JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 Vol.-15, No.-7, July (2020) pp 17-39 ISSN (Print) 0973-8975

[0,1] TRUNCATED LOMAX –INVERTED GAMMA DISTRIBUTION WITH PROPERTIES

Jumana A. Altawil[1], Saba N. Al-Khafaji[2], Ahmed Hadi Hussain[3], Sameer Annon Abbas[4]

[1]Iraqi Ministry of Education, The Directorate of Education of the Governorate Holy Karbala.

[2]Department of Mathematics, Faculty of Computer Science and Mathematics, University of Kufa.

[3]Department of Energy Engineering, College of Engineering Al-Musayab, University of Babylon.

[4]Iraqi Ministry of Education, The Directorate of Education of the Governorate, Babylon

[1]jumaa_abbas_alwan@karbala.edu.org, [2]sabanf.mc11p@uokufa.edu.iq, [3]met.ahmed.hadi@uobabylon.edu.iq, [4]sameer.annon@bab.epedu.gov.iq.

Corresponding Author: Ahmed Hadi Hussain

https://doi.org/10.26782/jmcms.2020.07.00003

Abstract

We proposed [0,1] truncated Lomax –Inverted Gamma ([0,1] TLIGD) distribution build on [0,1] truncated Lomax ([0,1] TLD) distribution. General expressions for the statistical properties are obtained, also The Shannon entropy , Relative entropy functions and Stress- Strength model of the ([0,1] TLIGD) are presented

Keywords : [0,1] TLIGD, Shannon entropy and Relative entropy functions, stress strength model.

I. Introduction

Using the work of Eugene and others, we will provide a generalized distribution that may profit us in other areas. Eugene et al. product the cdf for Beta-G distribution, by [II]

$$F(x) = \left(\frac{1}{\beta(a,b)}\right) \int_0^{G(x)} Z^{a-1}(1-Z)^{b-1}\, dZ,\ 0 < a,b < \infty \qquad (1)$$

Where $(a\,,b) = \int_0^1 Z^{a-1}(1-Z)^{b-1}\, dZ$. Jones [2][6] , he referred that $X = G^{-1}(N)$ is the X with CDF in (1) such that N$\sim beta(a,b)$. In addition , Eugene et al proposed the beta normal (BN) distribution by used G (x) to be the cumulative distribution function of the normal distribution and furthermore, general expressions for the statistical properties are obtained [II] [IV] . The (pdf) comparing to (1) can be found as follows,

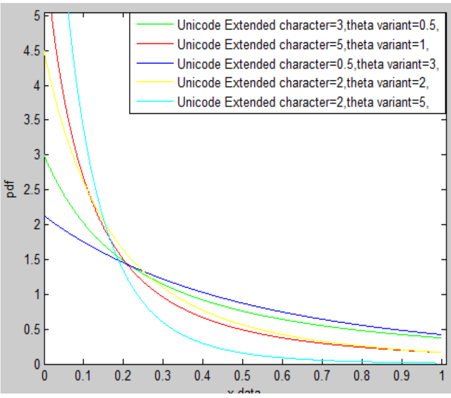
$$f(x) = \frac{1}{\beta(a,b)} G(x)^{a-1}\left(1 - G(x)\right)^{b-1} g(x) \tag{2}$$

Where $g(x) = \frac{\partial G(x)}{\partial x}$ is the (PDF)of the primary distribution [III].

The PDF and CDF of ([0,1] TLD) are given as follows,

$$s(x) = \frac{\eta\, \vartheta\, (1+\eta x)^{-(\vartheta+1)}}{(1-(1+\eta)^{-\vartheta})} \qquad 0 < x < 1 \tag{3}$$

$$S(x) = \frac{(1-(1+\eta x)^{-\vartheta})}{(1-(1+\eta)^{-\vartheta})} \tag{4}$$



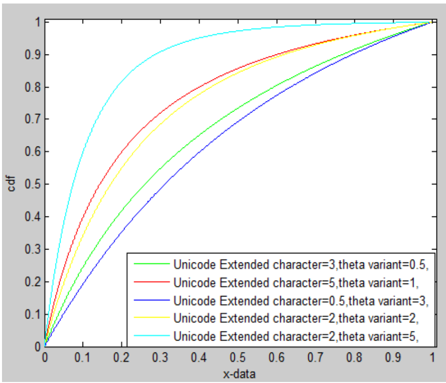


Fig. 1: PDF for ([0,1] TLD) distribution. **Fig. 2:** CDF for ([0,1] TLD) distribution.

We have two continuous cdfs, therefore we produce distribution F from configuring S with G , such

$F(x) = S(G(x))$become a CDF :

$$F(x) = \int_0^{G(x)} \frac{\left[\eta\vartheta\{1+\eta t\}^{-(\vartheta+1)}\right]}{[1-\{1+\eta\}^{-\vartheta}]}\, dt = \frac{1-\{1+\eta G(x)\}^{-\vartheta}}{1-\{1+\eta\}^{-\vartheta}} \tag{5}$$

While PDF :

$$f(x) = \frac{d}{dx} F(x) = \frac{\eta\vartheta[1+\eta G(x)]^{-(\vartheta+1)}}{[1-\{1+\eta\}^{-\vartheta}]} \,.\, g(x) \tag{6}$$

s.t $g(x) = \frac{dG(x)}{dx}$

We express in Eq (5) and (6) , a generalized class of distributions. Calling it([0,1] TLD - G) distribution. Assume G is Inverted Gamma distribution.

II. The ([0,1] TLIGD) Distribution

If it was $g(x) = \frac{\lambda^p}{\Gamma(p)} x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)}$ and $G(x) = \frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)} x > 0$ denoted PDF and CDF for Inverted Gamma distribution , respectively . Using (5)and (6) , obtain the CDF and PDF for ([0,1] TLIGD) distribution.

$$F(x) = \frac{1-\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-\vartheta}}{\left(1-(1+\eta)^{-\vartheta}\right)} \tag{7}$$

$$f(x) = \frac{\vartheta\eta\lambda^p}{\Gamma(p)\left(1-(1+\eta)^{-\vartheta}\right)} x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)} \left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)} \tag{8}$$

Therefore ,the reliability h (x) and hazard rate γ(x) functions, as follows

$$h(x) = 1 - F(x) = 1 - \left[\frac{1-\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-\vartheta}}{\left(1-(1+\eta)^{-\vartheta}\right)}\right] = \frac{\left[-(1+\eta)^{-\vartheta}+\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-\vartheta}\right]}{\left(1-(1+\eta)^{-\vartheta}\right)} \tag{9}$$

$$\gamma(x) = \frac{f(x)}{h(x)} = \frac{\left\{\vartheta\eta\lambda^p x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)}\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}\right\}}{\Gamma(p)\left\{-(1+\eta)^{-\vartheta}+\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-\vartheta}\right\}} \tag{10}$$

The r-thmoment is given as , [5].

$$E(x^r) = \int_0^\infty x^r f(x) dx$$

$$= \int_0^\infty x^r \frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})} x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)} \left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)} dx$$

$$E(x^r) = \frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})} \int_0^\infty x^r x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)} \left(1+\eta\left\{\frac{\Gamma\left(\alpha,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)} dx$$

Now , simplification $\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}$ through use of series expansion[VIII]

$$(1-z)^{-k} = \sum_{i=0}^{\infty} \frac{\Gamma(k+i)}{i!\,\Gamma(k)} z^i \ |z| < 1, \ k > 0 \tag{11}$$

we obtain

$$\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}=\left(1-\left[-\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right]\right)^{-(\vartheta+1)}$$

$$=\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^{d}\left(\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{d}$$

And then

$$E(x^{r})=\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^{d}\int_{0}^{\infty}x^{-(p-r+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{d}dx$$

Since $\Gamma(s,\tau)+\Upsilon(s,\tau)=\Gamma(s)\rightarrow\Gamma(s,\tau)=\Gamma(s)-\Upsilon(s,\tau)$, will thus,

$$\Gamma\left(p,\frac{\lambda}{x}\right)=\Gamma(p)-\Upsilon\left(p,\frac{\lambda}{x}\right)$$

And then

$$E(x^{r})=\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^{d}\int_{0}^{\infty}x^{-(p-r+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(\frac{\Gamma(p)-\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^{d}dx$$

$$E(x^{r})=\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^{d}\int_{0}^{\infty}x^{-(p-r+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^{d}dx$$

And again simplification $\left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^{d}$ and

using $(1-z)^{b}=\sum_{u=0}^{\infty}(-1)^{u}\frac{\Gamma(b+1)}{u!\Gamma(b-u+1)}z^{u}$, $|z|<1$, $b>0$ (12)

we get: $\left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^{d}=\sum_{u=0}^{\infty}(-1)^{u}\frac{\Gamma(d+1)}{u!\Gamma(d-u+1)}\left(\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^{u}$

And then

$$E(x^{r})=\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^{d}\sum_{u=0}^{\infty}(-1)^{u}\frac{\Gamma(d+1)}{u!\,\Gamma(d-u+1)}\int_{0}^{\infty}x^{-(p-r+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^{u}dx$$

By using incomplete gamma function

$$\frac{\Upsilon(p,\lambda x)}{\Gamma(p)}=\frac{(\lambda x)^{p}}{\Gamma(p)}\sum_{n=0}^{\infty}\frac{(-\lambda x)^{n}}{(p+n)n!} \quad (13)$$

we get :

$$=\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^{d}\sum_{l=0}^{\infty}(-1)^{l}\frac{\Gamma(d+1)}{l!\,\Gamma(d-l+1)}\int_{0}^{\infty}x^{-(p-r+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(\frac{\left(\frac{\lambda}{x}\right)^{p}}{\Gamma(p)}\sum_{m=0}^{\infty}\frac{\left(-\frac{\lambda}{x}\right)^{n}}{(p+n)n!}\right)^{l}dx$$

By use of a condition in part 0.314 for power series[VII] , we get for any l positive

$$\{\sum_{n=0}^{\infty} a_n(\lambda x)^n\}^l = \sum_{n=0}^{\infty} C_{l,n}(\lambda x)^n \quad (14)$$

Where, the coefficient $C_{l,n}(for\ n = 1,2, \dots)$ satisfy the recurrence relation

$C_{l,n} = (na_0)^{-1}\sum_{g=1}^{n}(lg - n + g)\, a_g C_{l,n}\ , C_{l,o} = a_0{}^l\ and\ a_g = \frac{(-1)^g}{(p+g)g!}$, we get :

$$\left(\frac{\left(\frac{\lambda}{x}\right)^p}{\Gamma(p)}\sum_{n=0}^{\infty}\frac{\left(-\frac{\lambda}{x}\right)^n}{(p+n)n!}\right)^l = \frac{\left(\frac{\lambda}{x}\right)^{pl}}{(\Gamma(p))^l}\left(\sum_{n=0}^{\infty}\frac{\left(-\frac{\lambda}{x}\right)^n}{(p+n)n!}\right)^l = \frac{\left(\frac{\lambda}{x}\right)^{pl}}{(\Gamma(p))^l}\sum_{n=0}^{\infty} C_{l,n}\left(\frac{\lambda}{x}\right)^n$$

And then

$$E(x^r) = \frac{\vartheta\eta\lambda^{n+p(l+1)}}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\,\Gamma(d-l+1)}\sum_{n=0}^{\infty} C_{l,n}$$
$$\times\int_0^{\infty} x^{-(n+p(l+1)-r+1)}e^{-\left(\frac{\lambda}{x}\right)}\,dx$$

$$E(x^r) = \frac{\vartheta\eta\lambda^r}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty} C_{l,n}\,\Gamma(n+p(l+1)-r) \quad (15)$$

And the characteristic function is

$$\Psi_x(t) = E\left[e^{ixt}\right]$$

$\because e^{ixt} = \sum_{r=0}^{\infty}\frac{\{it\}^r}{r!}x^r$, so $,\Psi_x(t) = E\left[\sum_{r=0}^{\infty}\frac{(it)^r}{r!}x^r\right] = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\,E(x^r)$

$$\Psi_x(t) = \frac{\vartheta\eta}{\{\Gamma(p)\}^{l+1}\{1-(1+\eta)^{-\vartheta}\}}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}\{-\eta\}^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\,\Gamma(d-l+1)}$$
$$\times\sum_{n=0}^{\infty} C_{l,n}\ \Gamma(n+p(l+1)-r) \quad (16)$$

The mean(μ) , variance (σ^2) , skewness(s_k) and the kurtosis (k_r) , can be find

$$\mu = E(x) = \frac{\vartheta\eta\lambda}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty} C_{l,n}\ \Gamma(n+p(l+1)-1) \quad (17)$$

$$\sigma^2 = E(x^2) - \left(E(x)\right)^2$$

$$\sigma^2 = \frac{\vartheta\eta\lambda^2}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty} C_{l,n}\,\Gamma(n+p(l+1)-2) - \left[\frac{\vartheta\eta\lambda}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty} C_{l,n}\ \Gamma(n+p(l+1)-1)\right]^2 \quad (18)$$

$$s_k = \frac{\left(E(x^3)\right)^2}{\sqrt{\left(E(x^2)\right)^3}} = \frac{E(x^3) - 3E(x)E(x^2) + 2\left(E(x)\right)^3}{(\sigma^2)^{\frac{3}{2}}}$$

Then :

$$s_k = \frac{\begin{bmatrix} \frac{\vartheta\eta\lambda^3}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-3) \\ -3\left(\frac{\vartheta\eta\lambda}{(\Gamma(p))^{u+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-1\right) \\ \left(\frac{\vartheta\eta\lambda^2}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-2)\right)+ \\ 2\left(\frac{\vartheta\eta\lambda}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-1)\right)^3 \end{bmatrix}}{\begin{bmatrix} \frac{\vartheta\eta\lambda^2}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-2) \\ -\left(\frac{\vartheta\eta\lambda}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-1)\right)^2 \end{bmatrix}^{\frac{3}{2}}} \quad (19)$$

$$k_r = \frac{E(x^3)}{\left(E(x^2)\right)^2} - 3 = \frac{E(x^4) - 4E(x)E(x^3) + 6\left(E(x)\right)^2 E(x^2) - 3\left(E(x)\right)^4}{(\sigma^2)^2} - 3$$

Then :

$$= \frac{\begin{bmatrix} \frac{\vartheta\eta\lambda^4}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-4) \\ -4\left(\frac{\vartheta\eta\lambda}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-1)\right) \\ \left(\frac{\vartheta\eta\lambda^3}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-3)\right)+ \\ 6\left(\frac{\vartheta\eta\lambda}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-1)\right)^2 \\ \left(\frac{\vartheta\eta\lambda^2}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-2)\right)- \\ 3\left(\frac{\vartheta\eta\lambda}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-1)\right)^4 \end{bmatrix}}{\begin{bmatrix} \frac{\vartheta\eta\lambda^2}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-2) \\ -\left(\frac{\vartheta\eta\lambda}{(\Gamma(p))^{l+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\,\Gamma(n+p(l+1)-1)\right)^2 \end{bmatrix}^2} - 3$$

$$
= \frac{\left[\frac{\eta^4\vartheta^4\lambda^4}{(\Gamma(p))^{4(l+1)}(1-(1+\eta)^{-\vartheta})^4}\left(\begin{array}{c}
\frac{(\Gamma(p))^{3(l+1)}(1-(1+\eta)^{-\vartheta})^3}{\vartheta^3\eta^3} \\
\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-4) \\
-\frac{4(\Gamma(p))^{2(l+1)}(1-(1+\eta)^{-\vartheta})^2}{\vartheta^2\eta^2} \\
\left\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-1)\right\} \\
\left\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-3\right\} \\
+\frac{6(\Gamma(p))^{(l+1)}(1-(1+\eta)^{-\vartheta})}{\vartheta\eta} \\
\left\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-1\right\}^2 \\
\left\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-2\right\} \\
-3 \\
\left\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-1\right\}^4
\end{array}\right)\right]}{\left[\frac{\eta^4\vartheta^4\lambda^4}{(\Gamma(p))^{4(l+1)}(1-(1+\eta)^{-\vartheta})^4}\left(\begin{array}{c}
\frac{(\Gamma(p))^{(l+1)}(1-(1+\eta)^{-\vartheta})}{\vartheta\eta} \\
\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-2) \\
-\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-1\}^2
\end{array}\right)\right]^2} - 3
$$

$$
kr = \frac{\left(\begin{array}{c}
\frac{(\Gamma(p))^{3(l+1)}(1-(1+\eta)^{-\vartheta})^3}{\vartheta^3\eta^3} \\
\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-4) \\
-\frac{4(\Gamma(p))^{2(l+1)}(1-(1+\eta)^{-\vartheta})^2}{\vartheta^2\eta^2} \\
\left\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-1)\right\} \\
\left\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-3\right\} \\
+\frac{6(\Gamma(p))^{(l+1)}(1-(1+\eta)^{-\vartheta})}{\vartheta\eta} \\
\left\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-1\right\}^2 \\
\left\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-2\right\} \\
-3 \\
\left\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-1\right\}^4
\end{array}\right)}{\left(\begin{array}{c}
\frac{(\Gamma(p))^{(l+1)}(1-(1+\eta)^{-\vartheta})}{\vartheta\eta} \\
\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-2) \\
-\{\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\Gamma(\vartheta+1)}(-\eta)^d\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(d+1)}{l!\Gamma(d-l+1)}\sum_{n=0}^{\infty}C_{l,n}\Gamma(n+p(l+1)-1\}^2
\end{array}\right)^2} - 3 \qquad (20)
$$

The Median can be computed numerically , since ,

$$
F(x) = \frac{\left(1-\left[1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right]^{-\vartheta}\right)}{(1-(1+\eta)^{-\vartheta})} = \frac{1}{2},
$$

Then

$$
\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\} = \frac{1}{\eta}\left(\left[1-\left(\frac{\{1-(1+\eta)^{-\vartheta}\}}{2}\right)\right]^{\frac{-1}{\vartheta}} - 1\right)
$$

And by solving the nonlinear equation

$$\Gamma\left(p,\frac{\lambda}{x}\right)-\frac{\Gamma(p)}{\eta}\left(\left[1-\left(\frac{\{1-(1+\eta)^{-\vartheta}\}}{2}\right)\right]^{\frac{-1}{\vartheta}}-1\right)=0$$

Additionally , we can derive quintile function $x_p$as

$$P(x\le x_p)=F_x(x_p)=\frac{\left[1-\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-\vartheta}\right]}{(1-\{1+\eta\}^{-\vartheta}\,)}\qquad ,x_p>0$$

$$\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)=\left(1-F_x(x_p).(1-(1+\eta)^{-\vartheta})\right)^{\frac{-1}{\vartheta}}$$

$$\Gamma\left(p,\frac{\lambda}{x}\right)-\frac{\Gamma(p)}{\eta}\left(\left(1-F_x(x_p).(1-(1+\eta)^{-\vartheta})\right)^{\frac{-1}{\vartheta}}-1\right)=0 \qquad (21)$$

Therefore , by using the inverse transform method, we make generate random variable for ([0,1] TLIGD) , by solving ,

$\Gamma\left(p,\frac{\lambda}{x}\right)-\frac{\Gamma(p)}{\eta}\left(\left(1-U.(1-(1+\eta)^{-\vartheta})\right)^{\frac{-1}{\vartheta}}-1\right)=0$ Numerically , where $0\le U\le 1$

III. Shannon Entropy and Relative Entropy Functions for ([0,1] TLIGD) Distribution [1].

The Shannon entropy of [0,1] TLIGD$(\eta,\vartheta,p,\lambda)$ is presented by,

$$H=-\int_0^\infty f(x)\ln(f(x))\,dx$$

$$H=-\int_0^\infty f(x)\ln\left[\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}\right]dx$$

$$=\int_0^\infty f(x)\left[-\ln\left(\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\right)+(p+1)\ln x+\frac{\lambda}{x}+(\vartheta+1)\ln\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)\right]dx$$

$$H = \ln\left(\frac{\Gamma(p)\left(1-(1+\eta)^{-\vartheta}\right)}{\vartheta\eta\lambda^{p}}\right) + (p+1)E(\ln x) + \lambda E\left(\frac{1}{x}\right) + (\vartheta+1)\boldsymbol{E}\left(\ln\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)\right)$$

Let $I_1 = (p+1)E(\ln x) = (p+1)\int_0^\infty \ln x\ f(x)dx$

$$I_1 = (p+1)\int_0^\infty \ln x \frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})} x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)}\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)} dx$$

Now , simplification $\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}$, applying equation (11) , we obtain

$$I_1 = (p+1)\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{v=0}^{\infty}\frac{\Gamma(\vartheta+1+v)}{v!\,\Gamma(\vartheta+1)}(-\eta)^{v}\int_0^\infty \ln x\, x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{v} dx$$

Since $\Gamma(v,\tau) + \Upsilon(v,\tau) = \Gamma(v) \rightarrow \Gamma(v,\tau) = \Gamma(v) - \Upsilon(v,\tau)$, will thus, $\Gamma\left(p,\frac{\lambda}{x}\right) = \Gamma(p) - \Upsilon\left(p,\frac{\lambda}{x}\right)$

And then

$$I_1 = (p+1)\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{v=0}^{\infty}\frac{\Gamma(\vartheta+1+v)}{v!\,\Gamma(\vartheta+1)}(-\eta)^{v}\int_0^\infty \ln x\, x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Gamma(p)-\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{v} dx$$

$$I_1 = (p+1)\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{v=0}^{\infty}\frac{\Gamma(\vartheta+1+v)}{v!\,\Gamma(\vartheta+1)}(-\eta)^{v}\int_0^\infty \ln x\, x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)}\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{v} dx$$

And again simplification $\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{v}$ by using equation (12) , we get

$$I_1 = (p+1)\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{v=0}^{\infty}\frac{\Gamma(\vartheta+1+v)}{v!\,\Gamma(\vartheta+1)}(-\eta)^{v}\sum_{b=0}^{\infty}(-1)^{b}\frac{\Gamma(v+1)}{b!\,\Gamma(v-b+1)}\int_0^\infty \ln x\, x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{b} dx$$

by using equation (13)and $\left(\sum_{n=0}^{\infty} a_n \left(\frac{\lambda}{x}\right)^n\right)^{b} = \sum_{n=0}^{\infty} C_{b,n}\left(\frac{\lambda}{x}\right)^n$,we get :

$$\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{b} = \left(\frac{1}{\Gamma(p)}\right)^{b}\left\{\Upsilon\left(p,\frac{\lambda}{x}\right)\right\}^{b} = \left(\frac{1}{\Gamma(p)}\right)^{b}\left(\frac{\lambda}{x}\right)^{pb}\left(\sum_{n=0}^{\infty}\frac{\left(-\frac{\lambda}{x}\right)^n}{(p+n)n!}\right)^{b}$$

$$= \left(\frac{1}{\Gamma(p)}\right)^{b}\left(\frac{\lambda}{x}\right)^{pb}\sum_{n=0}^{\infty} C_{b,n}\left(\frac{\lambda}{x}\right)^n$$

Where ,$C_{b,n} = (na_0)^{-1}\sum_{g=1}^{n}(bg-n+g)\, a_b C_{b,n}$, $C_{b,o} = a_0{}^{b}$ *and* $a_g = \frac{(-1)^g}{(p+g)g!}$

$$I_1 = (p+1)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{v=0}^{\infty}\frac{\Gamma(\vartheta+1+v)}{v!\,\Gamma(\vartheta+1)}(-\eta)^v\sum_{b=0}^{\infty}(-1)^b\frac{\Gamma(v+1)}{b!\,\Gamma(v-b+1)}$$

$$\left(\frac{1}{\Gamma(p)}\right)^{b}\sum_{n=0}^{\infty} C_{b,n}\int_0^{\infty}\ln x\, x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(\frac{\lambda}{x}\right)^{pb}\left(\frac{\lambda}{x}\right)^n dx$$

$$I_1 = -(p+1)\frac{\vartheta\eta\lambda^{n+p(b+1)}}{(\Gamma(p))^{b+1}(1-(1+\eta)^{-\vartheta})}\sum_{v=0}^{\infty}\frac{\Gamma(\vartheta+1+v)}{v!\,\Gamma(\vartheta+1)}(-\eta)^v\sum_{b=0}^{\infty}(-1)^b\frac{\Gamma(v+1)}{b!\,\Gamma(v-b+1)}$$

$$\sum_{b=0}^{\infty} C_{b,n}\int_0^{\infty}\ln(\frac{1}{x})\left(\frac{1}{x}\right)^{(n+p(b+1)+1)} e^{-\left(\frac{\lambda}{x}\right)} dx$$

Let $y = \frac{1}{x} \rightarrow x = y^{-1} \rightarrow dx = -y^{-2}dy$ then

$$I_1 = -(p+1)\frac{\vartheta\eta\lambda^{n+p(b+1)}}{(\Gamma(p))^{b+1}(1-(1+\eta)^{-\vartheta})}\sum_{v=0}^{\infty}\frac{\Gamma(\vartheta+1+v)}{v!\,\Gamma(\vartheta+1)}(-\eta)^v\sum_{b=0}^{\infty}(-1)^b\frac{\Gamma(v+1)}{b!\,\Gamma(v-b+1)}$$

$$\sum_{n=0}^{\infty} C_{b,n}\int_0^{\infty}\ln y\,(y)^{(n+p(b+1)-1)}e^{-(\lambda y)}\, dy$$

$$I_1 = -(p+1)\frac{\vartheta\eta}{(\Gamma(p))^{b+1}(1-(1+\eta)^{-\vartheta})}\sum_{v=0}^{\infty}\frac{\Gamma(\vartheta+1+v)}{v!\,\Gamma(\vartheta+1)}(-\eta)^v\sum_{b=0}^{\infty}(-1)^b\frac{\Gamma(v+1)}{b!\,\Gamma(v-b+1)}$$

$$\sum_{n=0}^{\infty} C_{b,n}\,\boldsymbol{\Gamma}(n+p(b+1))\,\{\Psi(n+p(b+1)) - \ln\lambda\}$$

Let $I_2 = \lambda E\left(\frac{1}{x}\right) = \lambda\int_0^{\infty}\left(\frac{1}{x}\right)f(x)\,dx = \lambda\int_0^{\infty}\left(\frac{1}{x}\right)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)} dx$

Now , simplification $\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}$ by using equation (11) , we obtain

$$I_2 = \lambda\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{w=0}^{\infty}\frac{\Gamma(\vartheta+1+w)}{w!\,\Gamma(\vartheta+1)}(-\eta)^w\int_0^{\infty}x^{-(p+2)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{w} dx$$

Since $\Gamma(s,\tau) + \Upsilon(s,\tau) = \Gamma(s) \rightarrow \Gamma(s,\tau) = \Gamma(s) - \Upsilon(s,\tau)$, will thus, $\Gamma\left(p,\frac{\lambda}{x}\right) = \Gamma(p) - \Upsilon\left(p,\frac{\lambda}{x}\right)$

$$I_2 = \lambda \frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{w=0}^{\infty}\frac{\Gamma(\vartheta+1+w)}{w!\,\Gamma(\vartheta+1)}(-\eta)^w \int_0^\infty x^{-(p+2)} e^{-\left(\frac{\lambda}{x}\right)}\left\{1 - \frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^w dx$$

And again simplification $\left\{1 - \frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^w$ by using equation (12) , we get

$$I_2 = \lambda \frac{\vartheta\eta\lambda^\alpha}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{w=0}^{\infty}\frac{\Gamma(\vartheta+1+w)}{w!\,\Gamma(\vartheta+1)}(-\eta)^w \sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(w+1)}{j!\,\Gamma(w-j+1)}\int_0^\infty x^{-(p+2)} e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^j dx$$

by using equation (11)and $\left(\sum_{n=0}^{\infty} a_n\left(\frac{\lambda}{x}\right)^n\right)^{b} = \sum_{n=0}^{\infty} C_{b,n}\left(\frac{\lambda}{x}\right)^n$,we get :

$$\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{b} = \left(\frac{1}{\Gamma(p)}\right)^{b}\left\{\left(\frac{\lambda}{x}\right)^p \sum_{n=0}^{\infty}\frac{\left(-\frac{\lambda}{x}\right)^n}{(p+n)n!}\right\}^{b} = \left(\frac{1}{\Gamma(p)}\right)^{b}\left(\frac{\lambda}{x}\right)^{pb}\sum_{n=0}^{\infty} C_{b,n}\left(\frac{\lambda}{x}\right)^n$$

Where $C_{b,n} = (na_0)^{-1}\sum_{g=1}^{n}(bg-n+g)\, a_g C_{b,n}\, , C_{b,o} = a_0{}^{b}$ $and\ a_g = \frac{(-1)^g}{(p+g)g!}$

$$I_2 = \frac{\vartheta\eta\lambda^{n+p(b+1)+1}}{(\Gamma(p))^{b+1}(1-(1+\eta)^{-\vartheta})}\sum_{w=0}^{\infty}\frac{\Gamma(\vartheta+1+w)}{w!\,\Gamma(\vartheta+1)}(-\eta)^w\sum_{b=0}^{\infty}(-1)^{b}\frac{\Gamma(w+1)}{b!\,\Gamma(w-b+1)}\sum_{n=0}^{\infty}C_{b,n}\int_0^\infty x^{-(n+p(b+1)+2)}e^{-\left(\frac{\lambda}{x}\right)}dx$$

$$I_2 = \frac{\vartheta\eta}{(\Gamma(p))^{b+1}(1-(1+\eta)^{-\vartheta})}\sum_{w=0}^{\infty}\frac{\Gamma(\vartheta+1+w)}{w!\Gamma(\vartheta+1)}(-\eta)^w\sum_{b=0}^{\infty}(-1)^{b}\frac{\Gamma(w+1)}{b!\Gamma(w-b+1)}\sum_{n=0}^{\infty}C_{b,n}\,\boldsymbol{\Gamma}(n+p(b+1)+1)$$

Let $I_3 = (\vartheta+1)E\left(\ln(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\})\right) = (\vartheta+1)\int_0^\infty \ln(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\})f(x)\,dx$

$$I_3 = (\vartheta+1)\int_0^\infty \ln(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\})\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}dx$$

By the same way simplification $\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}$ by using equation (11), we obtain

$$I_3 = (\vartheta+1)\frac{\vartheta\eta\beta^{\alpha}}{\Gamma(\alpha)(1-(1+\eta)^{-\vartheta})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\int_0^{\infty}\ln(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\})\,x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^k dx$$

We can simplification $\ln(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\})$ by using

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}, -1 < x < 1 \qquad (22)$$

We get:

$$\ln(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}) = \ln\left(1-\left(-\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)\right)$$

$$= -\frac{1}{r}\sum_{r=1}^{\infty}\left(-\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^r = -\frac{1}{r}\sum_{r=1}^{\infty}(-\eta)^r\left(\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^r$$

$$I_3 = -(\vartheta+1)\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^k\left(\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^r dx$$

Since $\Gamma(s,\tau)+\Upsilon(s,\tau)=\Gamma(s)\rightarrow\Gamma(s,\tau)=\Gamma(s)-\Upsilon(s,\tau)$, will thus, $\Gamma\left(p,\frac{\lambda}{x}\right) = \Gamma(p)-\Upsilon\left(p,\frac{\lambda}{x}\right)$

$$I_3 = -(\vartheta+1)\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^k\left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^r dx$$

By the same way simplification $\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^k$, $\left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^r$ by using equation (12) , we get

$$\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^k = \sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(k+1)}{l!\,\Gamma(k-l+1)}\left(\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^l, \left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^r = \sum_{d=0}^{\infty}(-1)^d\frac{\Gamma(r+1)}{d!\,\Gamma(r-d+1)}\left(\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^d$$

$$I_3 = -(\vartheta+1)\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(k+1)}{l!\,\Gamma(k-l+1)}\sum_{d=0}^{\infty}(-1)^d\frac{\Gamma(r+1)}{d!\,\Gamma(r-d+1)}\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^l\left(\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^d dx$$

$$I_3 = -(\vartheta+1)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(k+1)}{l!\,\Gamma(k-l+1)}$$

$$\sum_{d=0}^{\infty}(-1)^d\frac{\Gamma(r+1)}{d!\,\Gamma(r-d+1)}\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^{l+d}dx$$

by using equation (13)and $\left(\sum_{n=0}^{\infty}a_n\left(\frac{\lambda}{x}\right)^n\right)^l = \sum_{n=0}^{\infty}C_{b,n}\left(\frac{\lambda}{x}\right)^n$,we get :

$$\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{l+d} = \left(\frac{1}{\Gamma(p)}\right)^{l+d}\left\{\Upsilon\left(p,\frac{\lambda}{x}\right)\right\}^{l+d} = \left(\frac{1}{\Gamma(p)}\right)^{l+d}\left\{\left(\frac{\lambda}{x}\right)^p\sum_{n=0}^{\infty}\frac{\left(-\frac{\lambda}{x}\right)^n}{(p+n)n!}\right\}^{l+d}$$

$$= \left(\frac{1}{\Gamma(p)}\right)^{l+d}\left(\frac{\lambda}{x}\right)^{pl+pd}\sum_{n=0}^{\infty}C_{l+d,n}\left(\frac{\lambda}{x}\right)^n$$

Where $C_{l+d,n} = (na_0)^{-1}\sum_{g=1}^{n}([l+d]g-n+g)\,a_qC_{l+d,n}$, $C_{l+d,o} = {a_0}^l$ and $a_g = \frac{(-1)^g}{(p+g)g!}$

$$I_3 = -(\vartheta+1)\frac{\vartheta\eta\lambda^{n+p(l+d+1)}}{(\Gamma(p))^{l+d+1}(1-(1+\eta)^{-\vartheta})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}$$

$$\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(k+1)}{l!\,\Gamma(k-l+1)}\sum_{d=0}^{\infty}(-1)^d\frac{\Gamma(r+1)}{d!\,\Gamma(r-d+1)}\sum_{n=0}^{\infty}C_{l+d,n}\int_0^{\infty}x^{-(n+p(l+d+1)+1)}e^{-\left(\frac{\lambda}{x}\right)}dx$$

$$I_3 = -(\vartheta+1)\frac{\vartheta\eta}{(\Gamma(p))^{l+d+1}(1-(1+\eta)^{-\vartheta})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}$$

$$\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(k+1)}{l!\Gamma(k-l+1)}\sum_{d=0}^{\infty}(-1)^d\frac{\Gamma(r+1)}{d!\Gamma(r-d+1)}\sum_{n=0}^{\infty}C_{l+d,n}\,\boldsymbol{\Gamma}(n+p(l+d+1))$$

$$H = \ln\left(\frac{\Gamma(p)(1-(1+\eta)^{-\vartheta})}{\vartheta\eta\lambda^p}\right) - (p+1)\frac{\vartheta\eta}{(\Gamma(p))^{b+1}(1-(1+\eta)^{-\vartheta})}\sum_{v=0}^{\infty}\frac{\Gamma(\vartheta+1+v)}{v!\,\Gamma(\vartheta+1)}(-\eta)^v\sum_{b=0}^{\infty}(-1)^b\frac{\Gamma(v+1)}{b!\,\Gamma(v-b+1)}$$

$$\sum_{n=0}^{\infty}C_{b,n}\,\boldsymbol{\Gamma}(n+p(b+1))\ \{$$

$$\Psi(n+p(b+1)) - \ln\lambda\} + \frac{\vartheta\eta}{(\Gamma(p))^{b+1}(1-(1+\eta)^{-\vartheta})}\sum_{w=0}^{\infty}\frac{\Gamma(\vartheta+1+w)}{w!\Gamma(\vartheta+1)}(-\eta)^w$$

$$\sum_{b=0}^{\infty}(-1)^b\frac{\Gamma(w+1)}{b!\Gamma(w-b+1)}\sum_{n=0}^{\infty}C_{b,n}\,\boldsymbol{\Gamma}(n+p(b+1)+1) -$$

$$(\vartheta+1)\frac{\vartheta\eta}{(\Gamma(p))^{l+d+1}(1-(1+\eta)^{-\vartheta})}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(k+1)}{l!\,\Gamma(k-l+1)}\sum_{d=0}^{\infty}(-1)^d\frac{\Gamma(r+1)}{d!\,\Gamma(r-d+1)}\sum_{n=0}^{\infty}C_{l+d,n}$$

$$\boldsymbol{\Gamma}(n+p(l+d+1)) \tag{23}$$

The Relative entropy DKI(F//F^*) of [0,1] TLIGD$(\eta,\vartheta,p,\lambda)$ can be found as follows

$$\frac{f(x)}{f^*(x)}=\frac{\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}}{\frac{\vartheta_1\eta_1{\lambda_1}^{p_1}}{\Gamma(p_1)(1-(1+\eta_1)^{-\vartheta_1})}x^{-(p_1+1)}e^{-\left(\frac{\lambda_1}{x}\right)}\left(1+\eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right)^{-(\vartheta_1+1)}}$$

$$DKL=\int_0^\infty f(x)\ln\left(\frac{f(x)}{f^*(x)}\right)dx$$

$$=\int_0^\infty f(x)\ln\left[\frac{\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}}{\frac{\vartheta_1\eta_1{\lambda_1}^{p_1}}{\Gamma(p_1)(1-(1+\eta_1)^{-\vartheta_1})}x^{-(p_1+1)}e^{-\left(\frac{\lambda_1}{x}\right)}\left(1+\eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right)^{-(\vartheta_1+1)}}\right]dx$$

$$=\int_0^\infty f(x)\left[\ln\left(\frac{\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}}{\frac{\vartheta_1\eta_1{\lambda_1}^{p_1}}{\Gamma(p_1)(1-(1+\eta_1)^{-\vartheta_1})}}\right)+(p_1-p)\ln x+(\lambda_1-\lambda)\frac{1}{x}\right.$$
$$-(\vartheta+1)\ln\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)+(\vartheta_1$$
$$\left.+1)\ln\left(1+\eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right)\right]dx$$

$$=\ln\left(\frac{\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}}{\frac{\vartheta_1\eta_1{\lambda_1}^{p_1}}{\Gamma(p_1)(1-(1+\eta_1)^{-\vartheta_1})}}\right)+(p_1-p)E(\ln x)+(\lambda_1-\lambda)E\left(\frac{1}{x}\right)$$
$$-(\vartheta+1)E\left[\ln\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)\right]$$
$$+(\vartheta_1+1)E\left[\ln\left(1+\eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right)\right]$$

Let $I_{11}=(p_1-p)E(\ln x)=(p_1-p)\int_0^\infty \ln x\, f(x)\,dx$

$$I_{11}=(p_1-p)\int_0^\infty \ln x\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1\right.$$
$$\left.+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}dx$$

Now , simplification $\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}$ by using equation (11) , we obtain

$$I_{11}=(p_1-p)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\,\Gamma(\vartheta+1)}(-\eta)^{\delta}\int_0^{\infty}\ln x\,x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{\delta}dx$$

Since $\Gamma(s,\tau)+\Upsilon(s,\tau)=\Gamma(s)\rightarrow\Gamma(s,\tau)=\Gamma(s)-\Upsilon(s,\tau)$, will thus, $\Gamma\left(p,\frac{\lambda}{x}\right)=\Gamma(p)-\Upsilon\left(p,\frac{\lambda}{x}\right)$

$$I_{11}=(p_1-p)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\,\Gamma(\vartheta+1)}(-\eta)^{\delta}\int_0^{\infty}\ln x\,x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{\delta}dx$$

And again simplification $\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{\delta}$, Applying equation (12) , we obtain

$$I_{11}=(p_1-p)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\,\Gamma(\vartheta+1)}(-\eta)^{\delta}\sum_{v=0}^{\infty}(-1)^{v}\frac{\Gamma(\delta+1)}{v!\,\Gamma(\delta-v+1)}\int_0^{\infty}\ln x\,x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{v}dx$$

by using equation (13)and $\left(\sum_{n=0}^{\infty}a_n\left(\frac{\lambda}{x}\right)^{n}\right)^{v}=\sum_{n=0}^{\infty}C_{v,n}\left(\frac{\lambda}{x}\right)^{n}$,we get :

$$I_{11}=-(p_1-p)\frac{\vartheta\eta\lambda^{n+p(v+1)}}{(\Gamma(p))^{v+1}(1-(1+\eta)^{-\vartheta})}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\,\Gamma(\vartheta+1)}(-\eta)^{\delta}\sum_{v=0}^{\infty}(-1)^{v}\frac{\Gamma(\delta+1)}{v!\,\Gamma(\delta-v+1)}\sum_{n=0}^{\infty}C_{v,n}$$

$$\int_0^{\infty}\ln\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{(n+p(v+1)+1)}e^{-\left(\frac{\lambda}{x}\right)}\,dx$$

Let $y=\frac{1}{x}\rightarrow x=y^{-1}\rightarrow dx=-y^{-2}dy$, then

$$I_{11}=-(p_1-p)\frac{\vartheta\eta\lambda^{n+p(v+1)}}{(\Gamma(p))^{v+1}(1-(1+\eta)^{-\vartheta})}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\,\Gamma(\vartheta+1)}(-\eta)^{\delta}\sum_{v=0}^{\infty}(-1)^{v}\frac{\Gamma(\delta+1)}{v!\,\Gamma(\delta-v+1)}$$

$$\sum_{m=0}^{\infty}C_{v,n}\int_0^{\infty}\ln y\,(y)^{(n+p(v+1)-1)}e^{-(\lambda y)}\,dy$$

$$I_{11}=-(p+1)\frac{\vartheta\eta}{(\Gamma(p))^{j+1}(1-(1+\eta)^{-\vartheta})}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\,\Gamma(\vartheta+1)}(-\eta)^{\delta}\sum_{v=0}^{\infty}(-1)^{v}\frac{\Gamma(\delta+1)}{v!\,\Gamma(\delta-v+1)}$$

$$\sum_{m=0}^{\infty}C_{v,n}\,\boldsymbol{\Gamma}(n+p(k+1))\,\{\Psi(n+p(v+1))-\ln\lambda\}$$

Let $I_{22}=(\lambda_1-\lambda)E\left(\frac{1}{x}\right)=(\lambda_1-\lambda)\int_0^{\infty}\left(\frac{1}{x}\right)f(x)\,dx$

$$I_{22} = (\lambda_1 - \lambda) \int_0^\infty \left(\frac{1}{x}\right) \frac{\vartheta \eta \lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})} x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)} \left(1 + \eta \left\{ \frac{\Gamma\left(p, \frac{\lambda}{x}\right)}{\Gamma(p)} \right\} \right)^{-(\vartheta+1)} dx$$

Now , simplification $\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}$ by using equation (11) ,we obtain

$$I_{22} = (\lambda_1 - \lambda) \frac{\vartheta \eta \lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})} \sum_{i=0}^{\infty} \frac{\Gamma(\vartheta+1+i)}{i!\,\Gamma(\vartheta+1)} (-\eta)^i \int_0^\infty x^{-(p+2)} e^{-\left(\frac{\lambda}{x}\right)} \left\{ \frac{\Gamma\left(p, \frac{\lambda}{x}\right)}{\Gamma(p)} \right\}^i dx$$

Since $\Gamma(\upsilon,\tau) + \Upsilon(\upsilon,\tau) = \Gamma(\upsilon) \rightarrow \Gamma(\upsilon,\tau) = \Gamma(\upsilon) - \Upsilon(\upsilon,\tau)$, will thus, $\Gamma\left(p,\frac{\lambda}{x}\right) = \Gamma(p) - \Upsilon\left(p,\frac{\lambda}{x}\right)$

$$I_{22} = (\lambda_1 - \lambda) \frac{\vartheta \eta \lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})} \sum_{i=0}^{\infty} \frac{\Gamma(\vartheta+1+i)}{i!\,\Gamma(\vartheta+1)} (-\eta)^i \int_0^\infty x^{-(p+2)} e^{-\left(\frac{\lambda}{x}\right)} \left\{ 1 - \frac{\Upsilon\left(p, \frac{\lambda}{x}\right)}{\Gamma(p)} \right\}^i dx$$

And again simplification$\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^i$, Applying equation (12) , we obtain

$$I_{22} = (\lambda_1 - \lambda) \frac{\vartheta \eta \lambda^p}{\Gamma(\alpha)(1-(1+\eta)^{-\vartheta})} \sum_{i=0}^{\infty} \frac{\Gamma(\vartheta+1+i)}{i!\,\Gamma(\vartheta+1)} (-\eta)^i \sum_{t=0}^{\infty} (-1)^t \frac{\Gamma(i+1)}{t!\,\Gamma(i-t+1)} \int_0^\infty x^{-(p+2)} e^{-\left(\frac{\lambda}{x}\right)} \left\{ \frac{\Upsilon\left(p, \frac{\lambda}{x}\right)}{\Gamma(p)} \right\}^t dx$$

by using equation (13)and $\left(\sum_{n=0}^{\infty} a_n \left(\frac{\lambda}{x}\right)^n\right)^t = \sum_{n=0}^{\infty} C_{t,n} \left(\frac{\lambda}{x}\right)^n$,we get :

$$I_{22} = (\lambda_1 - \lambda) \frac{\vartheta \eta \lambda^{n+p(t+1)}}{\left(\Gamma(p)\right)^{t+1}(1-(1+\eta)^{-\vartheta})} \sum_{i=0}^{\infty} \frac{\Gamma(\vartheta+1+i)}{i!\,\Gamma(\vartheta+1)} (-\eta)^i \sum_{t=0}^{\infty} (-1)^t \frac{\Gamma(i+1)}{t!\,\Gamma(i-t+1)} \sum_{n=0}^{\infty} C_{t,n} \int_0^\infty x^{-(n+p(t+1)+2)} e^{-\left(\frac{\lambda}{x}\right)} dx$$

$$I_{22} = (\lambda_1 - \lambda) \frac{\vartheta \eta}{\left(\Gamma(p)\right)^{t+1}(1-(1+\eta)^{-\vartheta})} \sum_{i=0}^{\infty} \frac{\Gamma(\vartheta+1+i)}{i!\,\Gamma(\vartheta+1)} (-\eta)^i \sum_{t=0}^{\infty} (-1)^t \frac{\Gamma(i+1)}{t!\,\Gamma(i-t+1)} \sum_{n=0}^{\infty} C_{t,n} \boldsymbol{\Gamma}(n + p(t+1) + 1)$$

Let

$$I_{33} = -(\vartheta+1) E\left(\ln(1 + \eta \left\{ \frac{\Gamma\left(p, \frac{\lambda}{x}\right)}{\Gamma(p)} \right\}) \right) = -(\vartheta+1) \int_0^\infty \ln(1 + \eta \left\{ \frac{\Gamma\left(p, \frac{\lambda}{x}\right)}{\Gamma(p)} \right\}) f(x)\, dx$$

$$I_{33} = -(\vartheta+1)\int_0^\infty \ln\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}dx$$

By the same way simplification $\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}$ by using equation (11) , we obtain

$$I_{33} = -(\vartheta+1)\frac{\vartheta\eta\lambda^p}{\Gamma(\alpha)(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^d\int_0^\infty \ln\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^d dx$$

We can simplification $\ln\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)$, Applying equation (22) , we obtain

$$I_{33} = (\vartheta+1)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^d\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}\int_0^\infty x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^d\left(\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^r dx$$

Since $\Gamma(s,\tau)+\Upsilon(s,\tau)=\Gamma(s)\rightarrow\Gamma(s,\tau)=\Gamma(s)-\Upsilon(s,\tau)$, will thus, $\Gamma\left(p,\frac{\lambda}{x}\right)=\Gamma(p)-\Upsilon\left(p,\frac{\lambda}{x}\right)$

$$I_{33} = -(\vartheta+1)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^d\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}\int_0^\infty x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^d\left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^r dx$$

By the same way simplification $\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^d$, $\left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^r$, Applying equation (12) , we obtain

$$I_{33} = (\vartheta+1)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^d\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}\sum_{b=0}^{\infty}(-1)^b\frac{\Gamma(d+1)}{d!\,\Gamma(d-b+1)}$$

$$\sum_{s=0}^{\infty}(-1)^s \frac{\Gamma(r+1)}{s!\,\Gamma(r-s+1)} \int_0^{\infty} x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)} \left(\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^{b} \left(\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^{s} dx$$

$$I_{33} = (\vartheta+1)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^d \sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}\sum_{b=0}^{\infty}(-1)^{b}\frac{\Gamma(d+1)}{d!\,\Gamma(d-b+1)}$$

$$\sum_{s=0}^{\infty}(-1)^s \frac{\Gamma(r+1)}{s!\,\Gamma(r-s+1)} \int_0^{\infty} x^{-(p+1)} e^{-\left(\frac{\lambda}{x}\right)} \left(\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^{b+s} dx$$

by using equation (13)and $\left(\sum_{n=0}^{\infty} a_n \left(\frac{\lambda}{x}\right)^n\right)^{b} = \sum_{n=0}^{\infty} C_{b,n}\left(\frac{\lambda}{x}\right)^n$,we get :

$$\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{b+s} = \left(\frac{1}{\Gamma(p)}\right)^{b+s}\left\{\left(\frac{\lambda}{x}\right)^p \sum_{n=0}^{\infty}\frac{\left(-\frac{\lambda}{x}\right)^n}{(p+n)n!}\right\}^{b+s}$$
$$= \left(\frac{1}{\Gamma(p)}\right)^{b+s}\left(\frac{\lambda}{x}\right)^{pb+ps}\sum_{n=0}^{\infty} C_{b+s,n}\left(\frac{\lambda}{x}\right)^n$$

Where

$$C_{b+s,n} = (na_0)^{-1}\sum_{g=1}^{n}([b+s]g-n+g)\, a_g C_{b+s,n}\,,\ C_{b+s,o} = a_0{}^{b}\ and\ a_g = \frac{(-1)^g}{(p+g)g!}$$

$$I_{33} = -(\vartheta+1)\frac{\vartheta\eta}{(\Gamma(p))^{b+s+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^d\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}$$

$$\sum_{b=0}^{\infty}(-1)^{b}\frac{\Gamma(d+1)}{b!\Gamma(d-b+1)}\sum_{s=0}^{\infty}(-1)^s\frac{\Gamma(r+1)}{s!\Gamma(r-s+1)}\sum_{n=0}^{\infty} C_{b+s,n}\,\boldsymbol{\Gamma}(n+p(b+s+1))$$

$$I_{44} = (\vartheta_1+1)E\left(\ln\left[1+\eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right]\right) = (\vartheta_1+1)\int_0^{\infty}\ln\left[1+\eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right] f(x)dx$$

$$= (\vartheta_1+1)\int_0^{\infty}\ln\left[1+\eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right]\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)} dx$$

By the same way simplification$\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}$ by using equation (11) , we obtain

$$I_{44} = (\vartheta_1 + 1)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{Ь=0}^{\infty}\frac{\Gamma(\vartheta+1+Ь)}{Ь!\,\Gamma(\vartheta+1)}(-\eta)^{Ь}\int_0^{\infty}\ln\left[1 + \eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right]x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{Ь}dx$$

We can simplification $\ln\left(1+\eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right)$ by using equation (21) , we get

$$I_{44} = -(\vartheta_1 + 1)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{Ь=0}^{\infty}\frac{\Gamma(\vartheta+1+Ь)}{Ь!\,\Gamma(\vartheta+1)}(-\eta)^{Ь}\sum_{r=1}^{\infty}\frac{(-\eta_1)^r}{r}\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{Ь}\left(\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right)^{r}dx$$

Since $\Gamma(s,\tau) + \Upsilon(s,\tau) = \Gamma(s) \to \Gamma(s,\tau) = \Gamma(s) - \Upsilon(s,\tau)$, will thus, $\Gamma\left(p,\frac{\lambda}{x}\right) = \Gamma(p) - \Upsilon\left(p,\frac{\lambda}{x}\right)$

$$I_{44} = -(\vartheta_1 + 1)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{Ь=0}^{\infty}\frac{\Gamma(\vartheta+1+Ь)}{Ь!\,\Gamma(\vartheta+1)}(-\eta)^{Ь}\sum_{r=1}^{\infty}\frac{(-\eta_1)^r}{r}\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{Ь}\left(1-\frac{\Upsilon\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right)^{r}dx$$

By the same way simplification $\left\{1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{Ь}, \left(1-\frac{\Upsilon\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right)^{r}$, Applying equation (12) ,we obtain

$$I_{44} = -(\vartheta_1 + 1)\frac{\vartheta\eta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}\sum_{Ь=0}^{\infty}\frac{\Gamma(\vartheta+1+Ь)}{Ь!\,\Gamma(\vartheta+1)}(-\eta)^{Ь}\sum_{r=1}^{\infty}\frac{(-\eta_1)^r}{r}\sum_{v=0}^{\infty}(-1)^{v}\frac{\Gamma(Ь+1)}{v!\,\Gamma(Ь-v+1)}\sum_{d=0}^{\infty}(-1)^{d}\frac{\Gamma(n+1)}{d!\,\Gamma(n-d+1)}\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{v}\left(\frac{\Upsilon\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right)^{d}dx$$

by using equation (13)and $\left(\sum_{n=0}^{\infty}a_n\left(\frac{\lambda}{x}\right)^n\right)^v = \sum_{n=0}^{\infty}C_{v,n}\left(\frac{\lambda}{x}\right)^n$,we get :

$$\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^{v} = \left(\frac{1}{\Gamma(p)}\right)^{v}\left(\frac{\lambda}{x}\right)^{pv}\sum_{n=0}^{\infty}C_{v,n}\left(\frac{\lambda}{x}\right)^{n}, \left\{\frac{\Upsilon\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}^{d} = \left(\frac{1}{\Gamma(p_1)}\right)^{d}\left(\frac{\lambda_1}{x}\right)^{p_1 d}\sum_{l=0}^{\infty}C_{d,l}\left(\frac{\lambda_1}{x}\right)^{l}$$

$$I_{44} = -(\vartheta_1 + 1)\frac{\vartheta\eta\left(\frac{\lambda_1}{\lambda}\right)^{p_1 d+l}}{(\Gamma(p))^{v+1}(\Gamma(p_1))^{d}(1-(1+\eta)^{-\vartheta})}\sum_{Ь=0}^{\infty}\frac{\Gamma(\vartheta+1+Ь)}{Ь!\,\Gamma(\vartheta+1)}(-\eta)^{Ь}\sum_{r=1}^{\infty}\frac{(-\eta)^r}{r}$$

$$\sum_{v=0}^{\infty}(-1)^{v}\frac{\Gamma(\text{Ь}+1)}{v!\,\Gamma(\text{Ь}-v+1)}\sum_{d=0}^{\infty}(-1)^{d}\frac{\Gamma(r+1)}{d!\,\Gamma(r-d+1)}\sum_{n=0}^{\infty}C_{v,n}\sum_{l=0}^{\infty}C_{d,l}\boldsymbol{\Gamma}(n+p(v+1)+p_1 d+l)$$

$$KDL=\ln\left(\frac{\frac{\vartheta\eta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}}{\frac{\vartheta_1\eta_1\lambda_1^{p_1}}{\Gamma(p_1)(1-(1+\eta_1)^{-\vartheta_1})}}\right)-(p+1)\frac{\vartheta\eta}{(\Gamma(p))^{j+1}(1-(1+\eta)^{-\vartheta})}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\,\Gamma(\vartheta+1)}(-\eta)^{\delta}\sum_{v=0}^{\infty}(-1)^{v}\frac{\Gamma(\delta+1)}{v!\,\Gamma(\delta-v+1)}$$

$$\sum_{m=0}^{\infty}C_{v,n}\,\boldsymbol{\Gamma}(\text{n}+p(k+1))\,\{\,\Psi(\text{n}+p(v+1))-\ln\lambda\}+(\lambda_1-\lambda)\frac{\vartheta\eta}{(\Gamma(p))^{t+1}(1-(1+\eta)^{-\vartheta})}$$

$$\sum_{i=0}^{\infty}\frac{\Gamma(\vartheta+1+i)}{i!\Gamma(\vartheta+1)}(-\eta)^{i}\sum_{t=0}^{\infty}(-1)^{t}\frac{\Gamma(i+1)}{t!\Gamma(i-t+1)}\sum_{n=0}^{\infty}C_{t,n}\,\boldsymbol{\Gamma}(n+p(t+1)+1)\boldsymbol{\Gamma}(\text{m}+p(t+1)+1)$$

$$-(\vartheta+1)\frac{\vartheta\eta}{(\Gamma(p))^{\text{Ь}+s+1}(1-(1+\eta)^{-\vartheta})}\sum_{d=0}^{\infty}\frac{\Gamma(\vartheta+1+d)}{d!\,\Gamma(\vartheta+1)}(-\eta)^{d}\sum_{r=1}^{\infty}\frac{(-\eta)^{r}}{r}$$

$$\sum_{\text{Ь}=0}^{\infty}(-1)^{\text{Ь}}\frac{\Gamma(d+1)}{\text{Ь}!\Gamma(d-\text{Ь}+1)}\sum_{s=0}^{\infty}(-1)^{s}\frac{\Gamma(r+1)}{s!\Gamma(r-s+1)}\sum_{n=0}^{\infty}C_{\text{Ь}+s,n}\,\boldsymbol{\Gamma}(\text{n}+p(\text{Ь}+s+1))$$

$$-(\vartheta_1+1)\frac{\vartheta\eta\left(\frac{\lambda_1}{\lambda}\right)^{p_1 d+l}}{(\Gamma(p))^{v+1}(\Gamma(p_1))^{d}(1-(1+\eta)^{-\vartheta})}\sum_{\text{Ь}=0}^{\infty}\frac{\Gamma(\vartheta+1+\text{Ь})}{\text{Ь}!\,\Gamma(\vartheta+1)}(-\eta)^{\text{Ь}}\sum_{r=1}^{\infty}\frac{(-\eta)^{r}}{r}$$

$$\sum_{v=0}^{\infty}(-1)^{v}\frac{\Gamma(\text{Ь}+1)}{v!\,\Gamma(\text{Ь}-v+1)}\sum_{d=0}^{\infty}(-1)^{d}\frac{\Gamma(r+1)}{d!\,\Gamma(r-d+1)}\sum_{n=0}^{\infty}C_{v,n}\sum_{l=0}^{\infty}C_{d,l}\boldsymbol{\Gamma}(n+p(v+1)+p_1 d+l) \quad (24)$$

IV. Stress Strength Model of the [0,1] TLIGD Distribution

The stress -strength be presented by the form,

$$R=P(y<x)=\int_0^{\infty}f_x(x)F_Y(x)\,dx$$

$$R=\int_0^{\infty}\frac{\eta\vartheta\lambda^{p}}{\Gamma(p)(1-(1+\eta)^{-\vartheta})}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}\left(\frac{1-\left[1+\eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right]^{-\vartheta_1}}{(1-(1+\eta_1)^{-\vartheta_1})}\right)dx$$

Now , simplification $\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}$, $\left[1+\eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right]^{-\vartheta_1}$ by using equation (11), we obtain

$$\left(1+\eta\left\{\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}\right)^{-(\vartheta+1)}=\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^{k}\left(\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^{k}$$

$$\left[1+\eta_1\left\{\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}\right]^{-\vartheta_1} = \sum_{l=0}^{\infty}\frac{\Gamma(\vartheta_1+1+l)}{l!\,\Gamma(\vartheta_1+1)}(-\eta_1)^l\left(\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right)^l$$

And then

$$R=\frac{\eta\vartheta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})(1-(1+\eta_1)^{-\vartheta_1})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^k\left[1-\sum_{l=0}^{\infty}\frac{\Gamma(\vartheta_1+1+l)}{l!\,\Gamma(\vartheta_1+1)}(-\eta_1)^l\left(\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right)^l\right]dx$$

$$R=\frac{\eta\vartheta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})(1-(1+\eta_1)^{-\vartheta_1})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^k dx$$
$$-\frac{\eta\vartheta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})(1-(1+\eta_1)^{-\vartheta_1})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(\frac{\Gamma\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^k\sum_{l=0}^{\infty}\frac{\Gamma(\vartheta_1+1+l)}{l!\,\Gamma(\vartheta_1+1)}(-\eta_1)^l\left(\frac{\Gamma\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right)^l dx$$

Since $\Gamma(s,\tau)+\Upsilon(s,\tau)=\Gamma(s)\rightarrow\Gamma(s,\tau)=\Gamma(s)-\Upsilon(s,\tau)$, will thus,

$$\Gamma\left(p,\frac{\lambda}{x}\right)=\Gamma(p)-\Upsilon\left(p,\frac{\lambda}{x}\right)\quad,\Gamma\left(p_1,\frac{\lambda_1}{x}\right)=\Gamma(p_1)-\Upsilon\left(p_1,\frac{\lambda_1}{x}\right)$$

$$R=\frac{\eta\vartheta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})(1-(1+\eta_1)^{-\vartheta_1})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^k dx$$
$$-\frac{\eta\vartheta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})(1-(1+\eta_1)^{-\vartheta_1})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\sum_{l=0}^{\infty}\frac{\Gamma(\vartheta_1+1+l)}{l!\,\Gamma(\vartheta_1+1)}(-\eta_1)^l\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^k\left(1-\frac{\Upsilon\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right)^l dx$$

And again simplification $\left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^k,\left(1-\frac{\Upsilon\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right)^l$ by using equation (12) , we get :

$$\left(1-\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right)^k=\sum_{s=0}^{\infty}(-1)^s\frac{\Gamma(s+1)}{s!\,\Gamma(s-k+1)}\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^s,\left(1-\frac{\Upsilon\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right)^t$$
$$=\sum_{t=0}^{\infty}(-1)^t\frac{\Gamma(l+1)}{t!\,\Gamma(t-l+1)}\left\{\frac{\Upsilon\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}^t$$

And then

$$R=\frac{\eta\vartheta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})(1-(1+\eta_1)^{-\vartheta_1})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\sum_{s=0}^{\infty}(-1)^s\frac{\Gamma(s+1)}{s!\,\Gamma(s-k+1)}\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^s dx$$
$$-\frac{\eta\vartheta\lambda^p}{\Gamma(p)(1-(1+\eta)^{-\vartheta})(1-(1+\eta_1)^{-\vartheta_1})}\sum_{k=0}^{\infty}\frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k\sum_{l=0}^{\infty}\frac{\Gamma(\vartheta_1+1+l)}{l!\,\Gamma(\vartheta_1+1)}(-\eta_1)^l\sum_{s=0}^{\infty}(-1)^s\frac{\Gamma(s+1)}{s!\,\Gamma(s-k+1)}$$
$$\sum_{t=0}^{\infty}(-1)^t\frac{\Gamma(l+1)}{t!\,\Gamma(t-l+1)}\int_0^{\infty}x^{-(p+1)}e^{-\left(\frac{\lambda}{x}\right)}\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^s\left\{\frac{\Upsilon\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}^t dx$$

by using equation (13)and $\left(\sum_{n=0}^{\infty} a_n \left(\frac{\lambda}{x}\right)^n\right)^{\text{b}} = \sum_{n=0}^{\infty} C_{\text{b},n} \left(\frac{\lambda}{x}\right)^n$,we get :

$$\left\{\frac{\Upsilon\left(p,\frac{\lambda}{x}\right)}{\Gamma(p)}\right\}^s = \left(\frac{1}{\Gamma(p)}\right)^s \left(\frac{\lambda}{x}\right)^{ps} \sum_{n=0}^{\infty} C_{s,n} \left(\frac{\lambda}{x}\right)^n , \left\{\frac{\Upsilon\left(p_1,\frac{\lambda_1}{x}\right)}{\Gamma(p_1)}\right\}^t$$

$$= \left(\frac{1}{\Gamma(p_1)}\right)^t \left(\frac{\lambda_1}{x}\right)^{p_1 t} \sum_{d=0}^{\infty} C_{t,d} \left(\frac{\lambda_1}{x}\right)^d$$

And then

$$= \frac{\eta\vartheta\lambda^{p(s+1)+n}}{(\Gamma(p))^{(s+1)}(1-(1+\eta)^{-\vartheta})(1-(1+\eta_1)^{-\vartheta_1})} \sum_{k=0}^{\infty} \frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k \sum_{s=0}^{\infty}(-1)^s \frac{\Gamma(s+1)}{s!\,\Gamma(s-k+1)} \sum_{n=0}^{\infty} C_{s,n}$$

$$\int_0^{\infty} x^{-(n+p(s+1)+1)} e^{-\left(\frac{\lambda}{x}\right)} dx - \frac{\eta\vartheta\lambda^{n+p(s+1)}}{(\Gamma(p))^{(s+1)}\left(\Gamma(p_1)\right)^t(1-(1+\eta)^{-\vartheta})(1-(1+\eta_1)^{-\vartheta_1})} \sum_{k=0}^{\infty} \frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k$$

$$\sum_{l=0}^{\infty} \frac{\Gamma(\vartheta_1+1+l)}{l!\,\Gamma(\vartheta_1+1)}(-\eta_1)^l \sum_{s=0}^{\infty}(-1)^s \frac{\Gamma(s+1)}{s!\,\Gamma(s-i+1)} \sum_{t=0}^{\infty}(-1)^t \frac{\Gamma(l+1)}{t!\,\Gamma(t-l+1)}$$

$$\sum_{d=0}^{\infty} C_{t,d}\, {\lambda_1}^{d+p_1 t} \sum_{n=0}^{\infty} C_{\text{b},n} \int_0^{\infty} x^{-(p_1 t+d+n+p(s+1)+1)} e^{-\left(\frac{\lambda}{x}\right)} dx$$

$$R = \frac{\eta\vartheta}{\overset{7}{\left(\Gamma(p)\right)^{(s+1)}}(1-(1+\eta)^{-\vartheta})(1-(1+\eta_1)^{-\vartheta_1})} \sum_{k=0}^{\infty} \frac{\Gamma(\vartheta+1+k)}{k!\,\Gamma(\vartheta+1)}(-\eta)^k \sum_{s=0}^{\infty}(-1)^s \frac{\Gamma(s+1)}{s!\,\Gamma(s-k+1)} \sum_{n=0}^{\infty} C_{s,n}$$

$$(\text{n}+ p(s+1)) - \frac{\eta\vartheta\left(\frac{\lambda_1}{\lambda}\right)^{d+p_1 t}}{\left(\Gamma(p)\right)^{(s+1)}\left(\Gamma(p_1)\right)^l\left(1-(1+\eta)^{-\vartheta}\right)\left(1-(1+\eta_1)^{-\vartheta_1}\right)} \sum_{i=0}^{\infty} \frac{\Gamma(\vartheta+1+i)}{i!\Gamma(\vartheta+1)}(-\eta)^i$$

$$\sum_{l=0}^{\infty} \frac{\Gamma(\vartheta_1+1+l)}{l!\,\Gamma(\vartheta_1+1)}(-\eta_1)^l \sum_{s=0}^{\infty}(-1)^s \frac{\Gamma(s+1)}{s!\,\Gamma(s-k+1)} \sum_{t=0}^{\infty}(-1)^t \frac{\Gamma(l+1)}{t!\,\Gamma(t-l+1)}$$

$$\sum_{d=0}^{\infty} C_{t,d} \sum_{n=0}^{\infty} C_{s,n}\, \boldsymbol{\Gamma}(\text{n}+p(s+1)) \qquad (25)$$

V. Conclusions

In fact , we produced ([0,1] TLIGD) distribution build on ([0,1] TLD)distribution . We derived some important properties of ([0,1] TLIGD) distribution and Also , we studied stress strength model.

Jumana A. Altawil et al

References

I. Abid, Salah , K. Abdulrazak, Russul, "[0, 1] truncated fréchet-gamma and inverted gam-ma distributions", International Journal of Scientific World , 2017.

II. Eugene, N., Lee, C., & Famoye, F.,"Beta-normal distribution and its applications. Communications in Statistics-Theory and methods",vol. 31(4), pp: 497-512, 2002.

III. Gradshteyn, I. S., & Ryzhik, I. M., "Table of integrals, series, and products": Academic press,2014.

IV. Gupta, A. K., & Nadarajah, S., "On the moments of the beta normal distribution.Communications in Statistics-Theory and methods", vol. 33(1), pp: 1-13, 2005.

V. Jamjoom, A., & Al-Saiary, Z., "Computing the moments of order Statistics from independent nonidentically distributed exponentiated Frechet variables". Journal of Probability and Statistics, 2012.

VI. Jones, M., "Families of distributions arising from distributions of order statistics". Test, vol. 13(1),pp :1- 43,2004.

VII. Maria do Carmo, S. L., Cordeiro, G. M., & Ortega, E. M., "A new extension of the normal distribution. Journal of Data Science", vol. 13(2), pp: 385-408,2014.