

## A FIXED POINT THEOREM IN GENERALIZED METRIC SPACES

By

<sup>1</sup>M. K. Bose and <sup>2</sup>R. Tiwari

<sup>1</sup>Department of Mathematics, University of North Bengal, Siliguri, West Bengal-  
734013, India. E-mail: [manojkumarbose@yahoo.com](mailto:manojkumarbose@yahoo.com)

<sup>2</sup>Department of Mathematics, St. Joseph's College, Darjeeling, West Bengal-  
734104, India. E-mail: [tiwarirupesh1@yahoo.co.in](mailto:tiwarirupesh1@yahoo.co.in)

### **Abstract**

*In this article we prove a fixed point theorem in generalized metric spaces.*

### **সংক্ষিপ্তসার**

এই পত্রে আমরা সামান্যীকৃত ম্যাট্রিক্স দেশে হির বিন্দু উপপাদ্যকে প্রমান করেছি।

### **1. Introduction**

Branciari [1] introduced the idea of generalized metric spaces as follows.

**Definition 1.1** A generalized metric on a nonempty set  $X$  is a nonnegative real valued function  $d$  on  $X \times X$  such that for  $x, y \in X$  and for all distinct points  $\xi, \eta \in X$ , each of them different from  $x$  and  $y$ ,

$$\begin{aligned} d(x, y) &= 0 \text{ if and only if } x = y, \\ d(x, y) &= d(y, x), \\ d(x, y) &\leq d(x, \xi) + d(\xi, \eta) + d(\eta, y) \end{aligned}$$

If  $d$  is a generalized metric on  $X$ , then  $(X, d)$  is called a generalized metric space.

Clearly any metric space is a generalized metric space, but the converse is not true [1], [2]. A Cauchy sequence and a complete generalized metric space are defined in the usual way.

If  $(X, d)$  is a complete generalized metric space and  $F: X \rightarrow X$  is a contraction, i.e.  $d(Fx, Fy) \leq \alpha d(x, y)$ ,  $0 < \alpha < 1$  for all  $x, y \in X$ , then  $F$  has a unique fixed point. This is Banach's fixed point theorem in generalized metric spaces [1].

## 2. Theorem

Here we prove the following fixed point theorem in generalized metric spaces.

### Theorem 2.1.

Let  $(X, d)$  be a complete generalized metric space and  $F: X \rightarrow X$  satisfy the condition

$$d(Fx, Fy) \leq \alpha[d(Fx, x) + d(x, y) + d(y, Fy)] \quad (1)$$

for all  $x, y \in X$ , where  $0 < \alpha < \frac{1}{3}$ . Then  $F$  has a unique fixed point.

**Proof:** Let  $x \in X$ . If  $x$  is not a fixed point of  $F$  we write  $x_1 = Fx$ . In general, we write  $x_n = Fx_{n-1}$  if  $x_{n-1}$  is not a fixed point of  $F$ . Then

$$\begin{aligned} d(x_1, x_2) &= d(Fx, Fx_1) \leq \alpha[d(Fx, x) + d(x, x_1) + d(x_1, Fx_1)] \text{ [by (1)]} \\ &= \alpha[2d(x, x_1) + d(x_1, x_2)] \end{aligned}$$

$$\Rightarrow d(x_1, x_2) \leq rd(x, x_1),$$

where

$$r = \frac{2\alpha}{1-\alpha} < 1.$$

Similarly we get

$$\begin{aligned} d(x_2, x_3) &\leq r d(x_1, x_2) \\ &\leq r^2 d(x, x_1). \end{aligned}$$

In general, for any positive integer  $n$ , we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq r d(x_{n-1}, x_n) \\ &\leq r^n d(x, x_1). \end{aligned}$$

It then follows that  $x_n \neq x_m$  for all distinct positive integers  $n, m$ . We now prove that for all positive integers  $n$

$$d(x_n, x_{n+2k}) \leq \sum_{i=0}^{2k-3} r^i d(x_n, x_{n+1}) + \left[ 2 \sum_{i=0}^{2k-2} \alpha^i d(x_n, x_{n+1}) + \alpha^{2k-2} d(x_n, x_{n+2}) \right], \quad k = 2, 3, K. \quad (2)$$

$$d(x_n, x_{n+2k+1}) \leq \sum_{i=0}^{2k} r^i d(x_n, x_{n+1}), \quad k = 0, 1, 2, \dots \quad (3)$$

We prove (2) and (3) by mathematical induction. We have

$$\begin{aligned} d(x_n, x_{n+4}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\ &\leq (1+r)d(x_n, x_{n+1}) + d(Fx_{n+1}, Fx_{n+3}) \\ &\leq (1+r)d(x_n, x_{n+1}) + \alpha[d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \quad [\text{by (1)}] \\ &\leq (1+r)d(x_n, x_{n+1}) + \alpha r(1+r^2)d(x_n, x_{n+1}) + \alpha d(x_{n+1}, x_{n+3}) \\ &\leq (1+r)d(x_n, x_{n+1}) + \alpha r(1+r^2)d(x_n, x_{n+1}) + \alpha^2 \left[ d(x_n, x_{n+1}) + d(x_n, x_{n+2}) \right. \\ &\quad \left. + d(x_{n+2}, x_{n+3}) \right] \\ &\leq (1+r)d(x_n, x_{n+1}) + \alpha r(1+r^2)d(x_n, x_{n+1}) + \alpha^2(1+r^2)d(x_n, x_{n+1}) \\ &\quad + \alpha^2 d(x_n, x_{n+2}) \\ &< (1+r)d(x_n, x_{n+1}) + (\alpha + \alpha^2)2d(x_n, x_{n+1}) + \alpha^2 d(x_n, x_{n+2}) \quad [\text{since } r < 1]. \end{aligned}$$

Thus (2) is true for  $k=2$ . Let us now suppose that for some positive integer  $k_0$ , (2) is true for all positive integers  $k$  with  $2 \leq k \leq k_0$ . Then

$$d(x_n, x_{n+2k_0+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2k_0+2})$$

$$\begin{aligned} &< (1+r)d(x_n, x_{n+1}) + \left[ \sum_{i=0}^{2k_0-3} r^i d(x_{n+2}, x_{n+3}) + 2 \sum_{i=1}^{2k_0-2} \alpha^i d(x_{n+2}, x_{n+3}) \right. \\ &\quad \left. + \alpha^{2k_0-2} d(x_{n+2}, x_{n+4}) \right] \\ &\leq (1+r)d(x_n, x_{n+1}) + \left[ r^2 \sum_{i=0}^{2k_0-3} r^i d(x_n, x_{n+1}) + 2 \sum_{i=1}^{2k_0-2} \alpha^i r^2 d(x_n, x_{n+1}) \right. \\ &\quad \left. + \alpha^{2k_0-1} \{d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_{n+3}) + d(x_{n+3}, x_{n+4})\} \right] \\ &\leq (1+r)d(x_n, x_{n+1}) + \left[ r^2 \sum_{i=0}^{2k_0-3} r^i d(x_n, x_{n+1}) + 2 \sum_{i=1}^{2k_0-2} \alpha^i r^2 d(x_n, x_{n+1}) \right. \\ &\quad \left. + \alpha^{2k_0-1} r(1+r^2) d(x_n, x_{n+1}) + \alpha^{2k_0-1} d(x_{n+1}, x_{n+3}) \right] \\ &\leq (1+r)d(x_n, x_{n+1}) + \left[ + \alpha^{2k_0-1} r(1+r^2) d(x_n, x_{n+1}) + \alpha^{2k_0} \{d(x_n, x_{n+1}) \right. \\ &\quad \left. + d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3})\} \right] \end{aligned}$$

$$\begin{aligned} &\leq (1+r)d(x_n, x_{n+1}) + \left[ r^2 \sum_{i=0}^{2k_0-3} r^i d(x_n, x_{n+1}) + 2 \sum_{i=0}^{2k_0-2} \alpha^i r^2 d(x_n, x_{n+1}) \right. \\ &\quad \left. + \alpha^{2k_0-1} r(1+r^2) d(x_n, x_{n+1}) \right. \\ &\quad \left. + \alpha^{2k_0} (1+r^2) d(x_n, x_{n+1}) + \alpha^{2k_0} d(x_n, x_{n+2}) \right] \\ &< (1+r)d(x_n, x_{n+1}) + \left[ r^2 \sum_{i=0}^{2k_0-3} r^i d(x_n, x_{n+1}) + 2 \sum_{i=0}^{2k_0-2} \alpha^i d(x_n, x_{n+1}) \right. \\ &\quad \left. 2\alpha^{2k_0-1} d(x_n, x_{n+1}) + 2\alpha^{2k_0} d(x_n, x_{n+1}) + \alpha^{2k_0} d(x_n, x_{n+2}) \right] \\ &= \sum_{i=0}^{2k_0-1} r^i d(x_n, x_{n+1}) + 2 \sum_{i=0}^{2k_0} \alpha^i d(x_n, x_{n+1}) + \alpha^{2k_0} d(x_n, x_{n+1}). \end{aligned}$$

Thus (2) is true for  $k_0+1$ . Hence it is true for all  $k$ .

For  $k=0$ , (3) is obviously true. Let us suppose that for some nonnegative integer  $k_0$ , (3) is true for all nonnegative integers  $k$  with  $0 \leq k \leq k_0$ . Then

$$\begin{aligned} d(x_n, x_{n+2k_0+3}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2k_0+3}) \\ &\leq (1+r)d(x_n, x_{n+1}) + \sum_{i=0}^{2k_0} r^i d(x_{n+2}, x_{n+3}) \\ &\leq (1+r)d(x_n, x_{n+1}) + r^2 \sum_{i=0}^{2k_0} r^i d(x_n, x_{n+1}) \\ &= \sum_{i=0}^{2k_0+2} r^i d(x_n, x_{n+1}). \end{aligned}$$

So (3) is true for  $k_0+1$  and hence it is true for all  $k$ . Now

$$\begin{aligned} d(x_n, x_{n+2}) &\leq \alpha [d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ &\leq \alpha r^{n-1} (1+r^2) d(x, x_1) + \alpha^2 [d(x_{n-2}, x_{n-1}) + d(x_{n-2}, x_n) + d(x_n, x_{n+1})] \\ &\leq \alpha r^{n-1} (1+r^2) d(x, x_1) + \alpha^2 r^{n-2} (1+r^2) d(x, x_1) + \alpha^2 d(x_{n-2}, x_n) \\ &< 2\alpha r^{n-1} d(x, x_1) + 2\alpha^2 r^{n-2} d(x, x_1) + \alpha^2 d(x_{n-2}, x_n) \\ &< 2(\alpha r^{n-1} + \alpha^2 r^{n-2} + \dots + \alpha^n) d(x, x_1) + \alpha^n d(x, x_2) \\ &\leq 2n\mu^n d(x, x_1) + \alpha^n d(x, x_2) \quad [\mu = \max(\alpha, r) < 1]. \end{aligned}$$

Thus  $d(x_n, x_{n+2}) \rightarrow 0$  as  $n \rightarrow \infty$ . From (2) we get

$$\begin{aligned} d(x_n, x_{n+2k}) &< r^n \sum_{i=0}^{2k-3} r^i d(x, x_1) + 2r^n \sum_{i=0}^{2k-2} \alpha^i d(x, x_1) + \alpha^{2k-2} d(x_n, x_{n+2}) \\ &< \frac{r^n}{1-r} d(x, x_1) + \frac{2r^n}{1-\alpha} d(x, x_1) + \alpha^{2k-2} d(x_n, x_{n+2}), \quad k=2, 3, 4, \dots \end{aligned}$$

Therefore  $d(x_n, x_{n+2k}) \rightarrow 0$  as  $n \rightarrow \infty$  for  $k = 1, 2, 3, \dots$ . Again for  $k=0, 1, 2, \dots$

$$\begin{aligned} d(x_n, x_{n+2k+1}) &\leq \sum_{i=0}^{2k} r^i d(x_n, x_{n+i}) \quad [\text{by (3)}] \\ &\leq r^n \sum_{i=0}^{2k} r^i d(x, x_i) \\ &< \frac{r^n}{1-r} d(x, Tx) \end{aligned}$$

and so  $d(x_n, x_{n+2k+1}) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus it follows that  $\{x_n\}$  is a Cauchy sequence and so for some  $x_0 \in X$ ,  $\lim_{n \rightarrow \infty} x_n = x_0$ . Now

$$\begin{aligned} d(x_0, Fx_0) &\leq d(x_0, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Fx_0) \\ &\leq d(x_0, x_n) + d(x_n, x_{n+1}) + \alpha [d(x_n, x_{n+1}) + d(x_n, x_0) + d(x_0, Fx_0)] \\ \Rightarrow (1-\alpha)d(x_0, Fx_0) &\leq d(x_0, x_n) + d(x_n, x_{n+1}) + \alpha[d(x_n, x_{n+1}) + d(x_n, x_0)] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

So  $Fx_0 = x_0$  i.e.  $x_0$  is a fixed point of  $F$ . The uniqueness of the fixed point follows from the inequality (1).

#### References:

- 1) Branciari A., A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen, 57 (1-2) (2000), 31-37.
- 2) Lahiri B. K., Saha P. K. and Tiwari R. A generalized metric space is not Hausdorff, Rev. Bull. Cal. Math. Soc., 6(2) (2008), 177- 178.