



A NEW THIRD-ORDER DERIVATIVE-BASED ITERATIVE METHOD FOR NONLINEAR EQUATIONS

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Abstract

In this study, a new derivative-based cubically convergent iterative method is established for nonlinear equations, which is a modification of an existing method. The idea of difference quotient is used to arrive at a better formula than the existing one. The theorem concerning the order of convergence has been proved theoretically. Some examples of nonlinear equations have been solved to analyse convergence and competence of the PM against existing methods. High precision arithmetic has been used and graphs have been plotted using Ms Excel. Using standard test parameters: efficiency index, absolute error distributions, observed order of convergence, number of iterations and number of evaluations, the PM is compared against the existing methods, and is found to be a cost-efficient alternative with the higher order of convergence. From results, it has been detected that established technique is superior to the widely used Bisection (BM), Regula-Falsi (RFM) and Newton-Raphson (NRM) methods from iterations and accuracy perspectives. Moreover, the proposed method (PM) is cost-efficient than the original method used for modification as well as some other methods.

Keywords: Convergence, Efficiency, Nonlinear equation, Derivative-based, Precision.

I. Introduction

Nonlinear equations arise from the modelling of various disciplines. The exact solutions of nonlinear equations are not always possible to find out, so, mostly numerical methods are used. It is very important for a numerical method to be higher order accurate and cost efficient by using lesser number of iterations and evaluations.

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A huge number of problems arise in science and engineering [XII], [V], [IX] which require solving nonlinear equations of the general form:

$$f(x) = 0 \quad (1)$$

For example, in the load flow problems associated with the electrical power systems [XII], the equations for the unknown voltage buses are nonlinear, and demand numerical methods for the efficient solution that is helpful in full resolution studies of the power system stability and control [XXI]. Similarly, in studying the dynamics of fluid flow inside a pipe, on the basis of pipe roughness and Reynolds number associated with the fluid flow inside, often highly turbulent regimen have arrived in the rough pipes leading to the friction factor. This friction factor – also known as Darcy friction factor – is modelled in form of a nonlinear equation involving the implicit appearance of the friction factor [XXIII], [XXIV].

Using the mathematical modelling techniques, we often get nonlinear equations, and their solution is one of the most essential purposes to use the numerical analysis. Optimization problems are resolved using the numerical analysis, where it provides the optimal answer to the given problem. Due to having a similar repetitive structure, these solutions can be presented and inquired in a general framework. In numerical analysis, the ancient method for the nonlinear equations is the Bisection method (BM), which works when two initial guesses a and b are known bracketing the root, such that $f(a)$ and $f(b)$ are opposite signed numbers. Similarly, the Regula-Falsi method (RFM) is another method in literature for solving nonlinear equations, but RFM is more efficient and accurate than the BM [XII], [V]. Some improvements to the BM method have been discussed in [V], [XXVI].

On the other hand, one of the most effective methods is the Newton-Raphson method (NRM). The NRM is fast converging and has a quadratic rate of convergence. NRM uses two evaluations: one is $f(x)$ and another is $f'(x)$, and has the form :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots \quad (2)$$

However, the NRM has a pitfall [III] that the method suffers if the iterated approximations are in the vicinity of the singular points of $f(x)$ in (1). Nevertheless, NRM is the most useful and vigorous numerical technique to obtain the only solution of a nonlinear equation in the earlier times. In literature, there are several modifications to the methods including the NRM. In this zeal, in [III], [IV], [VIII], numerous iterated techniques were proposed for convergence and efficiency purpose by using different procedures such as Taylor series, quadrature formula, variational iteration method, and decomposition method [XV], [VII], [XXV], [XXI].

In 2010 [XVII], Noor et al. proposed two algorithms of third and fourth order convergence respectively, named as Noor1 and Noor2 here, and defined through (3)-(4) and (5)-(7) respectively.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (3)$$

$$x_{n+1} = y_n - \frac{4f(y_n)}{f'(x_n) + 2f'\left(\frac{x_n + y_n}{2}\right) + f'(y_n)} \quad (4)$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (5)$$

$$z_n = y_n - \frac{4f(y_n)}{f'(x_n) + 2f'\left(\frac{x_n + y_n}{2}\right) + f'(y_n)} \quad (6)$$

$$x_{n+1} = z_n - \frac{4f(z_n)}{f'(x_n) + 2f'\left(\frac{x_n + z_n}{2}\right) + f'(z_n)} \quad (7)$$

Similarly, in 2012 Noor et al. [XVII] proposed another method using first-order derivatives as a two-step algorithm, which we call Noor3 and is defined in (8)-(9).

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (8)$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)} - \frac{(y_n - x_n)f'(y_n)}{f'(x_n)} \quad (9)$$

From the algorithms of Noor1, Noor2 and Noor3, it appears that so many evaluations of the function and derivative have been used to achieve a lower order of accuracy, hence compromising too much on the efficiency index. While on other hand, some researchers [II],[XIX],[XI],[XIV] have efficiently utilized the forthcoming terms of the Taylor series to acquire higher order convergence while not compromising too much upon the efficiency index. The main characteristics of the sixth order method in [II] by Abro and Shaikh in 2019 have been the time and cost efficiency. Similarly, in [X],[XVI],[XXVII], few derivative-free approaches were discussed for improving the classical NRM. Receiving motivations from [II],[XIX],[XI],[XIV] studies, we have attempted to propose another iterative method based on derivatives to efficiently solve the nonlinear equations.

In this work, a modified method for solving nonlinear equations is suggested with an objective to reduce the computational cost with compromising upon the order of convergence of an existing method [XVIII] from literature. The proposed method (PM) is a derivative-based method, as it also utilizes tangent slopes at the iterates along with the functional evaluations. The PM is a cubically convergent method. The theorem concerning the order of convergence is proved, and several numerical experiments have been conducted to demonstrate the computational performance of the PM against other methods from literature using standard comparison parameters. It is expected that the PM will result in better accuracy and lesser errors than the NRM and others. Hence, the PM can be used for efficient and accurate solution of nonlinear equations.

II. The Proposed Method (PM)

We use $f(x) = 0$ to state the one variable nonlinear equation. Here, we take x as the quantity needed which may be related in $f(x)$ in the form of polynomials and transcendental equations on the left side. As the nonlinear equations for i cannot be solved in direct way mostly, so for the generation of sequences approximation solution that converges to the real solution, we use the numerical methods, and thus the proposed method can also be used.

The PM is established by modifying the Noor1 method [XVIII]. As a result, a new third-order convergent method is derived, the PM. The PM uses four evaluations: two

functions and two derivatives, instead of five as in the Noor 1 [XVIII]. PM algorithm can be described as in (10)-(11)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (10)$$

$$x_{n+1} = y_n - \frac{4(y_n - x_n)f(y_n)}{(y_n - x_n)[f'(x_n) + f'(y_n)] + 2[f(y_n) - f(x_n)]} \quad (11)$$

Where $n = 0, 1, 2, \dots$

The PM is established by using the finite difference quotient approximation of the mid-point derivative in the Noor1 method (4) as:

$$f' \left(\frac{x_n + y_n}{2} \right) = \frac{f(y_n) - f(x_n)}{2 \left\{ \left(\frac{y_i - x_i}{2} \right) \right\}} \quad (12)$$

Simplifying (12), we have:

$$f' \left(\frac{x_n + y_n}{2} \right) = \frac{f(y_n) - f(x_n)}{y_i - x_i} \quad (13)$$

Using (13) in (4) gives:

$$x_{n+1} = y_n - \frac{4f(y_n)}{f'(x_n) + 2 \left\{ \frac{f(y_n) - f(x_n)}{y_i - x_i} \right\} + f'(y_n)} \quad (14)$$

Simplifying (14) successively, we have:

$$x_{n+1} = y_n - \frac{4f(y_n)}{\frac{(y_i - x_i)f'(x_n) + 2f(y_n) - 2f(x_n) + f'(y_n)(y_i - x_i)}{y_i - x_i}} \quad (15)$$

$$\text{Or, } x_{n+1} = y_n - \frac{4(y_n - x_n)f(y_n)}{(y_n - x_n)f'(x_n) + 2f(y_n) - 2f(x_n) + f'(y_n)(y_n - x_n)} \quad (16)$$

$$\text{Finally, } x_{n+1} = y_n - \frac{4(y_n - x_n)f(y_n)}{(y_n - x_n)[f'(x_n) + f'(y_n)] + 2[f(y_n) - f(x_n)]} \quad (17)$$

$$\text{Where, } y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (18)$$

Hence, (17)-(18) define the algorithm of the PM for solving nonlinear equations.

We, now prove the theorem concerning the cubic order of convergence of the PM, and also derive the leading error term of the PM in Theorem 1.

Theorem 1. Let $\alpha \in Q$ be the zero of a differentiable function $f: Q \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval Q . Then, the three-step iterative method PM, i.e. equation (17) has third order convergence, and the consequent leading error term is: $(-c_2 - 6c_2^2 + 8c_3^2)e_n^3$ where, $e_i = x_i - \alpha$.

Proof of Theorem 1.

Let α be a simple zero of f . Then, by expanding $f(x_n)$ and $f'(x_n)$ in Taylor's series about α , we have

$$f(x_n) = f(\alpha)(e_n + c_2 e_n^2 + c_3 e_n^3 + \dots) \quad (19)$$

$$f'(x_n) = f'(\alpha)(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \dots) \quad (20)$$

By using $c_k = \frac{f^{(k)}(\alpha)}{k!f^{(k-1)}(\alpha)}$, for $k = 2, 3, 4, \dots$ and $e_n = x_n - \alpha$, and using (19) and (20), we have

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$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3) e_n^3 + \dots \quad (21)$$

Using (21) in (18) gives:

$$y_n = c_2 e_n^2 - 2(c_2^2 - c_3) e_n^3 + \dots \quad (22)$$

Expanding $f(y_n)$ and $f'(y_n)$ in Taylor's series about α by using (22), we have

$$f(y_n) = f(\alpha)[c_2 e_n^2 + 2(c_3 - c_2^2) e_n^3 + \dots] \quad (23)$$

$$f'(y_n) = f'(\alpha)[1 + 2c_2^2 e_n^2 + 4(c_2 c_3 - c_3^2) e_n^3 + \dots] \quad (24)$$

From(ii) and(v), we have

$$e_{n+1} = c_2 e_n^2 + \dots - \frac{4(c_2 e_n^2 - e_n + \dots) f'(\alpha) [1 + 2c_2^2 e_n^2 + \dots]}{\left[(c_2 e_n^2 - e_n + \dots) (f'(\alpha) (1 + 2c_2 e_n + 3c_3 e_n^2 + \dots)) + f'(\alpha) [1 + 2c_2^2 e_n^2 + \dots] \right] + 2(f'(\alpha) [c_2 e_n^2 + \dots] - f'(\alpha) (e_n + c_2 e_n^2 + \dots))} \quad (25)$$

Ignoring the higher order terms, and considering only those having an effect on the third order error term, we have:

$$e_{n+1} = c_2 e_n^2 - \frac{4[1 - c_2 e_n + 2c_2^2 e_n^2]}{4(1 + c_3 e_n^2)} \quad (26)$$

Simplifying the right side in (26) further, we have:

$$e_{n+1} = (-c_2 - 6c_2^2 + 8c_3^2) e_n^3 + o(e_n^4) \quad (27)$$

Equation (27), contains the leading error term associated with PM and shows that the PM is a third-order convergent nonlinear solver.

Using the formula for the efficiency index (E.I) [I], i.e.

$$E.I = S^{1/n} \quad (28)$$

where S shows the order of convergence of a method and n shows the number of evaluations required per iteration, the efficiency index of the PM is found to be 1.3160740129 which is higher than the efficiency indices 1, 1, 1.2457309396, 1.1892071150 and 1.2599210498 of the BM, RFM, Noor1, Noor2 and Noor3 methods, respectively. The PM is cubically convergent and is more efficient than lower order methods BM, RFM and Noor3. The PM is also more efficient than the accompanying cubically convergent method Noor1, and interestingly even better than the fourth-order convergent Noor2. However, it must be mentioned here that the efficiency index of the classical NRM is 1.4142135623, more than others as it is the optimal method of quadratic convergence. The PM is better than NRM from the point of views of the higher order of convergence as well as fewer iterations to reach a pre-specified error tolerance. The facts on the precedence of the PM over other methods are summarized and demonstrated in the next section.

III. Numerical Experiments, Results and Discussion

Several nonlinear equations have been solved from the literature [XXI],[XVIII],[XVII], [II], [XVIII], [XI], [XIV] to check and compare the performance of the PM with other most widely used methods: BM, RFM and NRM. The numerical results associated with the following 10 examples are provided here for brevity. For the computation of error and verification of accuracy and precision the 20 decimal places' accurate solutions obtained using the MATLAB and some of

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which were also reported in previous studies are also shown beside each example. Table 1 lists the intervals containing the roots for Examples 1-10 as used in the BM and RFM, and also the initial guesses, as the mid-points of the intervals, used in NRM and PM.

Example 1.	$3x + \sin(x) - \exp(x) = 0$	(0.36042170296032440136)
Example 2.	$\cos(x) - x \exp(x) = 0$	(0.51775736368245829832)
Example 3.	$3x + \cos(x) - 1 = 0$	(0.60710164810312263122)
Example 4.	$4\sin(x) - x + 1 = 0$	(2.70206137332604022180)
Example 5.	$\cos(x) - x = 0$	(0.73908513321516064165)
Example 6.	$\sin^2(x) - x^2 + 1 = 0$	(-1.4044916482153412260)
Example 7.	$x^2 - e^x - 3x + 2 = 0$	(0.25753028543986076045)
Example 8.	$(x - 1)^3 - 1 = 0$	(2)
Example 9.	$2x - \ln(x) - 7 = 0$	(4.21990648378038144162)
Example 10.	$\exp(x) - 5x = 0$	(0.25917110181907374505)

Using higher precision arithmetic in the MATLAB software, PM and other methods were implemented and it was desired to achieve at least 20 decimal places' accuracy between the exact and approximate solutions, or between any two successive approximations. The absolute error (A.E) formula used here are described in equation (29) for $i = 0, 1, 2, \dots$

$$A.E(i+1) = \begin{cases} |Exact\ solution - Approximate(i+1)|, & \text{if Exact is known} \\ |Approximate(i+1) - Approximate(i)|, & \text{if Exact is unknown} \end{cases} \quad (29)$$

The observed order of convergence (Q) [XX],[XVIII],[XVII], [II], [XVIII], [XI], [XIV] has been computed using (30):

$$Q_{j+1} = \frac{\ln\left(\frac{AE(j+1)}{AE(j)}\right)}{\ln\left(\frac{AE(j)}{AE(j-1)}\right)} \quad (30)$$

Using the formula (29) in every iteration, the absolute error distributions by all methods were obtained by all methods for Examples 1-10, and are shown in Fig. 1. It is evident from Fig. 1 that the PM always exhibits lower A.E. as compared to the BM, RFM and NRM for all examples. The A.E for first few iterations have been shown in Fig. 1, as the default lower limit in the Ms Excel for figures display is 1E-300, so errors below this were approximated by zero by Ms Excel. The errors are shown in reverse log scale for brevity, and the error distributions for the PM in Examples 1-10 are always on top of others. Using the formula (30), the observed orders of convergence were computed for all methods in Examples 1-10, and the comparison is shown in Fig. 2. It is evident from Fig. 2 that all methods converge to their theoretical orders of convergence, i.e. to 1, 1, 2 and 3 of BM, RFM, NRM and PM, respectively. The PM method, even after minimizing the use of one additional derivative as in Noor1 method, maintains the cubic convergence in all examples. The orders of convergence reported in Fig. 2 have been computed in the last iteration of each method here it achieves at least 50 decimal places' accuracy, i.e. A.E of atmost 1E-50.

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As seen in Fig. 1 the PM achieves the required A.E earlier than other methods. For instance, in Example 1 through Fig. 1, it can be seen that PM achieves the required error in 5 iterations, whereas it takes the BM, RFM and NRM to achieve same in lot more iterations, i.e. 165, 50 and 7 iterations, respectively. Such comparison for all Examples 1-10 regarding the number of iterations required to achieve A.E of atmost $1E-50$ is shown in Fig. 3, and it is quite clear from Fig. 3 that the PM achieves required error too earlier than all other methods in the discussion. As a consequence of the lower number of iterations, the computational cost in terms of total functional and derivative evaluations of the PM is also smaller than BM and RFM methods for all Examples as shown in Fig. 4. Since the PM uses 4 evaluations per iteration, so the number of evaluations for each example to achieve pre-specified A.E will be the number of iterations timed the cost per iteration, i.e. 4 for the PM. The results in Fig. 4 speak for themselves, as the evaluations in the Pm are smaller than those in BM and RFM as the latter requires much more operations than the PM. But, for the NRM the costs are lower and closer to the PM because NRM uses only 2 evaluations per iteration. Still the PM is better than the NRM from view-points being higher order convergent (Fig. 2), lower number of iterations (Fig. 3) and smaller error distributions (Fig. 1) for all examples. Thus, the PM exhibits strong precedence over other methods used in the comparison for all example nonlinear equations are taken from the literature.

Table 1: Intervals used in BM and RFM, and initial guesses in NRM and PM for Examples 1-10

Example	1	2	3	4	5	6	7	8	9	10
$[a, b]$	[0,1]	[0,1]	[0,1]	[8,9]	[0,1]	[-2,-1]	[0,1]	[0,1]	[3,5]	[0,0.5]
x_0	0.5	0.5	0.5	8.5	0.5	-1.5	0.5	0.5	4	0.25

IV. Conclusion

For solving scalar nonlinear equations efficiently with lesser evaluations without compromising upon the order of convergence, we modified the Noor` method in the literature to propose a new and efficient derivative-based method. The proposed method uses four evaluations instead of five as in Noor1 and maintains cubic convergence. The theorem concerning the order of convergence and derivation of the leading error term has been proved. The efficiency index of the proposed method is found to be higher than of the BM, RFM, Noor1, Noor2 and Noor3 methods. Several nonlinear equations from the literature were solved to compare the performance of the proposed method with other methods. The proposed method shows lower error distributions, lower number of iterations to reach a pre-specified error tolerance as compared to other methods and maintains the theoretical order of convergence in all examples. The proposed method can be used to solve the nonlinear equations more efficiently as compared to other discussed methods.

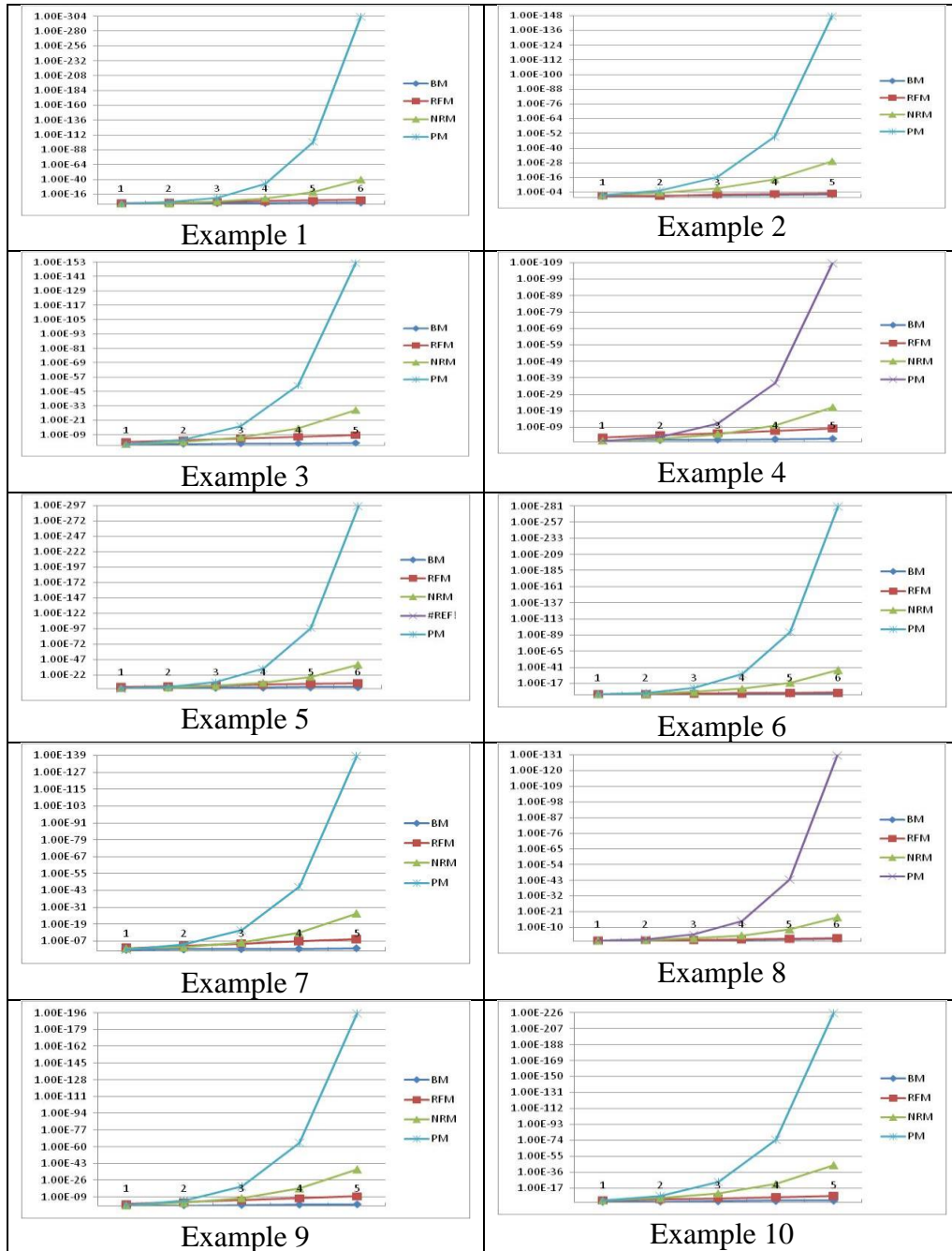


Fig 1. Absolute errors (along Y axis on log scale) for Examples 1-10 for starting iterations (along X-axis)

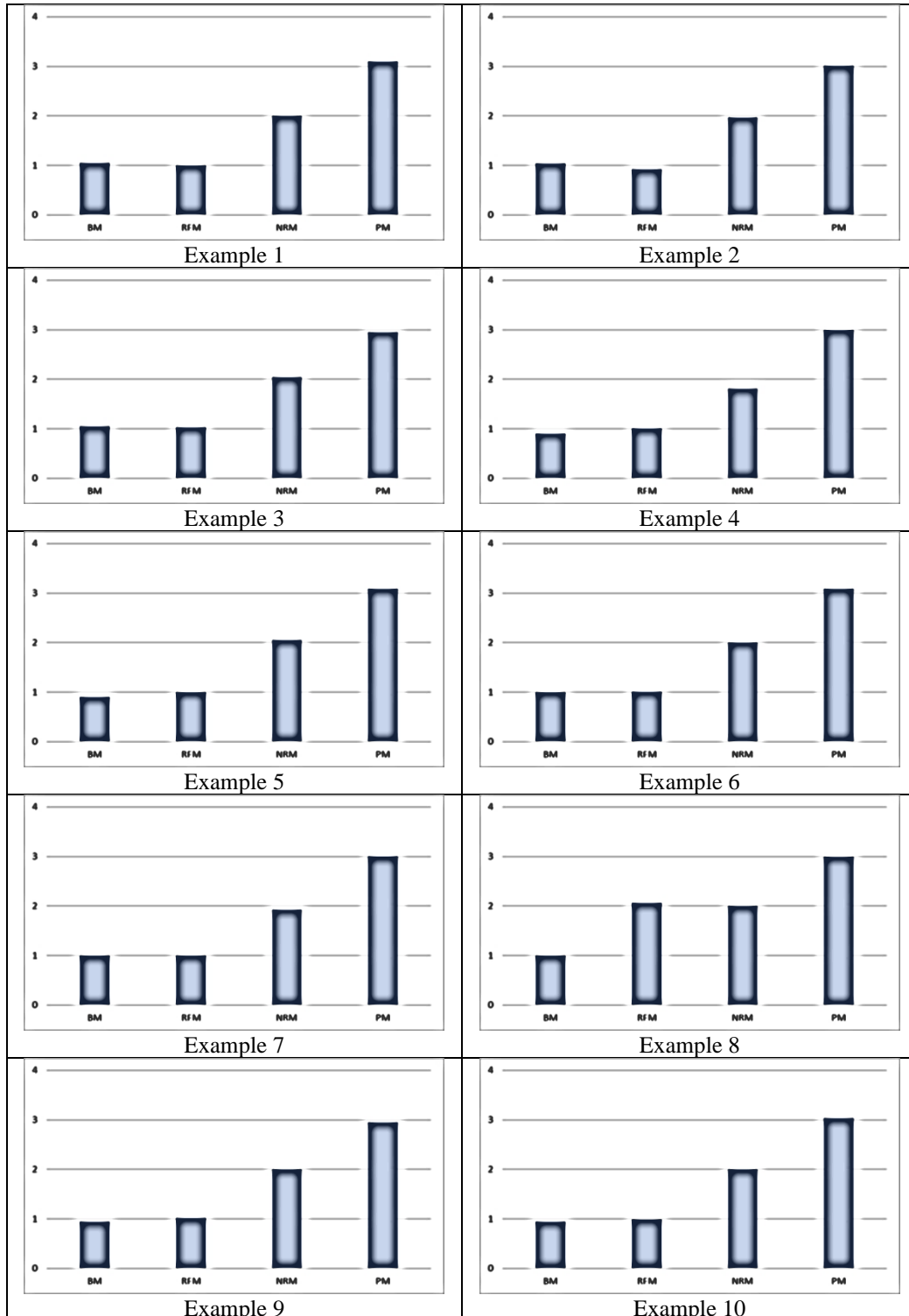


Fig 2. Comparison of the observed orders of convergence in last iteration for Examples 1-10

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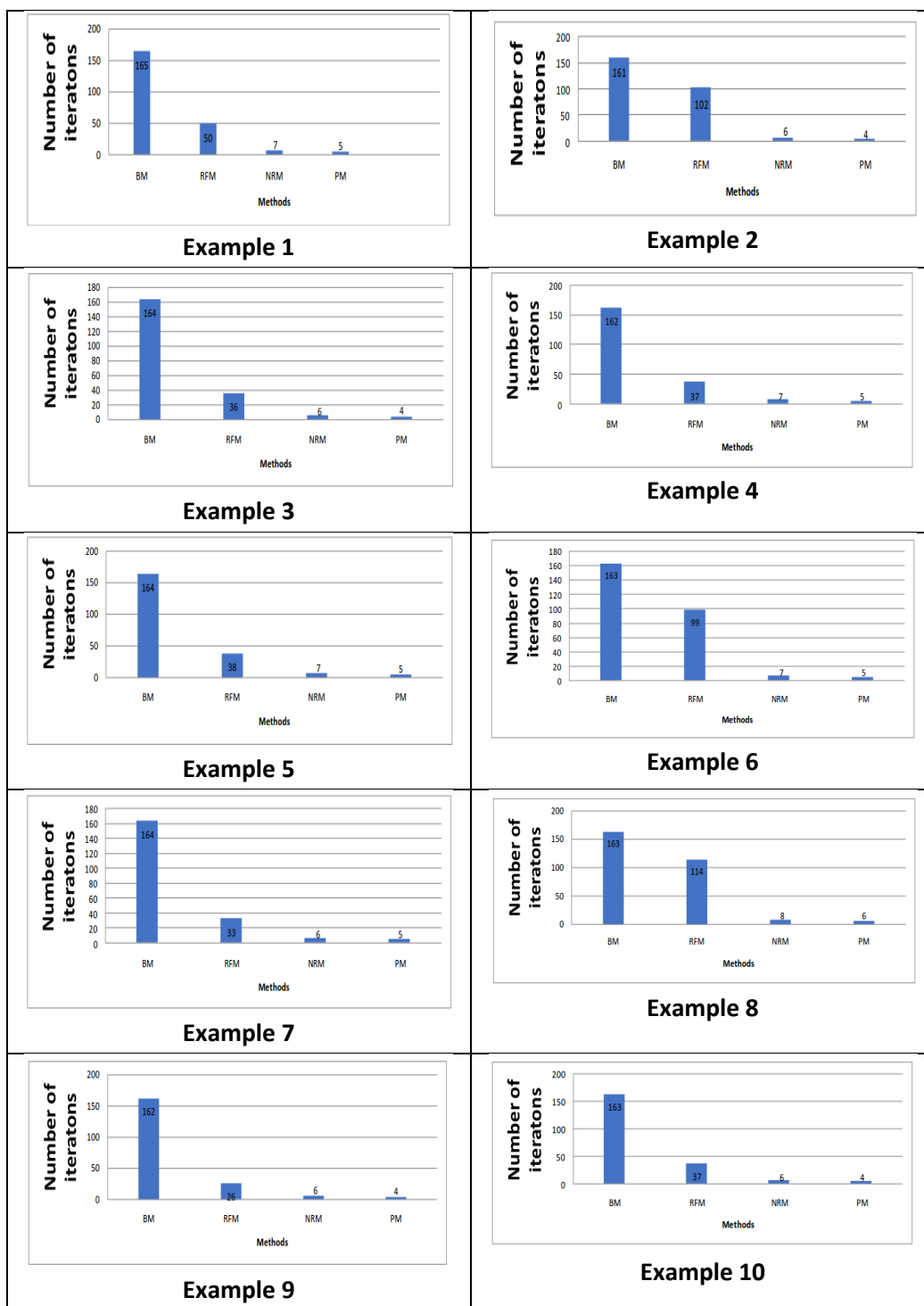


Fig 3. Comparison of number of iterations to achieve AE of atmost 1E-50 for Examples 1-10

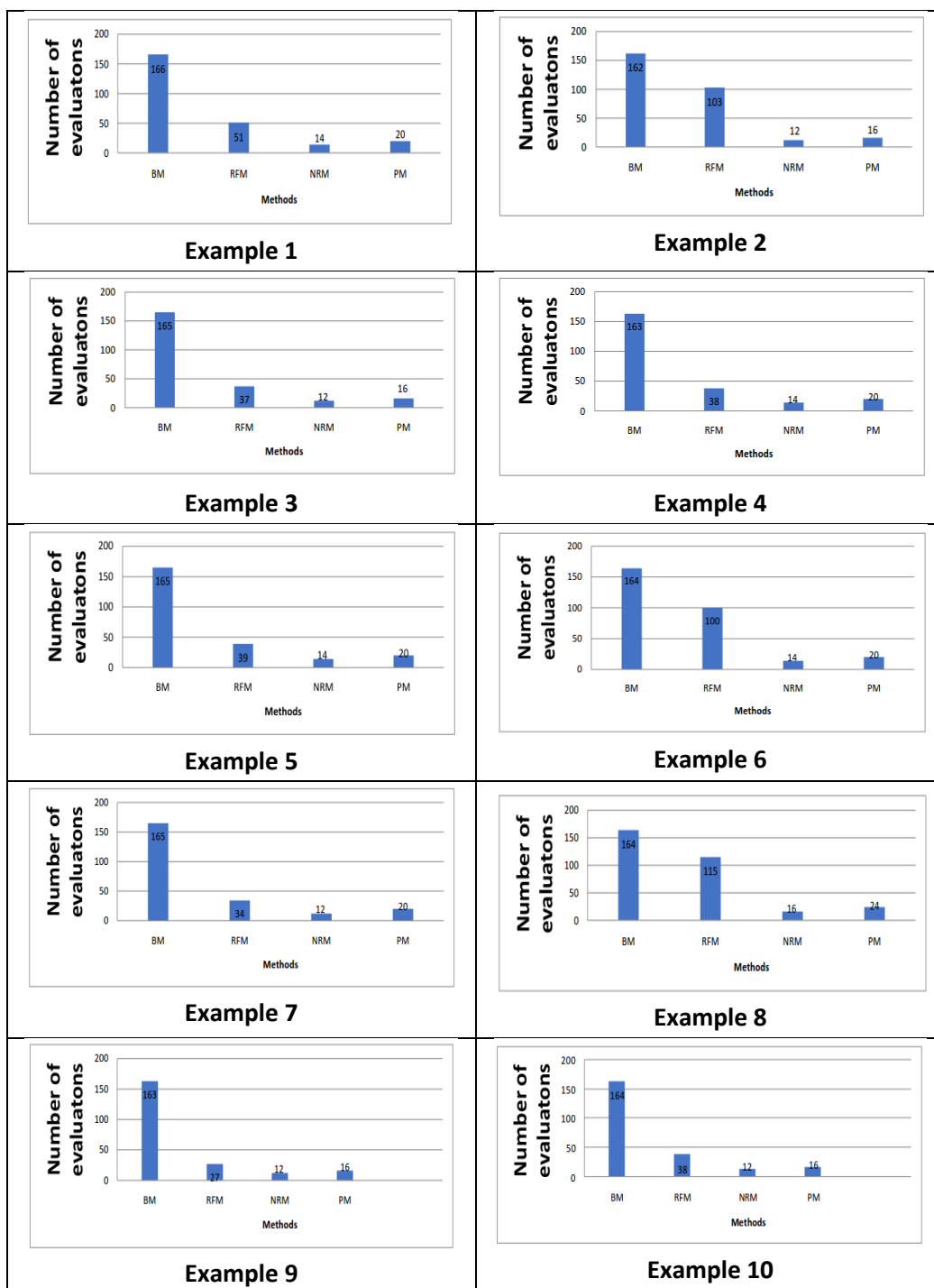


Fig 4. Comparison of number of evaluations to achieve AE of atmost $1E-50$ for Examples 1-10

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Conflict of Interest :

Authors declared : There is No conflict of interest regarding this article.

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