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SOME CRITERIA OF COMMUTATIVITY OF SEMIRINGS

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Abstract

In this article, we discuss some functional identities of certain semirings which enable us to induce commutativitiy in them. This will be helpful to extend some remarkable results of ring theory in the canvas of semirings. We also study some other useful functional identities which are trivial in ordinary rings.

Keywords: Semiring, Inversesemiring, MA-semiring, Derivation

I. Introduction

Semiring was introduced by Vandiver [XI] as an algebraic system (S,+,.) such that (S,+) is a commutative monoid, (S, .) is a semigroup in which distributive laws hold. A semiring S is known as additive inverse semirings if for each element u in S, there exits a unique element $u' \in S$ such that $u+u'+u = u$ and $u'+u+u' = u'$, where element u' is called pseudo-inverse of u [IX]. The condition of Bandlet and Petrich (A2) was attached to additive inverse semirings i.e. $u+u' \in Z$, where Z is the centre of S. This class of semirings was named as MA-semirings [VI]. Commutative inverse semirings and distributive lattices are examples of MA-semirings. An MA-subsemiring I of MA-semiring S is a left (right) MA-ideal if $ua(au)\in I$ for any $a\in I$ and $u\in S$. An ideal I of MA-semiring S is called an MA-ideal if it is both left and right MA-ideal. An MA-semiring S is called prime if for any two elements a and b of S, $aSb = 0$ implies either $a=0$ or $b=0$ and semiprime if $aSa=0$ implies $a=0$. A derivation of MA-semiring S is an additive mapping $\delta:S\rightarrow S$ satisfying $\delta(uv)=\delta(u)v+u\delta(v)$ $\forall u,v\in S$. The Lie and Jordan products of u and v in S are defined as $[u,v]=uv+(vu)'$ and $\langle u,v\rangle=uv+vu$ respectively. It should be noted that if $[u,v]=0$ $\forall u,v\in S$, then S is commutative but the converse is not true in general.

In this article, we will prove that if S is a prime MA-semiring, I is a left MA-ideal of S and $\delta:S\rightarrow S$ is a nonzero derivation satisfying one of the identities (i) $[[\delta(u), u], u]=0\forall u\in S$ with characteristic of not two; (ii) $[\delta(u),\delta(v)]=0$ $\forall u,v\in S$; (iii) $\delta([u,v])=0$ $\forall u,v\in I$, then S is commutative. This simultaneously generalises some important results of ordinary rings [I,V,XII]. We also explore some other useful functional identities in special class of MA-semirings which are otherwise trivial in ordinary rings.

Semirings have many applications in Artificial Intelligence and related fields [VIII]. Semiring based programming is popular in AI [II]. Some otherapplications of semirings are in the areas of optimization theory, graph theory, dynamical systems and automata theory [III,IV].

II. Preliminaries

We will use the following results in our main theorems.

Proposition 2.1 [V]. Let R be a prime ring and $d:R\rightarrow R$ be a nonzero derivation suchthat $[d(x),d(y)] = 0$ $\forall x,y\in R$, then R is commutative.

Proposition 2.2 [VII]. Let S be a prime MA-semiring and $\delta:S\rightarrow S$ be a nonzero derivationwith $[\delta(u), u]=0 \forall u \in S$, then S is commutative.

The proofs of following Lemmas are easy to follow.

Lemma 2.3. Let S be an inverse semiring and $a,b\in S$. If $a+b = 0$, then $a+a' = 0$.

Lemma 2.4. Let S be an inverse semiring and $\delta:S\rightarrow S$ be an additive map, then

$\delta(u') = \delta(u)'$ $\forall u\in S$.

Lemma 2.5. Let S be a MA-semiring and u,v,w be arbitrary elements in S, then following identities hold:

(i) $[u+v,w]=[u,w]+[v,w]$.

(ii) $[u,vw]=v[u,w]+[u,v]w$.

(iii) $[u,v]'=[v,u]=[u,v']=[u',v]$.

(iv) $uv=vu+[u,v]$.

(v) $[u,v]+[v,u]=v(u+u')=u(v+v')$.

(vi) $[[v,u],u]=[u,u]v+[[v,u],u]$.

(vii) $2[u,u]=[u,u]$.

III. MA-Semiring Case

Theorem 3.1. Let S be a prime MA-semiring, I be a left MA-ideal of S and $\delta:S\rightarrow S$ be a nonzero derivation with one of the identities (i) $[[\delta(u),u],u]=0$ $\forall u \in S$ with

char (S) $\neq 2$ (ii) $[\delta(u), \delta(v)]=0$ $\forall u,v\in S$; (iii) $\delta([u, v])=0$ $\forall u,v\in I$, then S is commutative.

Proof.(i). We are given that

$$[[\delta(u),u],u]=0 \forall u\in S. \quad (1)$$

Replce v by $\delta(u)$ in Lemma 2.5 (vi) and use (1) to get $[u, u]\delta(u)=0$. Invoking the primeness of S, we get

$$[u,u]= 0. \quad (2)$$

Linearize the last identity to get

$$[u,v]+[u,v]'=0 \forall u,v\in S. \quad (3)$$

First replace u by u + v and u + v′ successively in (1), then use Lemma 2.4 and (3) to get

$$[[\delta(u),v],v]+[[\delta(v),u],v]+[[\delta(v),v],u] = 0. \quad (4)$$

Substituting vu for u in last relation and using (2) to have

$$3[\delta(v),v][u,v]+\delta(v)[[u,v],v]=0. \quad (5)$$

Replace u by δ(v)u and simplify to get

$$3[\delta(v),v]\delta(v)[u,v]+\delta(v)([\delta(v)[[u,v],v]+2[\delta(v),v][u,v])=0. \quad (6)$$

By using (5), we have

$$3[\delta(v),v]\delta(v)[u,v]+\delta(v)(2[\delta(v),v][u, v]+3([\delta(v),v][u,v])')=0. \quad (7)$$

In view of identity (3), the last relation becomes

$$(3[\delta(v),v]\delta(v)+\delta(v)[\delta(v),v]')[u,v]=0. \quad (8)$$

Replace u by wu to get

$$(3[\delta(v),v]\delta(v)+\delta(v)[\delta(v),v]')w[u,v]=0. \quad (9)$$

As S is a prime MA-semiring, so either we have

$$3[\delta(v),v]\delta(v)+\delta(v)[\delta(v), v]'=0. \quad (10)$$

or [u,v] = 0. In later case S is commutative in view of Lemma 2.5 (iv). By replacing u with uδ(v) in (5), we obtain

$$3\delta(v)[\delta(v),v]+[\delta(v),v]\delta(v)'=0. \quad (11)$$

By using (3), (10) and (11), we get

$$[\delta(v),v]\delta(v)=0 \forall v \in S. \quad (12)$$

Now by replacing v with u+v and u+v′ successively in last relation to have

$$[\delta(v),v]\delta(u)+[\delta(u),v]\delta(v)+[\delta(v),u]\delta(v)=0. \quad (13)$$

Replace u by vu in above relation to get

$$[\delta(v),v]v\delta(u) + \delta(v)[u,v]\delta(v)+[\delta(v),v]u\delta(v)+v[\delta(u),v]\delta(v)+[v,v]\delta(u)\delta(v)+v[\delta(v),u]\delta(v) +[\delta(v),v]u\delta(v) = 0. \quad (14)$$

Use Lemma 2.5 (iv), (2) and (13) to get

$$2[\delta(v),v]u\delta(v)+\delta(v)[u,v]\delta(v)=0. \quad (15)$$

Replace u by uv to get

$$2[\delta(v),v]uv\delta(v)+\delta(v)[u,v]v\delta(v)=0. \quad (16)$$

Right multiply (15) by v′ to have

$$2[\delta(v), v]u\delta(v)v'+\delta(v)[u,v]\delta(v)v'=0. \quad (17)$$

From last two equations, we have

$$2[\delta(v),v]u[\delta(v),v]+\delta(v)[u,v][\delta(v),v]=0. \quad (18)$$

Replacing u by uδ(v) in (5) and simplifying, we get

$$3[\delta(v),v]u[\delta(v),v]+2\delta(v)[u,v][\delta(v),v]=0. \quad (19)$$

By using (3), the last two relations produce $[\delta(v),v]u[\delta(v),v]=0$. As S is a prime MA-semiring,we get $[\delta(v),v]=0$. Hence S is commutative in view of proposition 2.2.

(ii). By supposition, we have $[\delta(u),\delta(v)]=0\ \forall u,v\in S$. Replace u with uv to get $\delta(u)[v,\delta(v)]+[u,\delta(v)]\delta(v)=0$. By using Lemma 2.3, we have $\delta(u)([v,\delta(v)]+[\delta(v),v])=0$. Now invoking the primeness of S, we get $[v,\delta(v)]+[\delta(v),v]=0$. This becomes $\delta(v)(v+v')= 0$ in view of Lemma 2.5 (v). Again by using primeness of S, we get $u+u' = 0\ \forall u\in S$. So S becomes commutative in view proposition 2.1.

(iii). Replace v by vu in $\delta([u,v])=0\forall u,v\in I$ to get $\delta(v)[u,u]+[u,v]\delta(u)=0$. This can be written as $\delta(v)[u, u]+\delta(v)[u, u]'= 0$. This becomes $\delta(v)2[u, u]=0$. Using Lemma 2.5 (vii), we have $\delta(v)[u,u]=0$. As R is prime MA-semiring, so we have $[u,u]=0$. In this case, we get $[u,v]\delta(u)=0$. Replace v by wv, where $w \in S$, to have $[u,v]w\delta(u) = 0$. So we have $[u,v]=0\forall u,v\in I$. This implies $[u,v]=0\forall u,v\in S$, which implies $uv=vu\forall u,v\in S$. This completes the proof.

IV. Proper MA-Semiring Case

A nonzero MA-semiring S is called proper if it is not a ring. Let R be a set of real numbers with usual operations and W be set of whole numbers. Let $(W,\vee,\wedge)$ be a distributive lattice with $x\vee y = Sup(x,y)$ and $x\wedge y=Inf(x,y)\forall x,y\in W$. Take $R\times W$ with operations $(x_1, y_1)\oplus(x_2, y_2)=(x_1+x_2,y_1\vee y_2)$ and $(x_1, y_1)\otimes(x_2, y_2)=(x_1+x_2,y_1\wedge y_2)$. For $(x,y)\in R\times W$, we define $(x, y)'=(-x, y)$, then $(R\times W,\oplus,\otimes)$ is a proper MA-semiring.

Now we will study some commutator conditions in proper MA-semirings which are otherwise trivial in ordinary rings. We start with following Lemma which we will use in our main result.

Lemma 4.1. Let S be a proper prime MA-semiring and I be a left MA-ideal of S. If $\delta:S\to S$ is a nonzero derivation with $\delta([u,u])=0\forall u \in I$, then S is commutative.

Proof. Replace u by vu in $\delta([u,u])=0\ \forall u\in I$ to have

$$\delta(v)[u,v]u+\delta(v^2)[u,u]+v\delta([u,v])u+v[u,v]\delta(u)+\delta(v)[v,u]u+v\delta([u,v])v+ v[v,u]\delta(u)+[v,v]\delta(u^2)=0. \tag{20}$$

This reduces to $\delta(v)[u,v]u+(\delta(v)[u,v]u)' =0$ in view of Lemma 2.3. This implies $\delta(v)[u,v](u+u')=0$. As S is a proper prime MA-semiring, we get $\delta(v)[u,v]=0$. Replace u by wu, $w \in S$, to have $\delta(v)w[u,v]=0$. This results in $[u,v]=0\forall u \in I,\forall v\in S$. This implies $[v,u]=0\forall u,v\in S$, which gives $uv=vu$ in view of Lemma 2.5 (iv).So S is commutative.

Theorem 4.2. Let S be a proper prime MA-semiring and I be a left MA-ideal of S. Furthermore, let $\delta:S\to S$ be a nonzero derivation satisfying any one of the identities (i) $[\delta(u),\delta(u)]=0$; (ii) $[[\delta(u),u],[\delta(u),u]]=0$; (iii) $[\delta([u,u]),u]=0\ \forall u \in I$, then S is commutative.

Proof. (i). By the given hypothesis, we have $[\delta(u),\delta(u)] = 0\ \forall u \in I$. Replace u by vu to have

$$[\delta(v)u,v\delta(u)]+[\delta(v)u,\delta(v)u]+[v\delta(u),\delta(v)u]+[v\delta(u),v\delta(u)]=0. \tag{21}$$

This becomes

$$\delta(v)[u,v\delta(u)]+[\delta(v),v\delta(u)]u+[\delta(v)u,\delta(v)u]+[v\delta(u),\delta(v)u]+[v\delta(u),v\delta(u)]=0 \tag{22}$$

This can be written as

$$\delta(v)v[u,\delta(u)]+\delta(v)[u,v]\delta(u)+[\delta(v),v\delta(u)]u+[\delta(v)u,\delta(v)u]+[v\delta(u),\delta(v)u]+ [v\delta(u),v\delta(u)]=0. \tag{23}$$

This reduces to $\delta(v)[u,v]\delta(u) +\delta(v)[u,v]\delta(u)' = 0$, which becomes $\delta(v)[u,v](\delta(u)+\delta(u)')=0$. By invoking the primeness of S, we have either $\delta(v)[u, v]=0$ or $\delta(u)+\delta(u)'=0$. In the later case, we get $\delta([u,u])=0\forall u\in I$, which implies that S is commutative in view of Lemma 4.1. In case $\delta(v)[u,v]=0$, replace u by wu, for $w \in S$, and use the primeness of S to get $[u,v]=0$. So S is again commutative.

(ii). Replace u by vu in $[[\delta(u),u],[\delta(u),u]]=0\forall u\in I$ to get

$$[[\delta(v)u,vu]+[v\delta(u),vu],[\delta(v)u,vu]+[v\delta(u),vu]]=0. \tag{24}$$

This implies

$$[[\delta(v)u,vu],[\delta(v)u,vu]]+[[\delta(v)u,vu],[v\delta(u),vu]]+[[v\delta(u),vu],[\delta(v)u,vu]]+ [[v\delta(u),vu],[v\delta(u),vu]]=0. \tag{25}$$

This becomes

$$[\delta(v)[u,vu],\delta(v)[u,vu]]+[\delta(v)[u,vu],[\delta(v),vu]u]+[[\delta(v),vu]u,\delta(v)[u,vu]]+ [[\delta(v),vu]u,[\delta(v),vu]u]+[[\delta(v)u,vu],[v\delta(u),vu]]+[[v\delta(u),vu],[\delta(v)u,vu]]+ [[v\delta(u),vu],[v\delta(u),vu]]=0. \tag{26}$$

This can be written as

$$[\delta(v),\delta(v)][u,vu]^2+\delta(v)[\delta(v),[u,vu]][u,vu]+\delta(v)[[u,vu],\delta(v)[u,vu]]+[\delta(v)[u,vu],[\delta(v),vu]u]+[[\delta(v),vu]u,\delta(v)[u,vu]]+[[\delta(v),vu]u,[\delta(v),vu]u]+[[\delta(v)u,vu],[v\delta(u),vu]]+ [[v\delta(u),vu],[\delta(v)u,vu]]+ [[v\delta(u),vu],[v\delta(u),vu]]=0. \tag{27}$$

This reduces to $[\delta(v),\delta(v)][u,vu]^2+([\delta(v),\delta(v)][u,vu]^2)'=0$,which becomes $[\delta(v),\delta(v)][u,vu]^2=0$. As S is a proper MA-semiring, so we have either $[u,vu]^2 = 0$ or $[\delta(v),\delta(v)]=0$. In the later case, we proved that S is commutative in view of last result. Now $[u,vu]^2=0$ implies $v[u,u][u,vu]=0$. This reduces to $[u,vu]=0$, which implies $[u,v](u+u')=0$. This results in$[u,v]=0$. So S is commutative.

(iii). Replace u by vu in $[\delta([u,u]),u]=0\ \forall u\in I$ to have

$$[\delta([v,v])u^2,vu]+[\delta(v^2)[u,u],vu]+[v^2\delta([u,u]),vu]+[\delta(v)[u,v]u,vu]+[v\delta([u,v])u,vu]+ [v[u,v]\delta(u),vu]+ [\delta(v)[v,u]u,vu]+[v\delta([v,u])v,vu]+[v[v,u]\delta(u),vu]+[[v,v]\delta(u^2),vu]=0. \tag{28}$$

This implies

$$v\delta([v,v])[u^2,u]+v[\delta([v,v]),u]u^2+[\delta([v,v])u^2,v]u+[\delta(v^2)[u,u],vu]+[v^2\delta([u,u]),vu]+$$

$[\delta(v)[u,v]u,vu]+[v\delta([u,v])u,vu]+[v[u,v]\delta(u),vu]+[\delta(v)[v,u]u,vu]+[v\delta([v,u])v,vu]+$

$[v[v,u]\delta(u),vu]+[[v,v]\delta(u^2),vu]=0.$ (29)

This becomes

$2v\delta([v,v])u[u,u]+v[\delta([v,v]),u]u^2+[\delta([v,v])u^2,v]u+[\delta(v^2)[u,u],vu]+[v^2\delta([u,u]),vu]+[\delta(v)$
$[u,v]u,vu]+[v\delta([u,v])u,vu]+[v[u,v]\delta(u),vu]+[\delta(v)[v,u]u,vu]+[v\delta([v,u])v,vu]+$

$[v[v,u]\delta(u),vu]+[[v,v]\delta(u^2),vu]=0.$ (30)

This reduces to $v\delta([v,v])u[u,u]=0$. As S is a proper MA-semiring, so we get $v\delta([v,v])u=0$.This becomes $v\delta([v,v])=0$, which implies $\delta([v,v])=0$ $\forall v\in$ I. So S is commutative.

V. Conclusion

In this article, we studied one aspect of MA-semiring i.e. commutativity of MA-semirings under some functional identities. Many Lie and Jordan theoretic concepts in ordinary rings give rise to different kinds of functional identities which have many applications in linear algebra, operator theory, functional analysis and mathematical physics. These notions can easily be generalised in MA-semiring setup.

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