ISSN 0973-8975
J.Mech.Cont.& Math. Sci., Vol.-5, No.-1, July (2010) Pages 599-615

ON FOURTH ORDER MORE CRITICALLY DAMPED NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract:

In this article an analytical approximate solution has been investigated for obtaining the transient response of fourth order more critically damped nonlinear systems. The results obtained by the presented technique agree with the numerical results obtained by the fourth order Runge-Kutta method nicely. An example is solved to illustrate the method.

সংক্ষিপ্তসার

এই পত্রে চতুর্থক্রমের অতি-ক্রান্তিক অবমন্দিত অরৈখিক তন্ত্রের ক্ষণস্থায়ী প্রতিক্রিয়া নির্ণয়ের জন্য বৈশ্লেষিক আসন্ন মানে সমাধানকে অনুসন্ধান করা হয়েছে। পরিবেশিত কৃৎকৌশল দ্বারা নির্ণিত ফলাফলগুলি রুঙ্গো-কুট্টা (Runge-Kutta) পদ্ধতির সাহায্যে নির্ণিত সাংখ্যমানের ফলাফলগুলির সঙ্গে সঙ্গতিপূর্ণ। একটি উদাহরণ সমাধান করে উক্ত পদ্ধতিটিকে ব্যাখ্যা করা হয়েছে।

1. Introduction

The control of micro vibration has become a growing research field due to the demand of high-performance systems and the advent of micro and nanotechnology in various scientific and industrial fields, such as, semiconductor manufacturing, biomedical engineering, aerospace-equipments, and high-precision measurements. In micro and nanotechnology a small vibration has great significance, since due to a small vibration the produced equipment may be defective. So, in micro and nano-technological industries, vibration is avoidable rather than desirable. But it arises in different way, such as, earthquake, direct disturbance etc. Accordingly, the control of vibration in micro and nano-technological industries is very essential (see also Emdadul *et al.* [5], Mizuon *et al.* [8] for details). So, we should keep watch that the originated vibrations come to its equilibrium position within minimum time and the more critically damped systems have this characteristic. Therefore, more critically damped systems play an important role in micro and nano-technological industries.

To investigate the transient behavior of vibrating systems the Krylov-Bogoliubov-Mitropolskii (KBM) [4, 6] method is an extensively used tool. Originally, the method was developed for obtaining the periodic solutions of second order nonlinear differential systems with small nonlinearities. Later, the method extended by Popov [11] to investigate the solutions of nonlinear systems in presence of strong linear damping effects. Owing to physical importance Popov's results were rediscovered by Mendelson [7]. Murty *et al.* [9] developed a technique based on the method of Bogoliubov's to obtain the transient response of over-damped nonlinear systems. Later, Murty [10] presented a unified KBM method for second order nonlinear systems which covers the un-damped, damped and over-damped cases. Sattar [14] found an asymptotic solution of a second order critically damped nonlinear system. Shamsul [16] developed a new asymptotic solution for both over-damped and critically damped nonlinear systems.

Shamsul and Sattar [15] developed a technique based on the work of KBM for obtaining the solution of third order critically damped nonlinear systems. Later, Shamsul [17] investigated solutions of third order critically nonlinear systems whose unequal eigenvalues are in integral multiple. In article [17], Shamsul also investigated solutions of third order more critically damped nonlinear systems. Rokibul *et al.* [12] found a new technique for obtaining the solutions of third order critically damped nonlinear systems.

In article [9], Murty *et al.* also extended the KBM method for solving fourth order over-damped nonlinear systems. But their method is too much complex and laborious. Akbar *et al.* [1] presented an asymptotic method for fourth order over-damped nonlinear systems which is simple, systematic and easier than the method presented in [9], but the results obtained by [1] is identical as the results obtained in [9]. Later, Akbar *et al.* [2] extended the method presented in [1] for fourth order damped oscillatory nonlinear systems. Rokibul *et al.* [13] extended the KBM method for obtaining the response of fourth order critically damped nonlinear systems.

In the present article, we have investigated solutions for obtaining the transient response of fourth order more critically damped nonlinear systems. The results obtained by the presented technique match nicely with the results obtained by numerical method (fourth order Runge-Kutta method).

2. The method

Consider a fourth order weakly nonlinear ordinary differential system

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon f(x) \tag{1}$$

where $x^{(4)}$ denote the fourth derivative of x and over dots are used to denote the first, second and third derivative of x with respect to t; k_1, k_2, k_3, k_4 are characteristic parameters, ε is a small parameter and $f(x)$ is the given nonlinear function. As the equation is fourth order so there are four real negative eigenvalues, and three of the eigenvalues are equal (for more critically damped). Suppose the eigenvalues are $-\lambda,-\lambda,-\lambda,-\mu$. When $\varepsilon = 0$, the equation (1) becomes linear and the solution of the corresponding linear equation is

$$x(t,0) = (a_0 + b_0\, t + c_0\, t^2)e^{-\lambda t} + d_0\, e^{-\mu t} \tag{2}$$

where a_0, b_0, c_0, d_0 are constants of integration.

When $\varepsilon \neq 0$, following Shamsul [16] an asymptotic solution of the equation (1) is sought in the form

$$x(t,\varepsilon) = (a + b\,t + ct^2)e^{-\lambda t} + d\,e^{-\mu t} + \varepsilon\,u_1(a, b, c, d, t) + \cdots \tag{3}$$

where a, b, c, d the functions of t and satisfy the first order differential equations

$$\begin{aligned} \dot{a}(t) &= \varepsilon\, A_1(a,b,c,d,t) + \cdots \\ \dot{b}(t) &= \varepsilon\, B_1(a,b,c,d,t) + \cdots \\ \dot{c}(t) &= \varepsilon\, C_1(a,b,c,d,t) + \cdots \\ \dot{d}(t) &= \varepsilon\, D_1(a,b,c,d,t) + \cdots \end{aligned} \tag{4}$$

Now differentiating (3) four times with respect to t, substituting the value of x and the derivatives $\dot{x}, \ddot{x}, \dddot{x}, x^{(4)}$ in the original equation (1), utilizing the relations presented in (4) and finally extracting the coefficients of ε, we obtain

$$\begin{aligned} & e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2} + 3\frac{\partial B_1}{\partial t} + 6C_1 + t\left(\frac{\partial^2 C_1}{\partial t^2} + 6\frac{\partial C_1}{\partial t}\right) + t^2\,\frac{\partial^2 C_1}{\partial t^2}\right) \\ & + e^{-\mu t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)^3 D_1 + \left(\frac{\partial}{\partial t} + \lambda\right)^3\left(\frac{\partial}{\partial t} + \mu\right)u_1 = -f^{(0)}(a,b,c,d,t) \end{aligned} \tag{5}$$

where $f^{(0)}(a,b,c,d,t) = f(x_0, \dot{x}_0, \ddot{x}_0, \dddot{x}_0)$ and $x_0 = (a + bt + c\,t^2)e^{-\lambda t} + d\,e^{-\mu t}$.

In this article, the functional $f^{(0)}$ is expanded in the Taylor's series of the form (see also Sattar [14] and Shamsul [15-17] for details)

$$f^{(0)} = \sum_{l=0}^{\infty}\left((bt + ct^2)^l \sum_{i,\,j=0}^{\infty} F_l(a,d)e^{-(i\lambda + j\mu)t}\right) \tag{6}$$

Thus, using (6), the equation (5) becomes

$$\begin{aligned} & e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\left\{\frac{\partial^2 A_1}{\partial t^2} + 3\frac{\partial B_1}{\partial t} + 6C_1 + t\left(\frac{\partial^2 B_1}{\partial t^2} + 6\frac{\partial C_1}{\partial t}\right) + t^2\,\frac{\partial^2 C_1}{\partial t^2}\right\} \\ & + e^{-\mu t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)^3 D_1 + \left(\frac{\partial}{\partial t} + \lambda\right)^3\left(\frac{\partial}{\partial t} + \mu\right)u_1 = -\sum_{l=0}^{\infty}\left((bt + ct^2)^l \sum_{i,\,j=0}^{\infty} F_l(a,d)e^{-(i\lambda + j\mu)t}\right) \end{aligned} \tag{7}$$

KBM [4, 6], Murty *et al.* [9], Sattar [14], Shamsul and Sattar [15], Shamsul [17] imposed the condition that u_1 can not contains the fundamental terms (the solution (2) is called generating solution of (1) and its terms are called fundamental terms) of $f^{(0)}$. Therefore, equation (7) can be separated for the unknown functions u_1 and A_1, B_1, C_1 D_1 in the following way:

$$\begin{aligned}&e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left\{\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1+t\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)+t^2\,\frac{\partial^2 C_1}{\partial t^2}\right\}\\&+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1=-\sum_{l=0}^{1}\left((bt+ct^2)^l\sum_{i,j=0}^{\infty}F_l(a,d)e^{-(i\lambda+j\mu)t}\right)\end{aligned} \tag{8}$$

And

$$\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)u_1=-\sum_{l=2}^{\infty}\left((bt+ct^2)^l\sum_{i,j=0}^{\infty}F_l(a,d)e^{-(i\lambda+j\mu)t}\right) \tag{9}$$

Now equating the coefficients of t^0, t^1 and t^2; from equation (8), we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 C_1}{\partial t^2}=-c\sum_{i,j=0}^{\infty}F_1(a,d)e^{-(i\lambda+j\mu)t} \tag{10}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)=-b\sum_{i,j=0}^{\infty}F_1(a,d)e^{-(i\lambda+j\mu)t} \tag{11}$$

And

$$\begin{aligned}&e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)\\&+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1=-\sum_{i,j=0}^{\infty}F_0(a,d)e^{-(i\lambda+j\mu)t}\end{aligned} \tag{12}$$

Solving the equation (10), we obtain

$$C_1=\sum_{i,j=0}^{\infty}\frac{c\,F_1(a,d)e^{-((i-1)\lambda+j\mu)t}}{(i\lambda+(j-1)\mu)((i-1)\lambda+j\mu)^2} \tag{13}$$

Substituting the value of C_1 from (13) into equation (11) and solving, we obtain

$$B_1=-6\sum_{i,j=0}^{\infty}\frac{c F_1(a,d)e^{-((i-1)\lambda+j\mu)t}}{((i-1)\lambda+j\mu)^3(i\lambda+(j-1)\mu)}-\sum_{i,j=0}^{\infty}\frac{b F_1(a,d)\,e^{-((i-1)\lambda+j\mu)t}}{((i-1)\lambda+j\mu)^2(i\lambda+(j-1)\mu)} \quad (14)$$

Now substituting the value of C_1 from (13) and B_1 from (14) into equation (12), we obtain

$$\begin{aligned}&e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 A_1}{\partial t^2}+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1\\&=-12\sum_{i,j=0}^{\infty}\frac{c F_1(a,d)e^{-(i\lambda+j\mu)t}}{((i-1)\lambda+j\mu)^2}-3\sum_{i,j=0}^{\infty}\frac{b F_1(a,d)\,e^{-(i\lambda+j\mu)t}}{((i-1)\lambda+j\mu)}-\sum_{i,j=0}^{\infty}F_0(a,d)\,e^{-(i\lambda+j\mu)t}\end{aligned} \quad (15)$$

Now, we have only one equation (15) for obtaining the unknown functions A_1 and D_1. Therefore, to separate the equation (15) for obtaining the unknown functions A_1 and D_1, we need to impose some restrictions (see also Shamsul [17] and Akbar *et al.* [3] for details) and thus the value of A_1 and D_1 can be found subject to the condition that the coefficients in the solution of A_1 and D_1 do not become large. This completes the determination of A_1, B_1, C_1 and D_1.

Since $\dot{a}, \dot{b}, \dot{c}, \dot{d}$ are proportional to small parameter ε, so, they are slowly varying functions of time t and for first approximate solution, we may consider them as constants in the right hand side. This assumption was first made by Murty *et al.* [9]. Thus the solutions of the equation (4) become

$$\begin{aligned}a&=a_0+\varepsilon\int_0^t A_1(a_0,b_0,c_0,d_0,t)\,dt\\b&=b_0+\varepsilon\int_0^t B_1(a_0,b_0,c_0,d_0,t)\,dt\\c&=c_0+\varepsilon\int_0^t C_1(a_0,b_0,c_0,d_0,t)\,dt\\d&=d_0+\varepsilon\int_0^t D_1(a_0,b_0,c_0,d_0,t)\,dt\end{aligned} \quad (16)$$

Equation (9) is an inhomogeneous linear ordinary differential equation; therefore it can be solved by the well-known operator method.

Substituting the value of a, b, c, d and u_1 in the equation (3), we shall get the complete solution of (1).

Therefore, the determination of the first order improved solution is completed.

Example

As an example of the above method, in this article, we have considered the Duffing equation type fourth order nonlinear differential system

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon x^3 \tag{17}$$

Here $f(x) = x^3$. Therefore,

$$\begin{aligned} f^{(0)} &= a^3 e^{-3\lambda t} + 3a^2 d\, e^{-(2\lambda+\mu)t} + 3a\, d^2\, e^{-(\lambda+2\mu)t} + d^3 e^{-3\mu t} \\ &+ (bt + c\,t^2)^1 \left(3a^2 e^{-3\lambda t} + 6a\, d\, e^{-(2\lambda+\mu)t} + 3d^2\, e^{-(\lambda+2\mu)t}\right) \\ &+ (bt + c\,t^2)^2 \left(3a e^{-3\lambda t} + 3d\, e^{-(2\lambda+\mu)t}\right) + (bt + ct^2)^3 e^{-3\lambda t} \end{aligned} \tag{18}$$

For example equation (17), the equations (9)-(12) respectively become

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \lambda\right)^3 \left(\frac{\partial}{\partial t} + \mu\right) u_1 &= -\{b^3 t^3 e^{-3\lambda t} + 6abc\, t^3\, e^{-3\lambda t} + 3b^2\, ct^4\, e^{-3\lambda t} \\ &+ 3ac^2\, t^4\, e^{-3\lambda t} + 3bc^2\, t^5\, e^{-3\lambda t} + c^3\, t^6\, e^{-3\lambda t} + 6bcd\, t^3\, e^{-(2\lambda+\mu)t} \\ &+ 3c^2\, dt^4\, e^{-(2\lambda+\mu)t} + 3ab^2 t^2 e^{-3\lambda t} + 3d\, b^2 t^2\, e^{-(2\lambda+\mu)t}\} \end{aligned} \tag{19}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\frac{\partial^2 C_1}{\partial t^2} = -\{3a^2\, ce^{-3\lambda t} + 6acd\, e^{-(2\lambda+\mu)t} + 3cd^2\, e^{-(\lambda+2\mu)t}\} \tag{20}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\left(\frac{\partial^2 B_1}{\partial t^2} + 6\frac{\partial C_1}{\partial t}\right) = -\{3a^2 be^{-3\lambda t} + 6abd\, e^{-(2\lambda+\mu)t} + 3bd^2\, e^{-(\lambda+2\mu)t}\} \tag{21}$$

and

$$\begin{aligned} &e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2} + 3\frac{\partial B_1}{\partial t} + 6C_1\right) + e^{-\mu t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)^3 D_1 \\ &= -\{a^3 e^{-3\lambda t} + 3a^2 d\, e^{-(2\lambda+\mu)t} + 3ad^2\, e^{-(\lambda+2\mu)t} + d^3 e^{-3\lambda_4 t}\} \end{aligned} \tag{22}$$

The solution of the equation (20) is

$$C_1 = l_1\, a^2\, c e^{-2\lambda t} + l_2\, a c d\, e^{-(\lambda+\mu)t} + l_3\, c d^2\, e^{-2\mu t} \tag{23}$$

where

$l_1 = (3PL^2)/4,\quad l_2 = (3Q^2 L)/2,\quad l_3 = (3QM^2)/4,\quad P = 1/(3\lambda-\mu),$

$Q = 1/(\lambda+\mu),\quad L = 1/\lambda,\quad M = 1/\mu.$

Putting the value of C_1 from equation (23) into equation (21) and solving, we obtain

$$\begin{aligned} B_1 = {} & m_1\, a^2\, c e^{-2\lambda t} + m_2\, a c d e^{-(\lambda+\mu)t} + m_3\, c d^2\, e^{-2\mu t} \\ & + m_4\, a^2 b e^{-2\lambda t} + m_5 a b d\, e^{-(\lambda+\mu)t} + m_6 b d^2\, e^{-2\mu t} \end{aligned} \tag{24}$$

where

$m_1 = 9PL^3/4,\quad m_2 = 18Q^3 L,\quad m_3 = 9QM^3/4,\quad m_4 = 3PL^2/4,$

$m_5 = 3Q^2 L,\quad m_6 = 3QM^2/4.$

Substituting the values of B_1 and C_1 into equation (21), we shall get an equation for unknown functions A_1 and D_1. To separate the equation (22) for determining these unknown functions, in this article, we considered the relation $\lambda \approx 3\mu$ exists among the eigenvalues (see also Shamsul [15, 17] for details). *i. e.* the unequal eigenvalue λ is the multiple of μ. This type of relation ($\lambda \approx 3\mu$) appears intuitively in the symmetric problems. Since our problem (example equation (17)) is symmetric, therefore consideration of such type of relation is logical. Therefore, under this relation, we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 A_1}{\partial t^2} = 6m_1\,\lambda(\mu-3\lambda)a^2 c\, e^{-3\lambda t}$$
$$-12\,\lambda(\lambda+\mu)m_2\,acd\,e^{-(2\lambda+\mu)t}-6\,\mu(\lambda+\mu)m_3\,cd^2\,e^{-(\lambda+2\mu)t}$$
$$+6\lambda(\mu-3\lambda)m_4\,a^2be^{-3\lambda t}-6\,\lambda(\lambda+\mu)m_5 abd\,e^{-(2\lambda+\mu)t} \tag{25}$$
$$-6\,\mu(\lambda+\mu)m_6 bd^2\,e^{-(\lambda+2\mu)t}-6\,(\mu-3\lambda)l_1 a^2\,c\,e^{-3\lambda t}$$
$$+24\,\lambda l_2\,acd\,e^{-(2\lambda+\mu)t}+6(\lambda+\mu)l_3\,cd^2\,e^{-2\mu t}-a^3e^{-3\lambda t}$$
$$-3a^2d\,e^{-(2\lambda+\mu)t}-3ad^2\,e^{-(\lambda+2\mu)t}$$

$$e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1 = -d^3e^{-3\mu t} \tag{26}$$

The particular solutions of (25) and (26) respectively become

$$A_1 = n_1 a^2 ce^{-2\lambda t}+n_2\,acde^{-(\lambda+\mu)t}+n_3 cd^2\,e^{-2\mu t}+n_4\,a^2be^{-2\lambda t}$$
$$+n_5\,abd\,e^{-(\lambda+\mu)t}+n_6 bd^2\,e^{-2\mu t}+n_7\,a^2\,c\,e^{-2\lambda t}+n_8\,acde^{-(\lambda+\mu)t} \tag{27}$$
$$+n_9\,cd^2\,e^{-2\mu t}+n_{10}\,a^3e^{-2\lambda t}+n_{11}\,a^2d\,e^{-(\lambda+\mu)t}+n_{12}\,ad^2\,e^{-2\mu t}$$
$$D_1 = p_1\,d^3e^{-2\mu t} \tag{28}$$

where

$n_1 = 27PL^4/8,\quad n_2 = 18Q^4L,\quad n_3 = 27QM^4/8,\quad n_4 = 9PL^3/8$

$n_5 = 9Q^3L,\quad n_6 = 9QM^3/8,\quad n_7 = -9PL^4/8,\quad n_8 = -9Q^4L,$

$n_9 = -9QM^4/8,\quad n_{10} = -PL^2/4,\quad n_{11} = 3Q^2L/2,$

$n_{12} = 3QM^2/4,\quad p_1 = \dfrac{P^3\,Q^3}{(2P-Q)^3}.$

The solution of the equation (19) for u_1 is

$$u_1 = \left(r_1 t^3+r_2 t^2+r_3 t+r_4\right)\left(b^3+6abc\right)e^{-3\lambda t}+\left(r_5 t^4+r_6 t^3+r_7 t^2+r_8 t+r_9\right)$$
$$\times(b^2c+ac^2)e^{-3\lambda t}+\left(r_{10}t^5+r_{11}t^4+r_{12}t^3+r_{13}t^2+r_{14}t+r_{15}\right)bc^2\,e^{-3\lambda t}$$
$$+\left(r_{16}t^6+r_{17}t^5+r_{18}t^4+r_{19}t^3+r_{20}t^2+r_{21}t+r_{22}\right)c^3\,e^{-3\lambda t}$$
$$+\left(r_{23}t^3+r_{24}t^2+r_{25}t+r_{26}\right)bcde^{-(\mu+2\lambda)t} \tag{29}$$
$$+\left(r_{27}t^4+r_{28}t^3+r_{29}t^2+r_{30}t+r_{31}\right)c^2\,de^{-(\mu+2\lambda)t}$$
$$+\left(r_{32}t^2+r_{33}t+r_{34}\right)ab^2\,e^{-3\lambda t}+\left(r_{35}t^2+r_{36}t+r_{37}\right)b^2de^{-(\mu+2\lambda)t}$$

where

$$r_1 = -P L^3/8, \quad r_2 = r_1(3P + 9L/2), \quad r_3 = r_1(6P^2 + 9PL + 9L^2),$$

$$r_4 = r_1(6P^3 + 9P^2 L + 9PL^2 + 15L^3/2), \quad r_5 = -3PL^2/8,$$

$$r_6 = r_5(4P + 6L), \quad r_7 = r_5(12P^2 + 18PL + 18L^2),$$

$$r_8 = r_5(24P^3 + 36P^2 L + 36PL^2 + 30L^3),$$

$$r_9 = r_5(24P^4 + 36P^3 L + 36P^2 L^2 + 30PL^3 + 45L^4)/2), \quad r_{10} = -3PL^3/8,$$

$$r_{11} = r_{10}(5P + 15L/2), \quad r_{12} = r_{10}(20\,P^2 + 30PL + 30L^2),$$

$$r_{13} = r_{10}(60P^3 + 90P^2 L + 90PL^2 + 75L^3)$$

$$r_{14} = r_{10}(120P^4 + 180P^3 L + 180P^2 L^2 + 150PL^3 + 225L^4/2),$$

$$r_{15} = r_{10}(120P^5 + 180\,P^4 L + 180\,P^3 L^2 + 150P^2 L^3 + 150PL^4 + 315L^5/4),$$

$$r_{16} = -PL^3/8, \quad r_{17} = r_{16}(6P + 9L/2),$$

$$r_{18} = r_{16}(30P^2 + 45PL + 45L^2),$$

$$r_{19} = r_{16}(120P^3 + 180\,P^2 L + 180PL^2 + 150L^3),$$

$$r_{20} = r_{16}(360P^4 + 540P^3 L + 540P^2 L^2 + 450PL^3 + 675L^4/2),$$

$$r_{21} = r_{16}(720P^5 + 1080P^4 L + 1080P^3 L^2 + 900P^2 L^3 + 675PL^4 + 945L^5/2)$$

$$r_{22} = r_{16}(720P^6 + 1080\,P^5 L + 1080P^4 L^2 + 900\,P^3 L^3 + 675\,P^2 L^4 + 945P\,L^5/2 + 315\,L^5),$$

$$r_{23} = -3Q^3 L/2, \quad r_{24} = r_{23}(3L/2 + 9Q),$$

$$r_{25} = r_{23}(3L^2/2 + 9QL + 9Q^2),$$

$$r_{26} = r_{23}(3L^3/4 + 9QL/2 + 9Q^2 L + 60Q^3), \quad r_{27} = -3Q^3 L/2,$$

$$r_{28} = r_{27}(2L + 12Q), \quad r_{29} = r_{27}(3L^2 + 18QL + 72Q^2),$$

$$r_{30} = r_{28}(3L^3 + 18QL^2 + 72Q^2 L + 240Q^3),$$

$$r_{31} = r_{28}\left(3L^4/2 + 9QL^3 + 36Q^2L^2 + 120Q^3L + 360Q^4\right),\quad r_{32} = -3PL^3/8,$$

$$r_{33} = r_{32}\left(3L + 2P\right),\quad r_{34} = r_{32}\left(L^2 + 3PL + 2P^2\right),\quad r_{35} = -3Q^3L/2,$$

$$r_{36} = r_{35}\left(6Q + L\right),\quad r_{37} = r_{35}\left(12Q^2 + 3QL + L^2/2\right).$$

Substituting the values of A_1, B_1, C_1, D_1 from the equations (27), (24), (23) and (28) into equation (16), we obtain

$$
\begin{aligned}
a = a_0 + \varepsilon \Bigg\{ & \frac{n_1 a_0^2 c_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_2 a_0 c_0 d_0 \left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{n_3 c_0 d_0^2 \left(1 - e^{-2\mu t}\right)}{2\mu} \\
& + \frac{n_4 a_0^2 b_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_5 a_0 b_0 d_0 e^{-(\lambda+\mu)t}}{(\lambda+\mu)} + \frac{n_6 b_0 d_0^2 \left(1 - e^{-2\mu t}\right)}{2\mu} \\
& + \frac{n_7 a_0^2 c_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_8 a_0 c_0 d_0 \left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{n_9 c_0 d_0^2 \left(1 - e^{-2\mu t}\right)}{2\mu} \\
& + \frac{n_{10} a_0^3 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_{11} a_0^2 d_0 \left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{n_{12} a_0 d_0^2 \left(1 - e^{-2\mu t}\right)}{2\mu} \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
b = b_0 + \varepsilon \Bigg\{ & \frac{m_1 a_0^2 c_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{m_2 a_0 c_0 d_0 \left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{m_3 c_0 d_0^2 \left(1 - e^{-2\mu t}\right)}{2\mu} \\
& + \frac{m_4 a_0^2 b_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{m_5 a_0 b_0 d_0 \left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{m_6 b_0 d_0^2 \left(1 - e^{-2\mu t}\right)}{2\mu} \Bigg\}
\end{aligned}
\tag{30}
$$

$$c = c_0 + \varepsilon \left\{ \frac{l_1 a_0^2 c_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{l_2 a_0 c_0 d_0 \left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{l_3 c_0 d_0^2 e^{-2\mu t}}{2\mu} \right\}$$

$$d = d_0 + \varepsilon \frac{p_4 d_0^3 e^{-2\mu t}}{2\mu}$$

Therefore, we obtain the first order approximate solution of the equation (17) as

$$x(t,\varepsilon) = (a + b\,t + c t^2) e^{-\lambda t} + d\, e^{-\mu t} + \varepsilon u_1(a, b, c, d, t) \tag{31}$$

where a b, c, d are given by the equation (30) and u_1 given by (29).

4. Results and Discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we compare the approximate solution to the numerical solution. With regard to such a comparison concerning the presented technique of this article, we refer the work of Murty *et al.* [9]. First, we have considered the eigenvalues $\lambda = 3.1$, $\mu = 1.0$ $(\lambda \approx 3\mu)$. We have computed $x(t,\varepsilon)$ by (31) in which a, b, c, d are computed by equation (30) and u_1 is computed by equation (29), when $\varepsilon = 0.1$ together with two sets of initial conditions (i) $a_0 = 0.5$, $b_0 = 0.0$, $c_0 = 0.3$, $d_0 = 0.1$ [or $x(0) = 0.599982$, $\dot{x}(0) = -1.749542$, $\ddot{x} = 5.902058$, $\dddot{x}(0) = -21.860722$] and (ii) $a_0 = 0.4$, $b_0 = 0.0$, $c_0 = 0.4$, $d_0 = 0.1$ [or $x(0) = 0.499969$, $\dot{x}(0) = -1.439448$, $\ddot{x} = 5.140599$, $\dddot{x}(0) = -20.739441$] for various values of t and the results are presented in the second column of the Table-1 and Table-2 respectively. The corresponding numerical results (designated by x^*) have been computed by a fourth order Runge-Kutta method and the results are presented in the third column of the Table-1 and Table-2. The percentage errors have also been calculated and are presented in the fourth column of the Table-1 and Table-2. The first column represents various values of t.

Table-1

t	x	x^*	Errors%
0.0	0.599982	0.599982	0.00000
0.5	0.163559	0.163550	0.00550
1.0	0.056936	0.056904	0.05623
1.5	0.022951	0.022912	0.17021
2.0	0.010342	0.010310	0.31037
2.5	0.005155	0.005133	0.43711
3.0	0.002788	0.002774	0.50468
3.5	0.001592	0.001583	0.56854
4.0	0.000938	0.000932	0.64377
4.5	0.000561	0.000558	0.53763
5.0	0.000338	0.000336	0.59523

Initial values are (i) $a_0 = 0.5$, $b_0 = 0.0$, $c_0 = 0.3$, $d_0 = 0.1$ and $\varepsilon = 0.1$

x is computed by (31) and x^* is computed by *Runge-Kutta* method.

Table-2

t	x	x^*	Errors%
0.0	0.499969	0.499969	0.00000
0.5	0.147644	0.147639	0.00338
1.0	0.056938	0.056921	0.02986
1.5	0.024147	0.024127	0.08289
2.0	0.010951	0.010935	0.14631
2.5	0.005381	0.005370	0.20484
3.0	0.002861	0.002854	0.24526
3.5	0.001614	0.001609	0.31075
4.0	0.000944	0.000941	0.31880
4.5	0.000563	0.000561	0.35650
5.0	0.000339	0.000338	0.29585

Initial values are (ii) $a_0 = 0.4,\ \ b_0 = 0.0,\ \ c_0 = 0.4,\ \ d_0 = 0.1$ and $\varepsilon = 0.1$

x is computed by (31) and x^* is computed by *Runge-Kutta* method.

Secondly, we have considered $\lambda = 4.6,\ \ \mu = 1.5\ \ (\lambda \approx 3\mu)$ and $x(t,\varepsilon)$ is computed by (31) when $\varepsilon = 0.1$ together with two sets of initial conditions (i) $a_0 = 0.5,\ \ b_0 = 0.0,\ \ c_0 = 0.3,\ \ d_0 = 0.1$ [or $x(0) = 0.599999,\ \ \dot{x}(0) = -2.549924,\ \ \ddot{x} = 12.004210,\ \ \dddot{x}(0) = -60.204288$] and (ii) $a_0 = 0.4,\ \ b_0 = 0.0,\ \ c_0 = 0.4,\ \ d_0 = 0.1$ [or $x(0) = 0.499999,\ \ \dot{x}(0) = -2.089921,\ \ \ddot{x} = 10.088197,\ \ \dddot{x}(0) = -53.230522$] for various values of t and the results are presented in the second column of the Table-3 and Table-4 respectively. The corresponding numerical results (designated by x^*) have been computed by a fourth order Runge-Kutta method and the results are presented in the third column of the Table-3 and Table-4. The percentage errors have also been calculated and are presented in the fourth column of the Table-3 and Table-4. The first column represents various values of t

Table-3

t	x	x^*	Errors%
0.0	0.599999	0.599999	0.00000
0.5	0.092656	0.092655	0.00107
1.0	0.023288	0.023280	0.03436
1.5	0.008250	0.008244	0.07278
2.0	0.003495	0.003492	0.08591
2.5	0.001592	0.001591	0.06285
3.0	0.000744	0.000743	0.13485
3.5	0.000350	0.000350	0.00000
4.0	0.000165	0.000165	0.00000

Initial values are (i) $a_0 = 0.5,\quad b_0 = 0.0,\quad c_0 = 0.3,\quad d_0 = 0.1$ and $\varepsilon = 0.1$

x is computed by (31) and x^* is computed by *Runge-Kutta* method.

Table-4

t	x	x^*	Errors%
0.0	0.499999	0.499999	0.00000
0.5	0.085137	0.085138	0.00117
1.0	0.023288	0.023284	0.01717
1.5	0.008376	0.008373	0.03582
2.0	0.003525	0.003524	0.02837
2.5	0.001598	0.001597	0.06261
3.0	0.000745	0.000744	0.13440
3.5	0.000350	0.000350	0.00000
4.0	0.000165	0.000165	0.00000

Initial values are (ii) $a_0 = 0.4,\quad b_0 = 0.0,\quad c_0 = 0.4,\quad d_0 = 0.1$ and $\varepsilon = 0.1$

x is computed by (31) and x^* is computed by *Runge-Kutta* method.

From the above four Tables, we observe that the errors are much smaller than 1%.

5. Conclusion

An analytical approximate solution of fourth order more critically damped nonlinear systems is investigated in this article. The results obtained by the solution equation (31) for different sets of initial conditions as well as different damping forces show good coincidence with those results obtained by numerical method. The results may be used in various scientific and industrial fields where control of vibrations is needed.

Acknowledgement:

The author is grateful to Dr. M. Emdadul Hoque, Professor, Department of Mechanical Engineering, Rajshahi University of Engineering and Technology, Rajshahi-6205, Bangladesh, for his assistance to prepare the manuscript.

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