JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 Vol.-15, No.-8, August (2020) pp 109-127 ISSN (Print) 0973-8975

RELATIONSHIP BETWEEN COLORING, EMBEDDING AND DECYCLING NUMBER OF A GRAPH

Sajid Hussain[1], Ren Han[2], Nisar Hussain Khoja[3]

[1,2]School of Mathematical Science, East China Normal University
Shanghai 20062, China

[3]Karakoram International University, Gilgit-Baltistan, Pakistan

[1]sajid.khoja100@yahoo.com, [2]hrn@math.ecnu.edu.cn,
[3]khoja1210@gmail.com.

Corresponding Author: **Ren Han**

https://doi.org/10.26782/jmcms.2020.08.00011

(Received: May 10, 2020; Accepted: July 16, 2020)

Abstract

A set S of vertices of a graph G is said to be a decycling set if $G-S$ is acyclic. The size of a minimum decycling set of G is called the decycling number of G and it is denoted by $\nabla(G)$. In this paper, our chief objectives are to obtain the upper bound of the decycling number of a graph by using graph chromatics number and its order. The relation of the genus of the surface Σ and the decycling number of a graph embedded in surface Σ is studied. The decycling number of a planar graph with n vertices is conjectured to be $\frac{n}{2}$, which is shown in this paper if the girth of the graph is at least four. The decycling number of a graph with n vertices and maximum degree three is proved to be at most $\left\lceil\frac{n}{3}\right\rceil$. Also, we completely investigate the decycling number of the hypercube Q_n.

Keywords : Decycling number, Chromatic number, Maximum degree, Embedding, Girth, hypercube.

I. Introduction

Graphs throughout this paper are finite, undirected and simple. For general terminologies and theoretic notations, we follow [V]. Let $G=(V,E)$ be a graph and $S\subset V(G)$ if $G-S$ is forest, then S is said to be a decycling set of G. The cardinality of a smallest decycling set of G is called the *decycling number* and it is signified by $\nabla(G)$. The decycling sets are also called *feedback vertex sets*, which are heavily in the area of circuit design and deadlock prevention [IX]. Clearly, $\nabla(G) = 0$ if and only if G is acyclic, and $\nabla(G) = 1$ if and if G contains at least one cycle. In 1997 Beineke and Vandell [III] gives an expository result on the decycling number of the complete graph K_n is $n-2$ and the decycling number of the complete bipartite graph $K_{r,t}$ is $r-1$ if $r \leq t$ and they developed a relation between the maximum degree, the order, the size and the decycling number of a graph. And they also proved that $\nabla(G) = \frac{m-n+1}{\nabla-1}$ if G is a connected graph with n vertices, m edges, and maximum degree Δ.

In 2006 Punnim [XVII] explored the relationship between the maximum degree, the order and the decycling number of a graph under the definite condition. Now the question is that, what will be the relationship if the imposed conditions are ignored? In the second part of this paper, firstly we develop the relationship between the chromatic number and the decycling number of a graph. Then we determine the upper bound for the decycling number of a graph using its order and maximum degree. Also, we find the lower bound for the decycling number of a graph using its order and minimum degree.

The decycling number of a graph with maximum degree three has attracted much attention. Zheng et. al. [XXIII] shows that for any cubic graph G of order n without triangles, except for two cubic graphs with $n = 8$, $\nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil$. Punnim [XVII] proved that the decycling number of a graph G with maximum degree at most three, under the special conditions, for example, the condition says that $\boldsymbol{G}$ is not a cubic graph without triangles. In 2017 Ren et al [XIX] explored the relationship between the decycling number and maximum genus of a graph with maximum degree three. Whether the decycling number of a graph G with order n and maximum degree three is at most $\left\lceil \frac{n}{3} \right\rceil$? So, in the third part of this paper we give an affirmative solution.

Albertson and Berman [I] and Erdös et al [VIII] introduced a conjecture on the decycling number of a planer graph, which implies the following

Conjecture 1 *([I] [VIII]) If G is a connected planar graph with n vertices, then $\nabla(G) \leq \frac{n}{2}$.*

This conjecture is still open. In the fourth part of this paper, we prove that the conjecture holds if G is a planar graph with girth at least six. Also, we study the relationship between the decycling number of a graph and the genus of a surface Σ in which the graph is embedded.

In 1997 Beineke and Vandell [III] gives the lower bound for the decycling number of hypercubes graph Q_n and they determined the decycling number of Q_n when n is a smaller number. Pike [XVI] discovered the relation between the decycling number of Q_n and Hamming code. In the fifth part of this paper, we show that an upper bound for the decycling number of $Q_n$is the same as the lower bound in [III]. Thus, the decycling number of Q_n is determined.

Before moving on to our main discussion, we go through some basic terminologies and theoretic notations.

Preliminaries about Graph Theory

Let G be a graph. The degree of v in G, denoted by $d_G(v)$, is the number of edges incident with v in G. The maximum degree and the minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. If v is a vertex in G, then any other vertex which is adjacent to v is called a *neighbors* of v. The set of neighbors of v is denoted by $\mathrm{N}(v)$. A graph H is called an irreducible graph if H has not any cut edge. A *matching* M in a graph G is a set of edges in G such that any two edges in M have not a common end. A cycle with n vertices is said to be an $n-cycles$. The *girth* of a graph is the length of its shortest cycle and if a graph has no cycle, then it girth will be considered as infinity. $\lceil x \rceil$ is the minimum number which is not less than x. In the field of topological graph theory, the *graph embedding* is a special type of drawing of a graph in the surface Σ in which embedded graph's edges do not intersect to each other, except at their endpoints. A surface Σ is called *orientable* if a clockwise rotation can be made around all points consistently and is called non-orientable otherwise. All the orientable surfaces are homeomorphic to the sphere with k handles, denoted by S_k. And all the non-orientable surfaces are homeomorphic to the sphere with h crosscaps, denoted by N_h . Where k and h are said to be the genus of the orientable surface and non-orientable surface respectively.

II. The Chromatic Number and the Decycling Number of A Graph

A proper k-coloring of a graph G is a function f from $V(G)$ to the set $\{1,2, \dots k\}$ such that any two vertices are received different colors if they are adjacent to each other. The minimum number$k$ of colors needed to color a connected graph G is called its chromatic number and it is denoted by $\chi(G)$. A Subset of vertices receives the same color is called color class, each of the forms an independent set.

Theorem 2.1 *Suppose that G is a graph with n vertices. If $\chi(G) = \alpha$, then,*

$$\nabla(G) \leq \frac{\alpha-1}{\alpha} n - 1$$

Proof. Since $\chi(G) = \alpha$, G has a proper coloring with α colors. Let $X_1, X_2, \ldots, X_\alpha$ be α coloring classes. Without loss of generality, suppose that X_α is the largest coloring class. Then X_α contains $\frac{n}{\alpha}$ vertices. For any vertex v in $V(G)\backslash X_1, X_2, \ldots, X_\alpha \cup \{v\}$ does not contain any cycle. So $V(G)\backslash (X_\alpha \cup \{v\})$ is a decycling set of G. Since $X_\alpha \cup \{v\}$ has at least in $\frac{n}{\alpha} + 1$ vertices, we have that $\nabla(G) \leq \frac{\alpha-1}{\alpha} n - 1$. ■

By a result in [III], the decycling number of the complete graph $\nabla(K_n) = n - 2$. So, the upper bound in Theorem 2.1 is tight.

Theorem 2.2 *Suppose that G is a graph with n vertices, and suppose that α is a positive integer. If $\nabla(G) \leq \frac{\alpha-1}{\alpha} n - 1$, then $\chi(G) \geq \alpha$.*

Proof. Assume that $\chi(G) = \theta < \alpha$. Then $\nabla(G) \leq \frac{\theta-1}{\theta} n - 1$ by Theorem 2.1. Since $\frac{\theta-1}{\theta} < \frac{\alpha-1}{\alpha}$ if $< \alpha$, we have that $\nabla(G) \leq \frac{\alpha-1}{\alpha} n - 1$, a contradiction. So $\chi(G) \geq \alpha$.

■

It is important to say that $\chi(G)$ must not be α if $\nabla(G) \leq \frac{\alpha-1}{\alpha} n - 1$. Let us consider the following graphs.

Example.*1* Let H be the graph obtained from $K_{3,3}$ by adding an edge to connect two non-adjacent vertices, say x and y. Then $\nabla(H) = 2$ and $S = \{x, y\}$ is a decycling set of H, but $\chi(H) = 3$.

Lemma 2.3 *[VI] let G be a connected graph. If G is either complete graph nor an odd cycle, then $\chi(G) \leq \nabla(G)$.*

The upcoming result follows from theorem 2.1 and lemma 2.3.

Corollary 2.4 *Suppose that G is a graph which is neither complete graph nor an odd cycle. If G has n vertices with maximum degree Δ. Then*

$$\nabla(\mathrm{G}) \leq \frac{\Delta - 1}{\Delta} \mathrm{n} - 1$$

Punnim [XVII] shows that if G is a connected $r - regular$ graph of order $n > 2r + 2$, then $\nabla(G) \leq \frac{r-2}{r} n$ for all $r \geq 4$. We note that the condition $n > 2r + 2$ cannot be ignored in Punnim's result. Otherwise, it would be wrong if G is the graph K_n when $n \geq 4$. However, in our results, there is no restriction on order of the graph.

A forest k-coloring of a graph G is a function f from $V(G)$ to the set $\{1,2,\ldots,k\}$ such that each color class induces an acyclic subgraph. The *vertex-arboricity* of G, denoted by $va(G)$, is the minimum number of subsets in a partition of the vertex set of G in such a way that each subset induces an acyclic subgraph (or forest k-coloring). In fact, $va(G) \geq 1$ for all non-empty graph G and $va(G) = 1$ iff G itself a forest. We now consider the relationship between the vertex arboricity and the decycling number of a graph.

Theorem 2.5 *Suppose that G is a graph with n vertices. If $va(G) = \beta$, then*

$$\nabla(\mathrm{G}) \leq \frac{\beta - 1}{\beta} \mathrm{n}$$

Proof. Let us consider that f is a forest $\beta -$ coloring of G. Then there is a class X with at least $\frac{n}{\beta}$ vertices. So $\mathrm{V(G)} \backslash \mathrm{X}$ is a decycling set of G. Since $\mathrm{V(G)} \backslash \mathrm{X}$ contains at most $(1 - \frac{1}{\beta})n$ vertices, we have that $\nabla(\mathrm{G}) \leq \frac{\beta-1}{\beta}$ n.

■

Lemma 2.6 *[VII] For any graph G, $va(G) \leq \left\lceil \frac{1+\Delta(G)}{2} \right\rceil$*

The Theorem 2.7 obtained from Theorem 2.5 and Lemma 2.6

Theorem 2.7 *Suppose that G is a graph with n vertices and maximum degree Δ. Then*

$$\nabla(\mathrm{G}) \leq \frac{\left\lceil \frac{1+\Delta(\mathrm{G})}{2} \right\rceil - 1}{\left\lceil \frac{1+\Delta(\mathrm{G})}{2} \right\rceil} \mathrm{n}$$

The upper bound in Theorem 2.7 is tight. It is sufficient to consider the complete graph k_n, where n is a positive even number. Upon the relation between the minimum degree and the decycling number of a graph, we have the following theorem.

Theorem 2.8 *Suppose that G is a graph with n vertices. If $\delta(G) \geq 4$, then $\nabla(G) \leq \delta(G) - 1$.*

***Proof*.** Suppose on the contrary that $\nabla(\mathrm{G}) \leq \delta(\mathrm{G}) - 2$. Let S be a decycling set with $\delta(\mathrm{G}) - 2$ vertices of G. Then the minimum degree of $G - S$ is at least two. So $G - S$ has at least one cycle, a contradiction. Hence $\nabla(\mathrm{G}) \leq \delta(\mathrm{G}) - 1$.
■

The lower bound in Theorem 2.8 is also sharp. It is sufficient to consider for the complete graph k_n, where $n \geq 5$. Next, we determine the decycling number of two graphs.

Theorem 2.9 *Suppose that $n \geq 5$ is an integer. Let G be the graph obtained from the complete graph k_n by deleting a matching. Then $\nabla(G) = n - 3$.*

Proof. Suppose that $V(k_n) = \{v_1, v_2, \dots, v_n\}$. Without loss of generality, suppose that the matching is $\{v_1v_2, v_3v_4 \dots, v_{2k-1}v_{2k}\}$, where $2k \leq n$. Then $\{v_4, v_5, \dots, v_n\}$ is a decycling set of G. So $\nabla(\mathrm{G}) \leq n - 3$. Since $\delta(G) = n - 2$ we have that $\nabla(\mathrm{G}) \geq n - 3$ by Theorem 2.8. Hence $\nabla(\mathrm{G}) = n - 3$. ■

Theorem 2.10 *Suppose that $n \geq 6$ is an integer. Let G be the graph obtained from the complete graph k_n by deletion, all edges in a subgraph which is isomorphic to k_r, where $2 \leq r \leq n - 4$, then $\nabla(G) = n - r - 1$.*

Proof. Clearly, $\delta\,(G) = n - r$. So $\nabla(\mathrm{G}) \geq n - \mathrm{r} - 1$ by Theorem 2.8. Let H be a subgraph of k_n which is isomorphic to k_r. Let v be a vertex in $V\,(k_n) \backslash V\,(H)$. Then the graph induced by $V(H) \cup \{v\}$ does not contain any cycle. Since $V\,(H) \cup \{v\}$ has $r + 1$ vertices, we have that $\nabla\,(\mathrm{G}) \leq n - \mathrm{r} - 1$. Hence $\nabla\,(\mathrm{G}) = n - \mathrm{r} - 1$. ■

As we know that $\nabla(k_n) = n - 2$ if $n \geq 3$. At the end of the section, we consider the inverse proposition.

Theorem 2.11 *Let $n \geq 3$ be an integer. If G has n vertices and $\nabla(G) = n - 2$, then G is isomorphic to the complete graph K_n.*

***Proof*.** It is not difficult to find that G is isomorphic to k_n if $\nabla(G) = n - 2$ for $n = 3$ or $n = 4$. Now we suppose that $n \geq 5$, $\nabla(G) = n - 2$, and $V(G) = \{v_1, v_2, \dots, v_n\}$. If there are two vertices v_j and v_k belongs to graph G, in such a way that they are not adjacent to each other, then G is a subgraph of the graph obtained from K_n by removing the edge $v_j v_k$. Theorem 2.9 implies that, $\nabla(G) \leq n - 3$, a contradiction. So, any two vertices in G are adjacent to each other. Hence G is isomorphic to the complete graph K_n.

■

III. Graph with Maximum Degree Three

Let G be a graph with maximum degree three. If a 3-cycle of G contains a vertex with degree two, then the cycle is said to a singular 3-cycle. Let xy be an edge of G the following procedure is called that x is contracted into y if x is deleted and all neighbors of x other than y are adjacent to y newly.

Theorem 3.1 *Let G be a graph with n vertices and $\Delta(G) \leq 3$. Then*

$$\nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil$$

***Proof*.** Suppose that G is a connected irreducible graph. If G is non-connected, then we similarly consider its components. So $\delta(G) \geq 2$. If $\Delta(G) = 2$, then G is the union of several cycles. Since each cycle has at least three vertices, we have that $\nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil$.

We now suppose that G has at least one vertex with degree three. We use the induction on n to show the proposition. Clearly, $n \geq 4$ in this case. If $n = 4$, then G is one of the G looks like Figure 1.

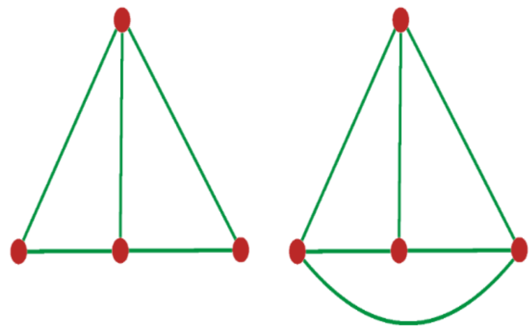

Fig. 1: Two graphs with maximum degree three

It easy to check that $\nabla(G_1) = 1$ and $\nabla(G_2) = 2$. So, the proposition holds.

Assume that the proposition is true if G has at most $n-1$ vertices, where $n \geq 5$. We now consider that G has n vertices. We claim that G has no any cut vertex. Otherwise, G has a cut edge, a contradiction. So G is a $2-connected$ graph.

Let us consider v is a vertex of G with degree three, and suppose that $N(v) = \{v_1, v_2, v_3\}$. If there are two vertices in $N(v)$, say $v_1 and\ v_2$, such that $d_G(v_1) = d_G(v_2) = 2$, then we delete v. Thus, each of v_1 and v_2 has degree one in the present graph. Next, they are deleted. Note that this procedure does not affect the decycling number.

Let G' be the obtained graph. Then G' is a graph with $\nabla(G') \leq 3$, and $\nabla(G) \leq \nabla(G')$+1. By the induction assumption, $\nabla(G') \leq \left\lceil \frac{n}{3} \right\rceil - 1$. So $\nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil$.

If there is only one neighbor of v, say v_1, which has degree two, then we consider two cases. If the neighbor of v_1 other than v is one of v_2 and v_3, say v_2, then we delete v at first. Thus, the degrees of $d(v_1) = 1$ and $d(v_2)$ =2, in the present graph. Next, we delete v_1, then v_2. Let G'' be the obtained graph. Then G'' is a graph with $\nabla(G'') \leq 3$. By the similar argument as that in the above paragraph, $\nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil$. If v_1 is not adjacent to anyone of v_2 and v_3, then there is no any singular 3-cycle passing through v_1. In this case v_1 is contracted into v. If the neighbor of v_1 other than v has degree two, then we proceed the similar argument, and so on. In the end, we have $\nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil$ or each neighbor of v has degree three.

We now suppose that $d_G(v_i) = 3$ for $i = 1,2,3$. Let H be the graph obtained from G by deleting v. Then $d_H(v_i) = 2$ for $i = 1,2,3$. If v_i is not in any 3-cycle in H for $i = 1,2,3$, then it is contracted it's a neighbor. Need to say that if v_i is adjacent to one of $\{v_1, v_2, v_3\}\backslash\{v_i\}$, say v_j, then v_i is contracted into v_j firstly, and v_j contracted into it's a neighbor, and so on. Let H' is a graph with $\nabla(H') \leq 3$. By the induction hypothesis $\nabla(H') \leq \left\lceil\frac{n}{3}\right\rceil - 1$. So $\nabla(G) \leq \nabla(H') + 1 \leq \left\lceil\frac{n}{3}\right\rceil$.

If there is a vertex in v_1, v_2 and v_3, say v_1, such that it is in some singular 3-cycle C, suppose that the neighbors of v_1 are x_1 and x_2. Then $C = x_1\, v_1\, x_2\, x_1$. We delete x_1 at first. Then the degree of v_1 is one and the degree of x_2. is at most two in the present graph. Next, v_1 and v_2 are deleted in this same order. Note that the deletion of v_1 and x_2 does not affect the decycling number. Let H'' be the obtain graph. If H'' also has a singular 3-cycle, then the cycle is dealt with as C, and so on. Suppose that k singular 3-cycles are being deleted. Let F be the obtained irreducible graph, i.e., F has not any cut edge.

If F has not any vertex, then $n = 3k + 1$. Considering that k vertices are deleted, we have that $\nabla(G) \leq \mathrm{k} + 1$, we have that $\nabla(G) \leq \left\lceil\frac{n}{3}\right\rceil$. If F has at least one vertex and any edge, then $\nabla(G) \leq \left\lceil\frac{n}{3}\right\rceil$ by a similar argument. If F has at least one edge, then $\delta(F) \geq 2$, since F is an irreducible graph. In this case, we claim that F is not a 3-regular graph. Otherwise, G is not connected, which violates the assumption. So F has at least one vertex with degree two. If F has only one vertex with degree two, then G has a cut edge or is not connected, a contradiction. Hence F has at least two vertices with degree two. Let y_1 and y_2 be two vertices with degree two if F. Then each of them is not in a 3-cycle. Otherwise, F has a singular 3-cycle. Let $\geq 4-cycle$ be a cycle whose length no less than four. If both y_1 and y_2 are not in the same $\geq 4-cycle$, then they are contracted respectively. Let F' be obtained graph. By the induction hypothesis, $\nabla(F') \leq \left\lceil\frac{n-3k-1-2}{3}\right\rceil$, that is $\nabla(F') \leq \left\lceil\frac{n}{3}\right\rceil - k - 1$. Thus $\nabla(\mathrm{G}) \leq \left\lceil\frac{n}{3}\right\rceil$.

Otherwise, suppose that C' is the $\geq 4-cycle$ which contains y_1 and y_2. If the length of C' is at least five, then y_1 and y_2 are contracted. By a similar argument as the above, we have that $\nabla(\mathrm{G}) \leq \left\lceil\frac{n}{3}\right\rceil$.

If C' is a 4-cycle, let z, w be two vertices in $V(C)\backslash\{y_1, y_2\}$. If $C' = zy_1\, y_2 z$, then $d_F(z) = d_F(w) = 2$ or $d_F(z) = d_F(w) = 3$. Otherwise, one of z and w is incident with a cut edge in F, a contraction. If $d_F(z) = d_F(w) = 2$, there are two cases to consider. If F is exactly the cycle C', then $\mathrm{n} \geq 3\mathrm{k} + 5$ and $\nabla(\mathrm{F}) = 1$. So $\nabla(\mathrm{G}) \leq k + 2$. Since $\left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{3k+5}{3} \right\rceil = k + 2$, we have that $\nabla(\mathrm{G}) \leq \left\lceil \frac{n}{3} \right\rceil$.

If $d_F(z) = d_F(w) = 3$, then we delete the vertex z the degrees of y_1, y_2 and w are 1,1, and 2, respectively, in the present graph. Next, y_1, y_2 and w are deleted in this order. The most interesting point is that this procedure does not affect the decycling number. Let B be the obtained irreducible graph. If B has not any edge, then $\nabla(\mathrm{G}) \leq \left\lceil \frac{n}{3} \right\rceil$. Otherwise, B has at least two vertices with degree two by a similar argument as F. Next, a vertex with degree two is contracted into its a neighbor. Let B' be the obtained graph. So $\nabla(\mathrm{G}) \leq \nabla(B') + \mathrm{k} + 2$. Since B' has at most $n - 3k - 6$ vertices, $\nabla(B') \leq \left\lceil \frac{n}{3} \right\rceil - k - 2$ by the induction hypothesis. Hence $\nabla(\mathrm{G}) \leq \left\lceil \frac{n}{3} \right\rceil$.

If $C' = zy_1 w\, y_2 z$, then we proceed the similar arguments as the above paragraph. So $\nabla(\mathrm{G}) \leq \left\lceil \frac{n}{3} \right\rceil$. ■

IV. Graphs on Surfaces

A surface Σ is a compact connected 2-dimensional manifold without boundary. The orientable surface S_{k} $(k \geq 0)$ can be obtained from the sphere by attaching k handles, where k is called the genus of the orientable surface. The non-orientable surfaces $N_h (h \geq 1)$ can be obtained from the sphere by attaching h crosscaps, where h is said to be the genus of the non-orientable surface. The classical examples of orientable and non-orientable surfaces are tours S_1 and projective plane N_1.

A graph embedding is a special type of drawing of a graph G on the surface in which embedded graphs edges do not intersect to each other, except at their endpoints. The chromatics number $\chi(\Sigma)$ of a surface Σ is the largest $\chi(G)$ such that G can be embedded in Σ.

Let Σ be a surface and γ be a simple closed curve in Σ. A simple closed curve

γ is said to be non-contractible if it is not null-homotopic. Let Ψ be an embedding of a graph G on a non-spherical surface Σ. The face-width of Ψ denoted by $fw(\Psi)$, is $min\{|\gamma \cap V(G)| : \gamma$ is a non-contactable simple close curve in Σ and $\gamma \cap G \subset V(G)\}$.

Theorem 4.1 *[X] (Heawood Map Color Theorem)*

(i) $\chi(S_k) = \left\lfloor \frac{7+\sqrt{1+18k}}{2} \right\rfloor, \ for \ k > 0,$

(ii) $(N_h) = \left\lfloor \frac{7+\sqrt{1+24h}}{2} \right\rfloor, \ for \ h = 1 \quad and \ h \geq 3, \ and \ (N_h) = 6.$

The coming up theorem follows from Theorem 2.1 and Theorem 4.1

Theorem 4.2 *Let G be a connected graph with n vertices.*

If G is embedded in the orientable surface S_k, where $k > 0$, then

$$\nabla(G) \leq \frac{\left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor - 1}{\left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor} n - 1$$

(1) Suppose that G is embedded in the non-orientable surface N_h. Then

$$\nabla(G) \leq \frac{\left\lfloor \frac{7+\sqrt{1+24h}}{2} \right\rfloor - 1}{\left\lfloor \frac{7+\sqrt{1+24h}}{2} \right\rfloor} n - 1, \quad if \ h = 0 \ or \ h > 3,$$

And $\nabla(G) \leq \frac{5}{6} n - 1, \quad if \ h = 2.$

Let G be a graph embedded on a surfaceΣ. A cycle $C$of G in the surface Σ is called *non-contractible* if it is not null-homotopic.

The length of the shortest non-contractible cycle is said to be the edge-width. Under the condition of the large edge-width, Thomasson [20] showed the following result.

Theorem 4.3 *[XX] Let G be a graph which is embedded in an orientable surface such that the edge-width is at least 2^{14k+6}. Then $\chi(G) \leq 5$.*

By combining Theorem 2.1 with Theorem 4.3, we get the following result.

Theorem 4.4 *Let G be a graph with n vertices is embedded in an orientable surface S_k in such a way that the edge-width is at least* 2^{14k+6}. *Then* $\nabla(G) \leq \frac{4}{5} n - 1$.

An embedding of a planar graph in the plane is said to be a plane graph. Let us give a lemma at first.

Lemma 4.5 *If G is a plane graph with girth at least six, then G has a vertex with a degree at most two.*

Proof. Suppose on the contrary that $\delta(G) \geq 3$. Let S_i be the number of vertices with degree $i \geq 3$. The G has $n = \sum_{i\geq 3} n_i$ vertices, and $m = \sum_{i\geq 3} in_i$ edges. Let f be the number of faces of G. By Euler's formula, we have that

$$n - m + f = \sum_{i\geq 3} n_i - \frac{1}{2}\sum_{i\geq 3} in_i + f = 2.$$

So $f = 2 + \frac{1}{2}\sum_{i\geq 3} in_i - \sum_{i\geq 3} n_i$. Thus

$$6f - 2m = 12 + \sum_{i\geq 3} (2i - 6)\, n_i > 0$$

Since the girth of G is at least six, we have that $6f \leq 2m$, a contradiction. ∎

We now consider the decycling number of a plane graph with girth at least six.

Theorem 4.6 *If* G *is a plane graph with* n *vertices and girth at least six, then*

$$\nabla(G) \leq \frac{n}{2}.$$

Proof. Assume that the theorem does not hold. Let G be a plane graph with the girth at least six and decycling number more than $\frac{|V(G)|}{2}$ which has as fewer vertices as possible. By lemma 4.5, G has a vertex v such that $d(v) \leq 2$. If $d(v) = 2$, then the vertex v will be deleted. This procedure does not affect the decycling number, there is a clear contradiction. So, we can suppose that $d(v) = 2$.

Let u be a neighbor of v and now we remove u at first, then v is removed, in such a way we obtain a new graph G'. If G' has not to cycle, then $\nabla(G) \leq 1$.

Obviously, $\nabla(G) \leq \frac{n}{2}$. Otherwise G' has at least one cycle, and the girth of $G'$is at least six. So $\nabla(G') \leq \frac{|V(G')|}{2}$. Since G' has $n-2$ vertices and $\nabla(G) \leq \nabla(G')+1$, we have that $\nabla(G) \leq \frac{n}{2}$, a contradiction. ■

For a graph with girth at least six which is embedded in the projective plane, we have the following result.

Theorem 4.7 *Let G be a graph with n vertices and girth at least six. If G is embedded into the projective plane with face-width k, then* $\nabla(G) \leq \frac{n+k-1}{2}$.

Proof. Let Σ contains a non-contractible closed curve that intersects the graph at k vertices. Then $k-1$ ones in those k vertices are eliminated. Thus, we obtained an embedding Π of a graph G' with $n-k+1$ vertices. Obviously, the face-width of Π is one. So G' is a planar graph with girth at least six (Π can be changed into an embedding in the plane by cutting the surface along a non-contractible closed curve passing the rest one of those k vertices and identifying the vertex with its copy). By Theorem 4.6, $\nabla(G') \leq \frac{n-k+1}{2}$. Since $\nabla(G) \leq \nabla(G') + (K-1)$, thus we have that $\nabla(G) \leq \frac{n+k-1}{2}$.

■

Theorem 4.8 *Let G be a graph with n vertices and girth at least six. If G is embedded in the torus with face-width k, then*

$$\nabla(G) \leq \frac{n+k}{2}.$$

Proof. Let ρ be a non-contractible closed curve such that it crosses k vertices in G. Then those k vertices are deleted. Thus, we obtain an embedding Π of a graph G' with $n-k$ vertices. Moreover, G' is a planar graph with girth at least six. Next, we proceed the similar argument as that in the proof of Theorem 4.7.

■

V. The Decycling Number of the Hypercube Q_n

The hypercube Q_n is an important regular graph which has numerous definitions. For $n \geq 2$, it can be reclusively defined as the Cartesian product of k_2 and Q_{n-1}, that is $Q_n = k_2 \times Q_{n-1}$. Beinke and Vandell [III] has been determined the decycling number of Q_n for $n \leq 8$, and they also showed that $\nabla Q_n \geq 112.2^{n-8}$ if $n \geq 8$. Since $112 = 2^7 - 2^4$, in such a way that we have the following result.

Lemma 5.1 *[III]* $\nabla(Q_n) \geq 2^{n-1} - 2^{n-4}$ $for\, n \geq 8$.

The hypercube Q_n is an n-regular graph with 2^n vertices. For the convenience, the vertices in Q_n can be treated as binary words of length n. If u is a binary word, the number of 1's is said to be the Hamming weight of u. For two binary words x and y, the hamming distance between x and y in Q_n, denoted by $\tau Q_n(x, y)$, is the number of bits in which x and y differ. Two vertices of $Q_n$are adjacent to each other if and only if Hamming distance between them is one.

Since $\chi(K_2) = 2$, it is easy to show that $\chi(Q_n) = 2$ by the induction on n. Since $Q_n = k_2 \times Q_{n-1}$, Q_n can obtain from Q_{n-1} and its copy Q'_{n-1} by adding 2^{n-1} edges. A vertex $(e_1, e_2, \ldots e_{n-1}, e_n)$ in Q_n is define as follows. If $(e_1, \ldots, e_{n-1})$ is in Q_{n-1}, then $e_n = 0$. if $(e_1, \ldots, e_{n-1})$ is in Q'_{n-1}, then $e_n = 1$. So a 2-coloring of Q_n can be obtained recursively. Any vertex in Q_{n-1} with even Hamming weight is colored by 1, and any vertex in Q_{n-1} with odd Hamming weight is colored by 2. However, any vertex in Q'_{n-1} with odd Hamming weight is colored by 1, and any vertex in Q'_{n-1} with even Hamming weight is colored by 2. For the convenience let A_n be the set of all vertices in Q_n in which each is colored by 1, and let B_n the set of all vertices in Q_n in which each is colored by 2. In the following discussion the 2-coloring $C(Q_n)$ is always the above coloring.

Lemma 5.2 *For any two words x and y in the same coloring class in the 2-coloring $C(Q_n)$, if $\tau Q_n(x, y) \geq 3$, then $N(x) \cap N(y) = \emptyset$.*

***Proof*.** Assume that $N(x) \cap N(y) = \emptyset$. Let z be a word in $N(x) \cap N(y)$. Then $\tau Q_n(x, z) = 1$ and $\tau Q_n(y, z) = 1$. So $\tau Q_n(x, y) \leq 2$, a contradiction. ■

Lemma 5.3 S*uppose that S is subset of A_n(or B_n) in the 2-coloring $C(Q_n)$. If $N(x) \cap N(y) = \emptyset$ for any two words x and y in S, then $A_n - S$ (or $B_n - S$) is a decycling set of Q_n.*

Proof. Let $H = Q_n - (A_n - S)$. The $V(H) = S \cup B_n$. Clearly, both S and B_n are independent set. If $N(x) \cap N(y) = \emptyset$ for any two words x and y in S. Similarly $B_n - S$ is a decycling set of Q_n. ■

We now give a subset S_1 in Q_1 and a subset S'_1 in Q'_8.

$$S_1 = \begin{Bmatrix} (1\ 1\ 0\ 0\ 0\ 0\ 0\ 0), S \\ (0\ 0\ 1\ 1\ 0\ 0\ 0\ 0), \\ (1\ 0\ 1\ 0\ 1\ 1\ 0\ 0), \\ (0\ 1\ 0\ 1\ 1\ 1\ 0\ 0), \\ (0\ 0\ 0\ 0\ 1\ 0\ 1\ 0), \\ (1\ 1\ 1\ 1\ 1\ 0\ 1\ 0), \\ (0\ 1\ 1\ 0\ 0\ 1\ 1\ 0), \\ (1\ 0\ 0\ 1\ 0\ 1\ 1\ 0), \\ (1\ 1\ 0\ 0\ 0\ 0\ 1\ 1), \\ (0\ 0\ 1\ 1\ 0\ 0\ 0\ 1), \\ (1\ 0\ 1\ 0\ 1\ 1\ 0\ 1), \\ (0\ 1\ 0\ 1\ 1\ 1\ 0\ 1), \\ (0\ 0\ 0\ 0\ 1\ 0\ 1\ 1), \\ (1\ 1\ 1\ 1\ 1\ 0\ 1\ 1), \\ (1\ 0\ 0\ 1\ 0\ 1\ 1\ 1), \\ (1\ 1\ 0\ 0\ 0\ 0\ 1\ 1), \end{Bmatrix}$$

$$S_1' = \begin{Bmatrix} (1\ 1\ 0\ 0\ 0\ 0\ 0\ 1), \\ (0\ 0\ 1\ 1\ 0\ 0\ 0\ 1), \\ (1\ 0\ 1\ 0\ 1\ 1\ 0\ 1), \\ (0\ 1\ 0\ 1\ 1\ 1\ 0\ 1), \\ (0\ 0\ 0\ 0\ 1\ 0\ 1\ 1), \\ (1\ 1\ 1\ 1\ 1\ 0\ 1\ 1), \\ (0\ 1\ 1\ 0\ 0\ 1\ 1\ 1), \\ (1\ 0\ 0\ 1\ 0\ 1\ 1\ 1), \\ (1\ 1\ 0\ 0\ 0\ 0\ 1\ 0), \\ (0\ 0\ 1\ 1\ 0\ 0\ 0\ 0), \\ (1\ 0\ 1\ 0\ 1\ 1\ 0\ 0), \\ (0\ 1\ 0\ 1\ 1\ 1\ 0\ 0), \\ (0\ 0\ 0\ 0\ 1\ 0\ 1\ 0), \\ (1\ 1\ 1\ 1\ 1\ 0\ 1\ 0), \\ (1\ 0\ 0\ 1\ 0\ 1\ 1\ 0), \\ (1\ 1\ 0\ 0\ 0\ 0\ 1\ 0), \end{Bmatrix}$$

We now recursively define a subset of $V(Q_1)$ (or $V(Q_1')$ based on S_1 ($or\ S_1'$) for $n \geq 9$. Suppose that S_{n-8} and S'_{n-8} have been constructed. For a word $(e_1, \dots . e_{n-1}, e_n)$ in S_{n-7} , if $(e_1, \dots . e_{n-1})$ in S_{n-8}, then $e_n = 0$; if $(e_1, \dots . e_{n-1})$ in S'_{n-8}, then $e_n = 1$. So $|S_{n-7}| = |S_{n-8}| + |S'_{n-8}|$ ($|X|$ is the cardinality of X). For a word $(e_1, \dots . e_{n-1}, e_n)$ in S'_{n-7} if $(e_1, \dots . e_{n-1})$ in S_{n-8}, then $e_n = 1$; if $(e_1, \dots . e_{n-1})$ in S'_{n-8}, then $e_n = 0$. So $|S'_{n-7}| = |S_{n-8}| + |S'_{n-8}|$.

Using the induction on n, it is easy to show the following result.

Lemma 5.4 *For* $n \geq 8$, $|S_{n-7}| = |S'_{n-7}| = 2^{n-4}$.

S_{n-7} and S'_{n-7} has the following properties.

Theorem 5.5 *Suppose that* $n \geq 8$.

(1) *For any two words* x *and* y *in* S_{n-7} *(or* S'_{n-7}*)*, $\tau Q_n(x,y) \geq 3$ *and* $\tau Q'_n(x,y) \geq 3$),

(2) *If* u *is a word in* S_{n-7}*, then there is only one-word* v *in* S'_{n-7} *such that* $\tau Q_{n+1}(u,v) = 2$. *For any word* w *in* $S'_{n-7} \backslash \{v\}$, $\tau Q_{n+}(u,w) \geq 3$.

(3) *If* u' *is a word* w S'_{n-7}*, then there is one word* v' *in* S_{n-7} *such that* $\tau Q_{n+1}(u',v') = 2$. *For any word* w' *in* $S_{n-7} \backslash \{v'\}$, $\tau Q_{n+1}(u',w') \geq 3$.

Proof. We use the induction on n. If $n = 8$, it can be checked that the proposition holds. Assume that the proposition is true for $n = k$. We now consider the case that $n = k + 1$.

(1) Suppose that $x = (e_1, \dots, e_k, e_{k-1})$ and $y = (f_1, \dots, f_k, f_{k-1})$ are two words in S_{k-6}. If $e_{k+1} = 0$, then $(e_1, \dots, e_k) \in S_{k-7}$. If $f_{k+1} = 0$, then $(f_1, \dots, f_k) \in S_{k-7}$. By the induction assumption, $\tau Q_k((e_1, \dots, e_k), (f_1, \dots, f_k)) \geq 3$. So $\tau Q_{k+1}(x,y) \geq 3$. If $f_{k+1} = 1$, then $(f_1, \dots, f_k) \in S'_{n-7}$. By the induction assumption, $\tau Q_k((e_1, \dots, e_k), (f_1, \dots, f_k)) \geq 2$. So $\tau Q_{k+1}(x,y) \geq 3$. If $e_{k+1} = 1$, then $(e_1, \dots, e_k) \in S'_{n-7}$. Proceeding the similar argument, we have that $\tau Q_{k+1}(x,y) \geq 3$. If x and y are two words in $S'_{k-6}(x,y) \geq 3$.

By the similar argument.

(2) Suppose that $u = (S_1, \dots, S_k, S_{k-1})$ and $v = (t_1, \dots, t_k, t_{k-1})$ are two words in S_{k-6} and S'_{n-6}, respectively. If $S_{k+1} = 0$, then $(S_1, \dots, S_k) \in S_{k-7}$. If $t_{k+1} = 0$, then $(t_1, \dots, t_k) \in S'_{n-7}$. By the induction assumption, $\tau Q_k((S_1, \dots, S_k), (t_1, \dots, t_k)) \geq 3$. So $\tau Q_{k+1}(u,v) \geq 3$ if $t_{k+1} = 1$, then $(t_1, \dots, t_k) \in S_{k-7}$. If $(S_1, \dots, S_k) = (t_1, \dots, t_k)$, then $\tau Q_{k+1}(u,v) = 2$. If $(S_1, \dots, S_k) \neq (t_1, \dots, t_k)$, then $\tau Q_k((S_1, \dots, S_k), (t_1, \dots, t_k)) \geq 3$. So $\tau Q_{k+1}(u,v) \geq 3$. If $t_{k+1} = 1$, then $(S_1, \dots, S_k) \in S'_{n-7}$. Continue the similar argument, we got that $\tau Q_{k+1}(u,v) = 2$ if $(S_1, \dots, S_k) = (t_1, \dots, t_k)$ and $Q_{k+1}(u,v) \geq 3$ otherwise.

(3) Suppose that u' *and* v' are two words in S'_{n-7} and S_{k-7}, respectively. By a similar argument, as we have given in (2), we get the desired result. ■

Theorem 5.6 $\nabla(Q_n) \leq 2^{n-1} - 2^{n-4}$ *for* $n \geq 8$.

Proof. We use the induction on n. The base case is that $n = 8$. Since each word in

S_1 has even Hamming weight, S_1 is a subset of A_8 in the 2-coloring $C(Q_8)$. By lemma 5., for any two words x And y in S_1, $\tau Q_{k+1}(x,y) \geq 3$. By lemma 5.2, $N(x) \cap V(y) = \emptyset$. So A_8-S_1 is a decycling set of Q_8 by lemma 5.3. Similar, B_8-S_1' is a decycling set of Q_8'. By lemma 5.4, A_8-S_1 has $2^7 - 2^4$ words. So, the proposition holds.

Assume that the proposition is true for $n = k$. We now consider the case that $n = k + 1$.

Since each word in S_{n-6} has even Hamming weight, S_{n-6} is a subset of A_{K+1} in the 2-coloring $C(Q_{k+1})$. By Lemma 5.5, for any two words u and v in S_{n-6}, $\tau Q_{k+1}(u,v) \geq 3$. By Lemma 5.2, $N(u) \cap N(v) = \emptyset$. So $A_{Q_{k+1}} - S_{k-6}$ is a decycling set of Q_{k+1} by lemma 5.3. Similarly, B_{k+1}- S_{k-6}' is a decycling set of Q_{K+1}'. By Lemma 5.4, $A_{Q_{k+1}} - S_{k-6}$ has $2^k - 2^{k+3}$ words. Hence $\nabla(Q_n) \leq 2^k - 2^{k+3}$. Thus, the proof is completed.

■

The theorem below follows from Theorem 5.1 and Theorem 5.6.

Theorem 5.7 $\nabla(Q_n) \leq 2^{n-1} - 2^{n-4}$ for n≥ 8.

VI. Conclusion

In this contribution, we successfully obtained the upper bound of the decycling number of a graph by using graph chromatics number and its order, and the relation of the genus of the surface and the decycling number of a graph embedded in surface has been studied. The decycling number of a planar graph with n vertices is conjectured to be $\frac{n}{2}$, which is shown in this paper if the girth of the graph is at least four. The decycling number of a graph with n vertices and maximum degree three is proved to be at most $\left\lceil \frac{n}{3} \right\rceil$. Also, we have been investigated the decycling number of the hypercube graphs.

Acknowledgements

The authors would like to express their gratitude to the referees for their kind guidance on the original manuscript, which make this work possible.

Reference

I Albertson M and Berman D., The acyclic chromatic number, Congr. Number., 17 (1976), 51-69.

II Bau S and Beineke L., The decycling number of graphs, Australas J. Combin., 25 (2002), 285-298.

III Beineke L and Vandell R., Decycling graphs, J. Graph Theory, 25(1997), No. 1:59-77.

IV Beineke L and Harary F., The genus of the n-cube, Canad. J.Math. 17 (1965), 494-496.

V Bondy J. A and Murty U.S.R., Graph Theory, Springer, 2008.

VI Brooks R.L, on coloring the nodes of a network, Proc. Cambridge Philos. Soc. 37 (1941), 194-197.

VII Chartrand G, Kronk H.V and Wall C.E., The point-arboricity of a graph, Israel J. Math. (1968) 6:169C175.

VIII Erd Ös P, Saks M and Sós V, Maximum induced trees in graphs, J. Combin. Theory Ser. B, 41 (1986), 61-79.

IX Festa P, Pardalos P.M and Reseude M.G.C., Feedback set problem, in Handbook of Combinatorial Optimization, Supplement Vol. A Kluwer, Dordrecht, (1999), 209-258.

X Heahood P.J., Map-colour theorem, Quart. J. Pure Appl. Math. 24 (1890), 332-338.

XI Karp R.M, Miller R.E., and Thatcher J.M., Reducibility among combinatorial problems. J. Symb. Log. 40 (1975) : 618C619.

XII Kirchhoff K, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme Gführt wird, Ann. Phy. Chem, 72 (1847) 497-508.

XIII Liu J.P and Zhao C, A new bound on the feedback vertex sets in cubic graphs, Discrete Math., 184(1996), 119-131.

XIV Long S.D., Ren H, The decycling number and maximum genus of cubic graphs, Journal of Graph Theory, 88 (2018), 375 C384.

XV Pike D.A, Decycling hypercubes, Graphs and Combin., 19 (2003), 547-550.

XVI Pike D.A and Zou Y., Decycling cartesian products of two cycles, SIAM J. Discrete Math. 19 (2005), No. 3:651-663.

XVII Punnim N, Decycling connected regular graphs, Austral. J. Combin., 35 (2006), 155 169.

XVIII Punnim N, The decycling number of regular graphs, Thai J. Math., 4 (2006), 145-161.

XIX Ren H, Yang C., and T.X Zhao, A new formula for the decycling number of regular graphs, Discrete Math. 340 (2017), 3020-3031.

XX Thomassen C, Five-coloring maps on surfaces, J. Combin. Theory Ser. B, 59(1993), 89-105.

XXI White A.T, Graphs of groups on surfaces, Elsevier, Amsterdam, London, New York, Oxford, Paris, Shannon, Tokyo, 2001.

XXII Wilson R.J, Introduction to graph theory 4th Ed., Addison –Wesley Longman, Reading MA. (1996).

XXIII Zheng M and Lu X, On the maximum induced forests of a connected cubic graph without triangles, Discrete Math., 85 (1990), 89-96.