



## MARKOV PROCESS AND DECISION ANALYSIS

R. Sivaraman

Associate Professor, Department of Mathematics, D.G. Vaishnav College,  
Chennai, India. National Awardee for Popularizing Mathematics among  
Masses.

Email: rsivaraman1729@yahoo.co.in

<https://doi.org/10.26782/jmcms.2020.07.00002>

### Abstract

*The need of proper medical diagnosis and treatment has been need of the day to deal with various infections caused by viruses and micro-organisms. To prevent the spread of the disease we need proper scientific approach and methods in place. This paper suggests one such method using Markov Process technique, in particular deciding how many patients should be allocated to respective doctors in a hospital.*

**Keywords :** Markov Process, Markov Decision Process, Transition Probabilities, Transition Matrix, Diagonalization of a matrix, Equilibrium Distribution

### I. Introduction

In modern times, health care has become essential and one of the important factors. In fact, if we are safe enough with our health care systems, then it will minimize our chance of getting infected by various diseases. In pandemic situation like COVID – 19, if someone gets affected due to virus, he/she need extensive medical care for about two to three weeks time. If the infected people get proper treatment through careful medical diagnosis, then their lives can be saved. This paper discuss about the mathematical procedure using Markov Process to provide methods for deciding number of patients to be treated periodically. The word Markov in “Markov Process” was named after the Russian mathematician Andrey Markov.

### II. Markov Models and Markov Process

We begin with a collection of objects and a list of possible states for each object. For example, the object may be an individual who can be assigned with one of two states namely healthy or sick. The collection of all possible states that an object can take is called the state space. At each time interval, the model assigns every object in the system to exactly one state from the state space. At the end of each time period, the objects move from one state to the next according to transition probabilities (or transition rates) that depend only on the current state of the system.

A process is said to be a “Markov Process” if the transition probabilities depend only on the current state of the system and not on the previous states. This is the key aspect of Markov models and is generally referred to as the Markov assumption. Due to the Markov assumption, Markov models are “forgetful” in the sense that knowledge of the past states of the system is not required to predict the future.

Even though the Markov assumption forces some level of forgetfulness on the models, it is nonetheless possible to build memory into a Markov model. The way this is done is to create new states that incorporate the memory for the desired trait. For example, in the case of modeling spread of disease through a population, one could create states labeled “susceptible”, “infected”, and “recovered”. The models considering these three states were called SIR models. The recovered state now effectively contains the memory that the individual was once infected. Markov models that incorporate memory in this manner are sometimes referred to as higher-order Markov models.

The Markov models discussed usually were Markov chains, meaning that all state transitions occur at fixed predefined time intervals. In the 1920s, a more general class of models, called

Markov processes, in which transitions occur at arbitrary times was also developed. In these models, time is viewed as a continuous variable, so time steps can occur at any point. One classic example of such a process is the “random walk of a drunkard”, in which a point stumbles in a random direction for a random distance. In this case, the concept of time is incorporated into distance, and so the point can be thought to be traveling in a random direction for a random length of time. In literature, the random walk of a drunkard is usually referred to as a Wiener process or Brownian motion.

### **III. Markov Decision Processes**

An extension of Markov processes and Markov chains, called Markov Decision Processes (MDPs) is a process in which the modeler is allowed to interact with the objects in the system by applying actions to the system. Applying an action to the system can be thought of as altering the transition matrix for a selection of time steps. MDPs are used extensively in business to help examine the effect of decision making in situations where outcomes are partly random and partly under the control of the decision maker.

### **IV. Mathematical Description of the Model**

For a given situation corresponding to Markov model, let  $X^0$  be the column vector of length  $N$  that represents the initial state of the system. Let  $X^1$  is the column vector in the first state of system and  $P$  is the transition probability matrix or simply transition matrix whose entries are probabilities (hence they are non-negative) representing transition from state 0 to state 1. Moreover, the transition matrix is time invariant. Thus in the Markov Model, the first state components of the system is obtained from initial state through the equation

*Copyright reserved © J. Mech. Cont.& Math. Sci.*  
*R. Sivaraman*

$$X^1 = PX^0 \quad (4.1)$$

By Markov Assumption the current state components of the system depends only on the immediately previous state and not on other states. Hence we have the following equations.

$$X^2 = PX^1, X^3 = PX^2, X^4 = PX^3, \dots, X^n = PX^{n-1} \quad (4.2)$$

These  $n - 1$  equation in (4.2) can be put together in a compact form as

$$X^n = P^n X^0 \quad (4.3)$$

In equation (4.3), we observe that  $P^n$  is  $n$  times multiplying the transition matrix  $P$ .

Now to identify the nature of the system for long period of time say a decade or several decades the computation from (4.3) becomes practically infeasible. Hence, we can think of what is called “Equilibrium Distribution” of the state system vector  $X$ .

In view of (4.3), for long period of time, for attaining Equilibrium Distribution, we should have

$$\lim_{n \rightarrow \infty} X^n = \lim_{n \rightarrow \infty} P^n X^0 \quad (4.4)$$

#### IV.i. Proposed Technique

To overcome the computational problem of  $P^n$ , I propose the idea of diagonalizing the transition matrix  $P$  by determining the Eigen Values of  $P$  and forming diagonal matrix  $D$  whose diagonal entries are Eigen Values of  $P$  and forming another invertible matrix  $Q$  whose columns are Eigen Vectors of the corresponding Eigen Values of the transition matrix  $P$ .

By matrix diagonalization theorem, we then have

$$P = QDQ^{-1} \quad (4.5)$$

Using (4.5) in the equation (4.4), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} X^n &= \lim_{n \rightarrow \infty} P^n X^0 = \lim_{n \rightarrow \infty} (QDQ^{-1})^n X^0 = \lim_{n \rightarrow \infty} (QDQ^{-1})(QDQ^{-1}) \times \dots \times (QDQ^{-1}) X^0 \\ &= \lim_{n \rightarrow \infty} (QD)(Q^{-1}Q)D(Q^{-1}Q)D(Q^{-1}Q) \times \dots \times (Q^{-1}Q)(DQ^{-1}) X^0 = \lim_{n \rightarrow \infty} QD^n Q^{-1} X^0 \end{aligned}$$

We thus have,

*Copyright reserved © J. Mech. Cont.& Math. Sci.  
R. Sivaraman*

$$\lim_{n \rightarrow \infty} X^n = \lim_{n \rightarrow \infty} P^n X^0 = \lim_{n \rightarrow \infty} QD^n Q^{-1} X^0 \quad (4.6)$$

In equation (4.6), we see the term  $D^n$ . This is easy to compute as  $D$  is a diagonal matrix. In particular if  $D$  is a diagonal matrix whose diagonal entries are the  $N$  Eigen Values say  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$  then  $D^n$  would also be a diagonal matrix whose diagonal entries are  $\lambda_1^n, \lambda_2^n, \lambda_3^n, \dots, \lambda_N^n$ .

Through equation (4.6) it will be possible for us to compute the Equilibrium Distribution of components of the process in a long run. To illustrate the advantages and application of the technique described here, let us consider a practical example by considering a typical hospital.

#### V. Doctor – Patients Allocation Model

Let us consider a hospital where appointments are not necessary for patients to meet the doctors in the hospital. Hence the patients may not see the same doctor on every visit. However, a patient who returns for further treatment is given an option of selecting a specific doctor of his/her choice. If that particular doctor is available on that day, the patient's waiting time increases but considerations are usually made, for example, if the patient is of age more than 80 years, they are preferred to go quickly and meet the desired doctor.

We assume that a patient's preference for a doctor is completely determined by the doctor that he or she visited in his or her last visit and the random factor of when that doctor will be available. Let the probabilities of visiting a given doctor, given the doctor seen during the previous visit, are given as in Table 1. Notice that some doctors inspire more patient loyalty than others.

**Table 1: Assignment of Transition Probabilities of Doctor – Patients Model**

Previous Visit	Next Visit choose Doctor 1	Next Visit choose Doctor 2	Next Visit choose Doctor 3
Treated by Doctor 1	0.72	0.09	0.21
Treated by Doctor 2	0.18	0.85	0.15
Treated by Doctor 3	0.10	0.06	0.64

We notice that the total probability for each doctor in the respective columns add up to 1. We also assume the patients arriving at the hospital choose the three doctors in

equally likely fashion, as they do not have any idea about their way of treatment during their first visit.

We will also assume that 300 patients return on a regular basis for their continuous treatment and we wish to see how these patients impact each doctor's workload in long period of time assuming Table 1 probability assignments for each of them. Since initially all the three doctors were equally likely to be chosen, we assume that each doctor would treat exactly 100 of the returning patients. Hence the initial state column vector of our system is given by

$$X^0 = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}.$$

The transition matrix for this Markov chain is time invariant (because of long run consideration). Hence, the transition matrix for our Doctor – Patients Model through Table 1, is given by

$$P = \begin{bmatrix} 0.72 & 0.09 & 0.21 \\ 0.18 & 0.85 & 0.15 \\ 0.10 & 0.06 & 0.64 \end{bmatrix} \quad (5.1)$$

The first state column vector of our model through equation (4.1) is given by

$$X^1 = PX^0 = \begin{bmatrix} 0.72 & 0.09 & 0.21 \\ 0.18 & 0.85 & 0.15 \\ 0.10 & 0.06 & 0.64 \end{bmatrix} \times \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 102 \\ 118 \\ 80 \end{bmatrix}$$

Using Matrix multiplication operation successively, using equations in (4.2), we have the following results.

$$X^1 = \begin{bmatrix} 102 \\ 118 \\ 80 \end{bmatrix}, X^2 = \begin{bmatrix} 100.86 \\ 130.66 \\ 68.48 \end{bmatrix}, X^3 = \begin{bmatrix} 98.7594 \\ 139.4878 \\ 61.7528 \end{bmatrix}, \dots, X^{50} = \begin{bmatrix} 89.6414 \\ 158.9641 \\ 51.3944 \end{bmatrix}, \dots, X^{100} = \begin{bmatrix} 89.6414 \\ 158.9641 \\ 51.3944 \end{bmatrix} \quad (5.2)$$

One can easily multiply a  $3 \times 3$  matrix with a single column vector to get the results displayed in (5.2) or we can also use computer programming to perform matrix multiplication recursively to finally obtain the hundredth state Markov Chain  $X^{100}$  for our system.

Now, from the final Markov Chain  $X^{100}$ , we can infer that in the long run, out of 300 patients visiting the hospital regularly, Doctor 1 treats 90 patients, Doctor 2 will have

159 patients and Doctor 3 will treat 51 patients approximately. This is the usual observation done by actual available method.

Now to confirm this result as per the proposed technique presented in section 4.1, we proceed as follows:

The Eigen values of the transition matrix  $P$  obtained through the equation  $|P - \lambda I| = 0$  are 0.680, 0.531 and 1. The Eigen vectors for these Eigen values forming the invertible matrix  $Q$  given by

$$Q = \begin{bmatrix} -0.583 & -0.867 & -0.481 \\ 0.820 & 0.154 & -0.854 \\ -0.237 & 0.713 & -0.276 \end{bmatrix} \quad (5.3)$$

The diagonal matrix  $D$  is such that its diagonal entries are the Eigen Values of  $P$  (due to diagonalization theorem), given by

$$D = \begin{bmatrix} 0.680 & 0 & 0 \\ 0 & 0.531 & 0 \\ 0 & 0 & 1.000 \end{bmatrix} \quad (5.4)$$

Therefore from equation (4.6), we have

$$X^{100} = \lim_{n \rightarrow 100} X^n = \lim_{n \rightarrow 100} P^n X^0 = \lim_{n \rightarrow 100} QD^n Q^{-1} X^0 = QD^{100} Q^{-1} X^0 \quad (5.5)$$

Now to compute  $D^{100}$ , we first notice that both 0.680 and 0.531 are real numbers between 0 and 1. Hence, their hundredth powers will be very small and arbitrarily

close to zero. Also,  $1^{100} = 1$ . Hence we have  $D^{100} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . Now computing the

inverse of the matrix in equation (5.3) and finding the value of  $QD^{100}Q^{-1}$  we get

$$QD^{100}Q^{-1} = \begin{bmatrix} 0.30 & 0.30 & 0.30 \\ 0.53 & 0.53 & 0.53 \\ 0.17 & 0.17 & 0.17 \end{bmatrix} \quad (5.6)$$

Hence from equations (5.5) and (5.6), we have

$$X^{100} = QD^{100}Q^{-1}X^0 = \begin{bmatrix} 0.30 & 0.30 & 0.30 \\ 0.53 & 0.53 & 0.53 \\ 0.17 & 0.17 & 0.17 \end{bmatrix} \times \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 90 \\ 159 \\ 51 \end{bmatrix} \quad (5.7)$$

We notice that (5.7) provides with us the conclusion that Doctors 1, 2, 3 will have 90, 159, 51 patients in long run respectively, ) exactly coincide with that of direct matrix multiplication performed hundred times to arrive (5.2).

As a bonus and distinctive advantage of the proposed method, we consider the following situation: If we assume instead of equally likely chance of 100 patients for each doctor initially, that the doctors 1, 2 and 3 were treating say  $p$ ,  $q$ ,  $r$  patients respectively, then we get  $p + q + r = 300$ . Hence, in this situation, using equations (5.5) and (5.6), we see that

$$X^{100} = QD^{100}Q^{-1}X^0 = \begin{bmatrix} 0.30 & 0.30 & 0.30 \\ 0.53 & 0.53 & 0.53 \\ 0.17 & 0.17 & 0.17 \end{bmatrix} \times \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 0.30(p+q+r) \\ 0.53(p+q+r) \\ 0.17(p+q+r) \end{bmatrix} = \begin{bmatrix} 90 \\ 159 \\ 51 \end{bmatrix} \quad (5.8)$$

Thus from equations (5.7) and (5.8) we notice that the initial assignment of patients to respective doctors is unimportant as through the transition matrix  $P$  provided in Table 1, the system always settle down to the conclusion that Doctor 1 treats 90 patients, Doctor 2 treats 159 patients and Doctor 3 treats 51 patients amounting to 300 patients totally who visits the hospital for periodic treatment.

## VI. Conclusion

Using the concept of Markov process, in the long run, we proved that the initial distribution of patients is immaterial; eventually the system will always settle down to equilibrium distribution of patients to respective doctors. This computation, allows us to decide many factors beyond this decision. For example, in the Doctor – Patients model discussed in section 5, we find that Doctor 2 is most preferred by many visiting patients and Doctor 1 is preferred next followed by the least preferred Doctor 3. This analysis can help the hospital management to provide additional benefits for Doctor 2 who does a very good job in treating patients. In a way, this method provides an idea of efficiency between doctors in the hospital.

We can easily replace Doctor – Patient model to any other preferred practical life scenario like deciding which fertilizer is best suitable for crops, which player is best possible among available players and so on. Using this concept, we can come make a meaningful decision in choosing which among the available resource is best and how to allocate the available resources in long run. This kind of analysis will eventually enhance good performance of any system.

## References

- I 49(10):1021–1025, 1998.
- II Amanda A. Honeycutt, James P. Boyle, Kristine R. Broglio, Theodore J. Thompson, Thomas J. Hoerger, Linda S. Geiss, and K. M. Venkat Narayan, A dynamic Markov model for forecasting diabetes prevalence in the United States through 2050. Health Care Management
- III Behavioral Sciences, pages 9242–9250, 2004.
- IV Chih-Ming Liu, Kuo-Ming Wang, and Yuh-Yuan Guh. A Markov chain model for medical
- V Distribution under treatment. Mathematical and Computer Modeling, 19(11):53–66, 1994.
- VI For discrete-time longitudinal data on human mixed-species infections. In Some Mathematical
- VII J. E. Cohen and B. Singer. Malaria in Nigeria: Contrained continuous-time Markov models
- VIII L. Billard. Markov models and social analysis, International Encyclopedia of the Social and
- IX Questions in Biology, pages 69–133. Providence: American Mathematical Society, 1979.
- X Record analysis. The Journal of the Operational Research Society, 42(5):357–364, 1991.
- XI S. I. McClean, B. McAlea, and P. H. Millard. Using a Markov reward model to estimate
- XII Science, 6:155–164, 2003.
- XIII Spend-down costs for a geriatric department. The Journal of the Operational Research Society,
- XIV Y. W. Tan. First passage probability distributions in Markov models and the HIV incubation