THE RELIABLE ESTIMATION FOR THE LASER WELD BY THE H- AND P- REFINEMENT OF THE FINITE ELEMENT METHOD

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Abstract

The finite element (FE) solutions are different from the exact ones due to the presence of various error sources, such as computer round-off error, error due to discrete of the displacement field, etc. This paper uses the h- and p-refinement of the finite element method for the laser butt weld problem, with the base metal is AISI 1018 steel highness 8 mm. The objective is to present estimation techniques the strain energy relative error η and evaluate its reliability through two indices: the affectivity index θ and the uniformity index SD. The numerical results achieve to meet the conditions for reliability assessment. Specifically, the η, $\bar{\theta}$, SD values of h-refinement, and p- refinement respectively: less than 6%, 0.535667, 0.019528, and less than 4%, 0.506616, 0.103834.

Keywords: Finite element method (FEM), Laser butt weld, Relative error, Reliability, h- refinement, p- refinement

I. Introduction

In actual engineering analysis, some studies have developed error estimates, but calculations are very intensive, and the accuracy needs to be verified [V], [VI], [VII]. The finite element method (FEM) is also one of the common methods in this evaluation. Because the FEM based on the minimization of the total potential energy,

the error estimate of this method with refinement mesh is done in the form of energy norm. On this basis, to increase the efficiency of the refinement process, the evaluation of error for stress to achieve accurate results is essential [VII], [XIV], [XV]. This gives useful information regarding the convergence rate of stress. Within the scope of this article, we present two common techniques and have been published by many studies: h- and p- refinement of the FEM.

[IX] Proposed error estimation of h-convergent approximations calculates only for element level and measure strain energy related to an element. [III]Continued to evaluate the strain energy error of p- convergence for two-dimensional LEFM problems, and the result of p-convergent was substantially faster.

[VIII] Presented the convergence rate of the FEM. Some general conclusions relative computational efficiencies and reliability of the modes of convergence: the smoothness of the function affected the rate of convergence, the rate of convergence of h- refinement is slower of p- refinement sequences of meshes and p-distributions can be optimally designed at singularities, the p-version meets both efficient and reliable better than the h –version.

[XVI] Presented local error evaluation for two 2D models (the Poisson problem in the plane and the plane elasticity problem) using the adaptive h-type FEM based on interpolation estimates and the 'extraction formulas' of BabuSka and Miller. The results compared with those obtained by the 'ad-hoc' methods. However, these formulas made more widespread for certain nonlinear cases, general elliptic boundary value problems.

[II] Presented the meshes design guide for p-extension of the FEM in the evaluation and controlling error. This is very convenient, inexpensive, and the establishment of engineering data is calculated with accuracy and reliability. The results were proved by calculating stress at singularities in three applications: L-shaped plane-elastic body, plane-elastic body with many stress singularities, elliptic arch.

[I] Compared the results of p-version with both h-version and adaptive h-refinement through the benchmarking problem for the deformation theory. The superiority and accuracy of p-version are very promising for more complex material models.

[XX] Pointed out that the guidance of creating a finite element solution with high precision by the h-p-r-refinement through the problem of the planar four-bar mechanism undergoing high-speed motion but the cost of the computation can increase. Therefore, based on the accuracy and computational cost, users can choose the appropriate refinement methods.

[XVIII] used h- and p- refinement of the FEM in estimating error and convergence rate for two-dimensional and three-dimensional elastic mechanical problems. Besides, based on the enrichment indicator or the ratio of error indicator, the refinement criteria for the adaptive strategy have also established. A correction for the technique was made to enhance the energy norm assessment, and the results were compared with the least square method. The results seem to suggest that the p-

refinement and the h-p-refinement have reduced the best approximation error for problems without singularities, and the h-refinement is more effective at singularities.

[XVII] has overcome the disadvantages of isogeometric analysis based on NURBS, and the effect was significantly improved. In this paper, an adaptive algorithm, h-refinement with T-splines, has been combined with a posteriori error estimation technique. The potential of IGA to T-splines is evident in numerical examples and paves the way for further developments.

[IV] have approached Binev's algorithm to prove the convergence and design an optimal method through two examples: Lacunary Function and Non-degenerate Function, respectively were the self-adjoint elliptic problem (one dimension) and the Poisson problem (two-dimensions). The hinges of Claudio Canuto et al. is the hp-AFEM. The algorithm of the hp-AFEM consists of two repetition routines: hp-NEARBEST and REDUCE. The near-best hp-approximation of the current discrete solution and data is detected by hp-NEARBEST to achieve the desired accuracy. The error tolerances are reduced by REDUCE but are acceptable. This concatenation has created the converging sequence.

II. The General Theory of the Reliable Estimation the H- and P- Refinement of the Finite Element Method (FEM)

As we know in the FEM, a continuum model presented by a certain number of elements with a simple approximation field causes the presence of discretization error in solutions. This error depends on many factors such as a number of elements, type of element, an order of interpolation functions, the shape of elements, singular points in a problem domain, and the representation of applied loads and support conditions.

Thus, in practical engineering analysis, it is necessary to check the discretization error by observing the convergence of results. Of course, it must be sufficiently reliable. In order to evaluate its reliability, two simple indices can be introduced: *the effectively index* θ *and the uniformity index* **SD**.

The equilibrium equation (the strong form):

$$Lu = f \Rightarrow Lu - f = 0; \qquad u \in \Omega \tag{1}$$

The principle of virtual work (weak form):

$$\int_{\Omega} v(Lu - f)d\Omega = 0; \qquad u, v \in \Omega \tag{2}$$

The homogenous Dirichlet's boundary condition:

$$u = 0 \qquad on \qquad \Gamma_D \tag{3}$$

Find $u \in V$ such that it satisfies equation (3):

$$B(u,v) = L(v) \qquad \forall v \in V \tag{4}$$

where V is the infinite – dimension space, $B(u,v)$ is the virtual work of internal stresses, $L(v)$ is the virtual work of external forces.

The energy norm:

$$\|u\|_{E(\Omega)} = \sqrt{B(u,u)} \tag{5}$$

The strain energy:

$$U(u) = \frac{1}{2} B(u,u) \tag{6}$$

Denote by u the exact displacement, σ is the stress fields, and by u_h and σ_h the ones obtained by the finite element method using a mesh of size h, then the point-wise error in the displacement field is:

$$(e_h)_u = u - u_h \tag{7}$$

and the point-wise error in the stress field is:

$$(e_h)_\sigma = \sigma - \sigma_h \tag{8}$$

Since it is not convenient to describe such point-wise error, it is always hopeful of measuring the error by certain norms for practical considerations, such as for the determination of global precision and elemental ones.

For problems for which a functional called total energy exists, the finite element solution can be shown to be the one which minimizes the global stress error of the structure in the energy sense [XIX], i.e., the solution u_h corresponds to:

$$\underset{u_h \in U_h}{Minimise} \int_\Omega (\sigma - \sigma_h)^T H^{-1} (\sigma - \sigma_h) d\Omega \tag{9}$$

where Ω is the domain, H is the Hooke's elastic matrix, and U_h is the finite element subspace. Consequently, the energy norm of the error:

$$\|e_h\|_{E(\Omega)} = \left[\int_\Omega (\sigma - \sigma_h)^T H^{-1} (\sigma - \sigma_h) d\Omega \right]^{1/2} \tag{10}$$

In most cases, the exact solution is unknown, so the error must be estimated. The a-posteriori error estimation consists in finding a procedure that can offer a reliable estimate of the exact energy norm of the error not only at the global level but also at the elemental level. Denote by ε such as to estimate:

$$\varepsilon \approx \|e_h\|_{E(\Omega)} = \|u - u_h\|_{E(\Omega)} \tag{11}$$

which is often called an *error estimator.*

The algebraic rate of convergence:

$$\Box e_h \Box_{E(\Omega)} = \Box u - u_h \Box_{E(\Omega)} \le \frac{k}{N^{\beta}} \tag{12}$$

where k is a positive problem's dependent constant, N the number of degrees of freedom, and β the asymptotic rate of convergence.

The error estimation using Richardson extrapolation technique:

$$\Box e_h \Box^2_{E(\Omega)} = 2 \Box U - U_h \Box_{E(\Omega)} = \frac{k^2}{N^{2\beta}} \tag{13}$$

The exact relative error:

$$\eta_{FEM}(\%) = \sqrt{\frac{U - U_h}{U}} \times 100\% \tag{14}$$

The estimated relative error (***using the Richardson extrapolation technique***):

$$\eta_{extra}(\%) = \sqrt{\frac{U_h - U_{h-1}}{U_h}} \times 100\% \tag{15}$$

The effectivity index:

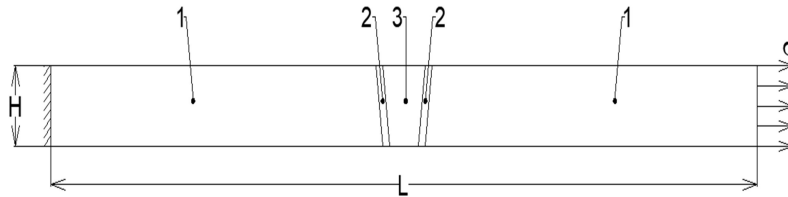$$\theta = \frac{\eta_{extra}}{\eta_{FEM}} \tag{16}$$

The uniformity index:

$$SD = \left[\frac{1}{n} \sum_{i=1}^{n} \left(\theta_i - \overline{\theta} \right)^2 \right]^{\frac{1}{2}} \tag{17}$$

where θ_i is the index for elemental level, $\overline{\theta}$ is the average index for global level

For the range $1 \le \eta(\%) \le 10$, ***an estimator is said reliable if:*** $0.8 \le \theta \le 1.2$ ***and*** $SD \le 0.2$ [VI]

III.   The Problem of Laser-Welded butt Joints under Tensile Stress

We consider the model of the laser butt weld (as shown in Fig. 1)

1. Base Metal (BM)   2. HAZ 3. Weld Zone (WZ)

Fig.1: The scheme of the weld

The base metal is AISI 1018 steel. The modulus of elasticity E = 205 GPa and the Poissons ratio ν = 0.29 ([X], [XIII])

The length L = 100 mm, the high H = 8 mm, and the thick t = 1 mm

The tensile strength of the laser weld after the test σ = 562 MPa

The volume V = L×H×t = 800 mm^3

The exact strain energy U by given ([XI], [XII]):

$$U = \frac{1}{2}\frac{\sigma^2 V}{E} \tag{18}$$

⇒*The value of the exact strain energy*

$U = 0.616281$ kJ (19)

The finite element analyses were done in the case of plane strain.

The analysis is implemented by Matlab code for not only the finite element analyses but also error estimation. The detail of the Matlab code program structure is shown in Fig.2.
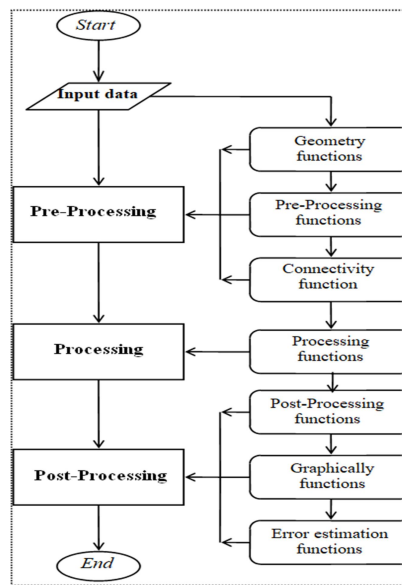

Fig.2: The Matlab code program structure

IV.   Results and Discussion

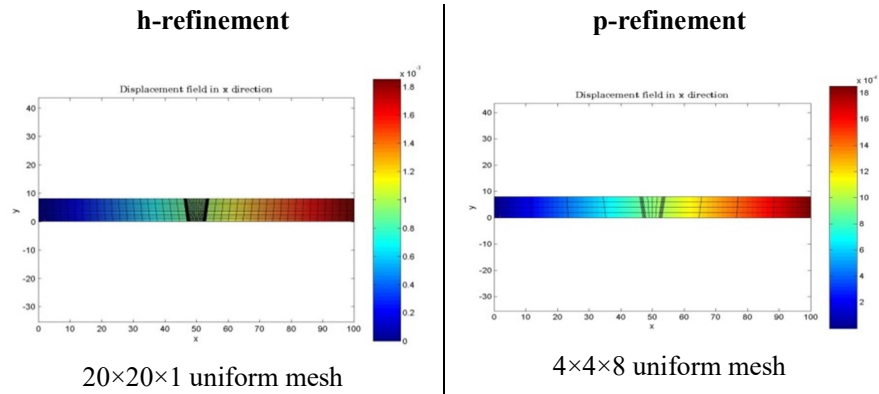The results of the finite element analysis are shown in Fig. 3 and Fig. 4

h-refinement   **p-refinement**

20×20×1 uniform mesh   4×4×8 uniform mesh

Fig.3: The displacement field in the x-direction

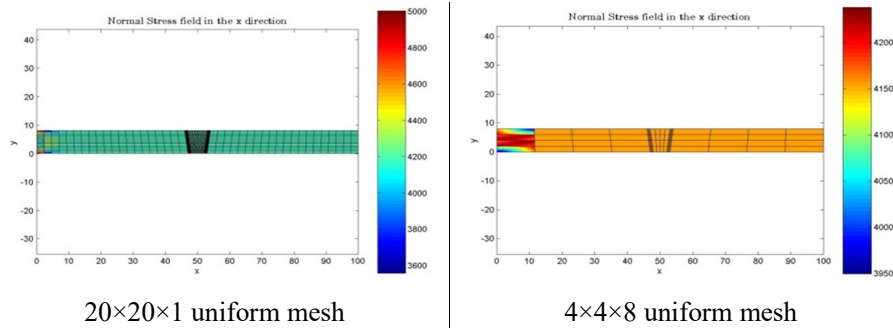20×20×1 uniform mesh   4×4×8 uniform mesh

Fig.4: The stress field in the x-direction

Corresponds to mesh, element numbers, and Dofs, the values of the strain energy and the error are presented in Tab. 1 and Tab. 2

Table 1: The h- refinement estimation results with the uniform mesh

Mesh	Element numbers	Dofs	The FEM strain energy (kJ)	The extra strain energy (kJ)	CPU time (s)
10×10	500	1122	0.614324438	0.6151587149	4.311
11×11	605	1344	0.614415158	0.6151357316	6.466
12×12	720	1586	0.614489235	0.6151203606	10.007
13×13	845	1848	0.614550606	0.6151100296	15.431
14×14	980	2130	0.614602102	0.6151028451	23.057
15×15	1125	2432	0.614645795	0.6150978253	33.682
16×16	1280	2754	0.614683239	0.6150942646	49.702
17×17	1445	3096	0.614715611	0.6150917687	66.657
18×18	1620	3458	0.614743821	0.61508999289	92.058
19×19	1805	3840	0.614768580	0.6150885422	122.959
20×20	2000	4242	0.614790452	0.6150875415	166.091

Dofs	η_{FEM} (%)	η_{extra} (%)	θ	$\bar{\theta}$	SD
1122	5.634525995	3.682658519	0.653588		
1344	5.502347809	3.422580574	0.622022		
1586	5.392015867	3.203154843	0.594055		
1848	5.298867385	3.015739461	0.569129		
2130	5.219426836	2.853210237	0.546652		
2432	5.151060698	2.710887540	0.526278	0.535667	0.019528
2754	5.091743887	2.585019044	0.507688		
3096	5.039898373	2.472948899	0.490674		
3458	4.994279531	2.372119294	0.474967		
3840	4.953894817	2.280763997	0.460398		
4242	4.917944624	2.197735303	0.446881		

Table 2: The p- refinement estimation results with 4×4 uniform mesh

Degree p	Element numbers	Dofs	The FEM strain energy (kJ)	The extra strain energy (kJ)	CPU time (s)
3		946	0.6153716271	0.6162412117	1.338
4		1474	0.6155020466	0.6155621713	2.509
5	80	2162	0.6155502623	0.6155911812	5.415
6		3010	0.6155853998	0.6157916617	11.768
7		4018	0.6156130668	0.6158877709	24.952
8		5186	0.6156326251	0.6157169343	49.870

Dofs	η_{FEM} (%)	η_{extra} (%)	θ	$\bar{\theta}$	SD
946	3.841329993	3.756475407	0.977910103		
1474	3.555218998	0.988304002	0.277986814		
2162	3.443430762	0.815296901	0.23676878	0.506616	0.103834
3010	3.359622269	1.830174929	0.544756161		
4018	3.292131168	2.111941071	0.641511824		
5186	3.243573294	1.170165056	0.360764179		

The values of the exact relative error η_{FEM} (%) are calculated from Eq. 14 between the exact $U = 0.616281\ kJ$ (Eq. 20) and the FEM (in Tab. 1 and Tab. 2) strain energy values. The values of the estimated relative error η_{extra} (%)in Tab. 1 and Tab. 2 are calculated from Eq. 15 between the FEM and the extra strain energy values.

This value ranges of the exact relative error:

$$5.634525995 \geq \eta_{h-FEM}(\%) \geq 4.917944624 \,\&$$
$$3.841329993 \geq \eta_{p-FEM}(\%) \geq 3.243573294$$

and the estimated relative error:

$$3.682658519 \geq \eta_{h-extra}(\%) \geq 2.197735303 \,\&$$
$$3.756475407 \geq \eta_{p-extra}(\%) \geq 0.815296901$$

The relationships between the number of Dofs and (the strain energy U, the relative error η, the convergence rate θ) of the h- and p- refinement are shown in Fig. 5, Fig. 6 and Fig. 7

Although the convergence curve of h- refinement is more smooth, the advantage of p-refinement shows that the convergence rate is much faster with only fewer element numbers and degrees of freedom, and lower computational costs.
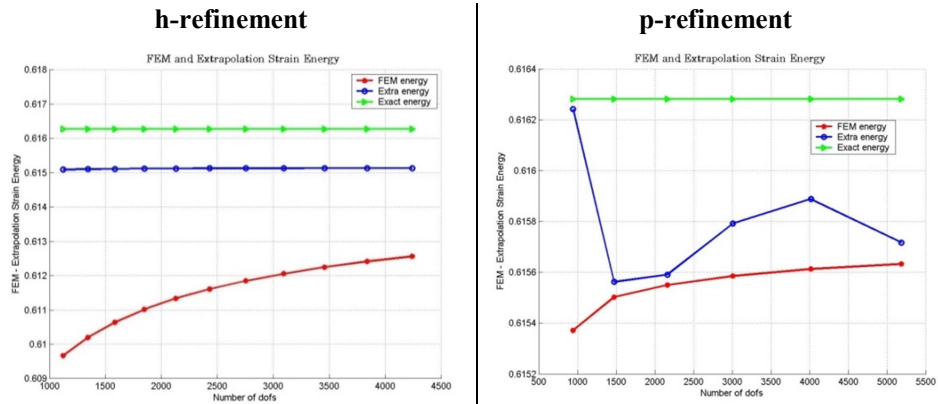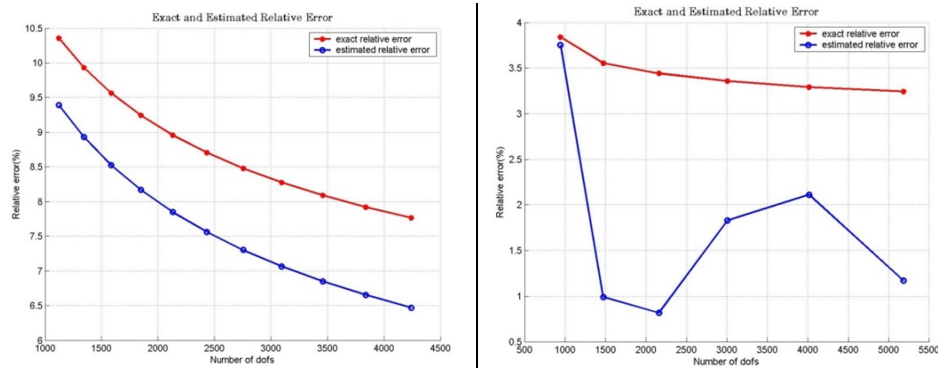
h-refinement   **p-refinement**

Fig.5: The Dofs and U graph

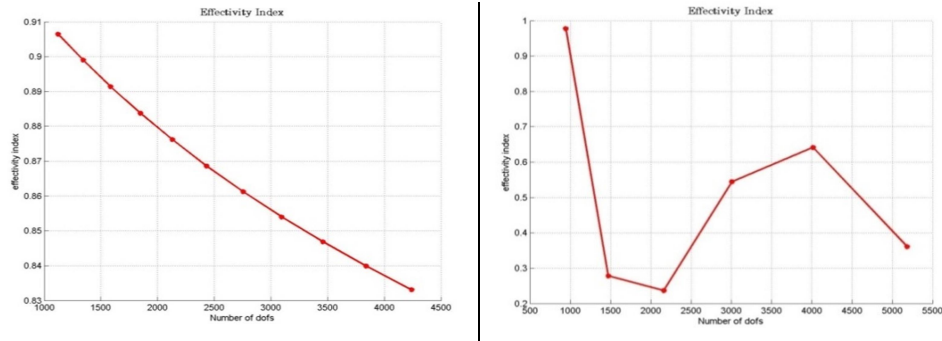

Fig.6: The Dofs and η graph

Fig.7: The Dofs and θ curve

V.   Conclusion

In this paper, the reliability assessment for the h- and p-refinement of the finite element method with the quadrilateral element has performed. The two-dimension laser-welded butt joints under tensile stress for the AISI 1018 steel highness 8mm has considered. The number of mesh surveyed for h- and p-refinement were 11 (degree p is 1) and 6 (degree p is 3 to 8). The relative error value in assessing the error is within the permitted range, less than 10%. Besides, ***using the Richardson extrapolation technique*** has brought very feasible error values: $\eta_{extra}^{max}(\%) = 3.756475407$ and $\eta_{extra}^{min}(\%) = 0.815296901$. Moreover, with the values of two indicators: the effectivity index $\bar{\theta}_{h-refinement} = 0.535667\ \&\ \bar{\theta}_{p-refinement} = 0.506616$ and the uniformity index $SD_{h-refinement} = 0.019528\ \&\ SD_{p-refinement} = 0.103834$ (satisfying Eq. 18 and ***can extend the lower bound of θ to 0.5***), the goal of the paper is confirmed in the specific technical problem.

Abbreviations

FEM	Finite Element Method
BM	Base Metal
HAZ	Heat-Affected Zone
WZ	Weld Zone
AISI	American Iron and Steel Institute
Dof(s)	Degree(s) of freedom
LEFM	Linear Elastic Fracture Mechanics
IGA	IsoGeometric Analysis
NURBS	Non-Uniform Rational B-Splines
hp-AFEM	Adaptive hp-Finite Element Method

VI. Acknowledgment

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