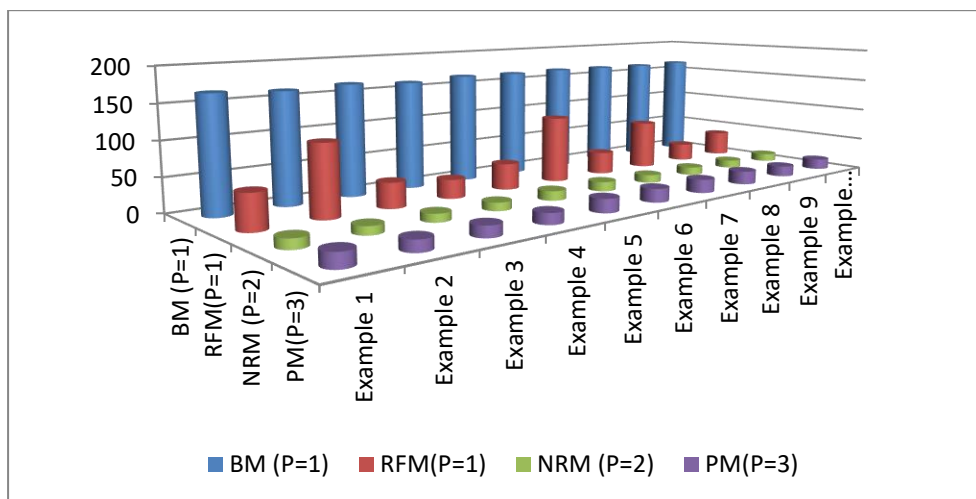


Fig. 3. Number of evaluations for Examples 1-10.

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Conflict of Interest :

No conflict of interest regarding this article

Sehrish Umar et al

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