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GENERALIZATION OF SOME WEIGHTED ČEBYŠEV-TYPE INEQUALITIES

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Abstract

In present paper, we give generalisation of inequalities of Čebyšev type involving weights for absolutely continuous functions whose derivatives belong to $L_r(c,d)$ *(Lebesgue space), where* $r \geq 1$*. Our results recapture many established results of different authors. Applications are also given in probability theory.*

Keywords: ČebyševInequality, Probability Density Function, Cumulative Density Function.

I. Introduction

In [XV], P. L. Čebyšev discovered one of the important and classical integral inequality (can also be seen in [XIII]). i. e.

$$|T(g,h)| \leq \tfrac{1}{12}(d-c)^2\|g'\|_\infty\|h'\|_\infty , \tag{1.1}$$

where functions $g, h : [c,d] \to \mathbb{R}$, absolutely continuous whose derivatives $g', h' \in L_\infty[c,d]$ and

$$|T(g,h)| = \frac{1}{d-c}\int_c^d g(\mathrm{T})\,h(\mathrm{T})\,d\,\mathrm{T} - \left(\frac{1}{d-c}\int_c^d g(\mathrm{T})\,d\,\mathrm{T}\right)\left(\frac{1}{d-c}\int_c^d h(\mathrm{T})\,d\,\mathrm{T}\right) \tag{1.2}$$

is calledČebyšev functional, provided that the above integrals exist.

Many researchers have paid considerable attention on integral Čebyšev inequality due to fundamental importance of inequality (1.1) in analysis and applications andfor this reason, in the literature a number of generalizations, extensions and variants have made on it (see [I, II, III, IV, VI, VII, IX, X, XII, XIII]) and the references given therein. For more study about Čebyšev functional see [V].

It is well known that the aboveČebyšev functional is used in several branches of mathematics such as special functions, transformation theory, probability and statistical and numerical integration (see [XIV]).

The Čebyšev-type inequality for r-norm is given by B. G. Pachpatte in his article[VIII] that is;

Proposition1.1. Let functions $g, h : [c, d] \to \mathbb{R}$ be absolutely continuous and theirderivatives$g', h' \in L_r(c, d),\ r > 1$, then

$$|T(\alpha, \beta, g, h)| \leq \frac{1}{(d-c)^2} M^{\frac{2}{s}} \|g'\|_r \|h'\|_r, \qquad (1.3)$$

where

$$\alpha = \frac{1}{3}\left[\frac{g(c) + g(d)}{2} + 2g\left(\frac{c+d}{2}\right)\right]$$

$$\beta = \frac{1}{3}\left[\frac{h(c) + h(d)}{2} + 2h\left(\frac{c+d}{2}\right)\right]$$

$$M = \frac{(d-c)^{s+1}(2^{s+1}+1)}{3(s+1)6^s}$$

with $\frac{1}{r} + \frac{1}{s} = 1$,

$$\|g\|_r = \left(\int_c^d |g(\mathrm{T})|^r \, d\,\mathrm{T}\right)^{\frac{1}{r}}$$

and

$$T(\alpha, \beta, g, h) = \alpha\beta - \frac{1}{d-c}\left(\alpha \int_c^d h(\mathrm{T}) d\,\mathrm{T} + \beta \int_c^d g(\mathrm{T}) d\,\mathrm{T}\right) + \left(\frac{1}{d-c}\int_c^d g(\mathrm{T}) d\,\mathrm{T}\right)\left(\frac{1}{d-c}\int_c^d h(\mathrm{T}) d\,\mathrm{T}\right). \qquad (1.4)$$

The generalization of (1.3) is stated by Zheng Liu in his article [XVII] which is as follows.

Proposition 1.2. Let the assumptions of Proposition 1.1 be valid, where $l \in [0,1]$. Then

$$|T(\Gamma_l, \Delta_l, g, h)| \leq \frac{1}{(d-c)^2} M_l^{\frac{2}{s}} \|g'\|_r \|h'\|_r\,, \qquad (1.5)$$

where

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$$M_l = \frac{l^{s+1} + (1-l)^{s+1}(d-c)^{s+1}}{(q+1)2^s}$$

and

$$\Gamma_l = \frac{l}{2}[g(c) + g(d)] + (1-l)\, g\left(\frac{c+d}{2}\right)$$

$$\Delta_l = \frac{l}{2}[h(c) + h(d)] + (1-l) h\left(\frac{c+d}{2}\right).$$

In upcoming main section, we would obtain generalization of above inequalities (1.3) and (1.5) involving weights and then obtained inequalities would apply on probability density functions.

II. Generalization of WeightedČebyšev-Type Inequality

From now onwards we define $w: [c,d] \to [0,1]$ be a probability density function.In the following, we define some notations for simplicity of the expression for suitable functions $g \; and \; h : [c,d] \to \mathbb{R} \; and \; l \in [0,1]$.

$$\Gamma_{l,x} = g(x)\left(\int_0^{(1-l)(x-c)} w(\mathrm{T})d\,\mathrm{T} - \int_0^{(1-l)(x-d)} w(\mathrm{T})d\,\mathrm{T}\right) - g(c)\int_0^{l(c-x)} w(\mathrm{T})\,d\,\mathrm{T} + g(d)\int_0^{l(d-x)} w(\mathrm{T})\,d\,\mathrm{T},$$

$$\Delta_{l,x} = h(x)\left(\int_0^{(1-l)(x-c)} w(\mathrm{T})d\,\mathrm{T} - \int_0^{(1-l)(x-d)} w(\mathrm{T})d\,\mathrm{T}\right) - h(c)\int_0^{l(c-x)} w(\mathrm{T})\,d\,\mathrm{T} + h(d)\int_0^{l(d-x)} w(\mathrm{T})\,d\,\mathrm{T} \qquad (2.1)$$

and$T(\Gamma_{l,x}, \Delta_{l,x}, g, h)$ is defined as follows.

$$T(\Gamma_{l,x}, \Delta_{l,x}, g, h) = \left(g(x)\left(\int_0^{(1-l)(x-c)} w(\mathrm{T})d\,\mathrm{T} - \int_0^{(1-l)(x-d)} w(\mathrm{T})d\,\mathrm{T}\right) - g(c)\int_0^{l(c-x)} w(\mathrm{T})\,dT + g(d)\int_0^{l(d-x)} w(\mathrm{T})\,d\,\mathrm{T} - \int_c^d g(\mathrm{T})w(\mathrm{T})d\,\mathrm{T}\right) \times \left(h(x)\left(\int_0^{(1-l)(x-c)} w(\mathrm{T})d\,\mathrm{T} - \int_0^{(1-l)(x-d)} w(\mathrm{T})d\,\mathrm{T}\right) - h(c)\int_0^{l(c-x)} w(\mathrm{T})\,d\,\mathrm{T} + h(d)\int_0^{l(d-x)} w(\mathrm{T})\,d\,\mathrm{T} - \int_c^d h(\mathrm{T})w(\mathrm{T})d\,\mathrm{T}\right) \qquad (2.2)$$

The upcoming result holds for the generalization of weighted Čebyšev-type inequality.

Theorem 2.1. Let the suppositions of Proposition 1.1 be valid, where $x \in [c,d] \; and \; l \in [0,1]$. Then we have for $\frac{1}{r} + \frac{1}{s} = 1 \; (r \geq 1)$

$$\left|T(\Gamma_{l,x}, \Delta_{l,x}, g, h)\right| \leq M_{l,x}^{\frac{2}{s}} \|g'\|_r \|h'\|_r\,, \qquad (2.3)$$

where$T(\Gamma_{l,x}, \Delta_{l,x}, g, h)$ is stated as in (2.2) and

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$$\mathrm{M}_{l,x} = \int_c^x \left| \int_0^{\mathrm{T}-(1-l)c-lx} w(u)du \right|^s d\,\mathrm{T} + \int_x^d \left| \int_0^{\mathrm{T}-(1-l)d-lx} w(u)du \right|^s d\,\mathrm{T}\,. \quad (2.4)$$

Proof. We begin the proof of this theorem by defining the required kernal

$$k(x,\mathrm{T};l) = \begin{cases} \int_0^{\mathrm{T}-(1-l)c-lx} w(u)du\,, & if \ \mathrm{T} \in [c,x], \\ \int_0^{\mathrm{T}-(1-l)d-lx} w(u)du\,, & if \ \mathrm{T} \in [x,d], \end{cases}$$

where $x \in [c,d]$.

Then the following identities are obtained

$$\Gamma_{l,x} - \int_c^d w(\mathrm{T})\, g(\mathrm{T}) d\,\mathrm{T} = \int_c^d k(x,\mathrm{T};l)\, g'(\mathrm{T}) d\,\mathrm{T}, \quad (2.5)$$

$$\Delta_{l,x} - \int_c^d w(\mathrm{T})\, h(\mathrm{T}) d\,\mathrm{T} = \int_c^d k(x,\mathrm{T};l)\, h'(\mathrm{T}) d\,\mathrm{T}. \quad (2.6)$$

By multiplying the L.H.S and R.H.S of (2.5) and (2.6), we get

$$T\left(\Gamma_{l,x}, \Delta_{l,x}, g, h\right) = \left(\int_c^d k(x,\mathrm{T};l)\, g'(\mathrm{T}) d\,\mathrm{T}\right)\left(\int_c^d k(x,\mathrm{T};l)\, h'(\mathrm{T}) d\,\mathrm{T}\right),$$

now by applying absolute value property we obtain

$$\left|T\left(\Gamma_{l,x}, \Delta_{l,x}, g, h\right)\right| \le \left(\int_c^d |k(x,\mathrm{T};l)|\, |g'(\mathrm{T})|\, d\,\mathrm{T}\right)\left(\int_c^d |k(x,\mathrm{T};l)|\, |h'(\mathrm{T})| d\,\mathrm{T}\right). \quad (2.7)$$

Thus, by applying the integral Hölder's inequalityfor $r \ \geq \ 1$ we get

$$\begin{aligned} &\left|T\left(\Gamma_{l,x}, \Delta_{l,x}, g, h\right)\right| \\ &\le \left[\left(\int_c^d |k(x,\mathrm{T};l)|^s\, d\,\mathrm{T}\right)^{\frac{1}{s}} \left(\int_c^d |g'(\mathrm{T})|^r d\,\mathrm{T}\right)^{\frac{1}{r}}\right] \\ &\times \left[\left(\int_c^d |k(x,\mathrm{T};l)|^s\, dt\right)^{\frac{1}{s}} \left(\int_c^d |h'(\mathrm{T})|^r d\,\mathrm{T}\right)^{\frac{1}{r}}\right] \\ &= \left(\int_c^d |k(x,\mathrm{T};l)|^s\, d\,\mathrm{T}\right)^{\frac{2}{s}} \|g'\|_r \|h'\|_r, \end{aligned} \quad (2.8)$$

where the integral of the function$|k(x,\mathrm{T};l)|^s$ is defined as

$$\mathrm{M}_{l,x} = \int_c^d |k(x,\mathrm{T};l)|^s\, d\,\mathrm{T} = \int_c^x \left| \int_0^{\mathrm{T}-(1-l)c-lx} w(u)du \right|^s d\,\mathrm{T} + \int_x^d \left| \int_0^{\mathrm{T}-(1-l)d-lx} w(u)du \right|^s d\,\mathrm{T}. \quad (2.9)$$

Using (2.8) and (2.9), we obtain our desired result (2.3).

Remark 2.2. For $w(t) = \frac{1}{d-c}$ in Theorem 2.1, Theorem 4.3 of [XI] is recaptured.

Remark 2.3. For $w(t) = \frac{1}{d-c}$, $x = \frac{c+d}{2}$, $l = \frac{1}{3}$in (2.3),inequality(1.3) is recaptured.

Remark 2.4. For $w(t) = \frac{1}{d-c}$, $x = \frac{c+d}{2}$ in (2.3), inequality (1.5) is recaptured.

In the following corollaries, we present some special cases of Theorem 2.1.

Corollary 2.5. Let the suppositions of Theorem 2.1 be valid. Then for $s = 1$ we get

$$\left|T(\Gamma_{l,x}, \Delta_{l,x}, g, h)\right| \leq \left[\int_c^d |k(x,\mathrm{T}; l)| \, d\,\mathrm{T}\right]^2 \|g'\|_\infty \|h'\|_\infty.$$

Corollary 2.6. Let the suppositions of Proposition 1.1 be valid, then

$$\left|T(\Gamma_{1,\frac{c+d}{2}}, \Delta_{1,\frac{c+d}{2}}, g, h)\right| \leq M_{1,\frac{c+d}{2}}^{\frac{2}{s}} \|g'\|_r \|h'\|_r , \tag{2.10}$$

where

$$M_{1,\frac{c+d}{2}} = \int_c^d \left|\int_0^{\mathrm{T}-\frac{c+d}{2}} w(u)\, du\right|^s d\,\mathrm{T} \tag{2.11}$$

and

$$\Gamma_{1,\frac{c+d}{2}} = -g(c) \int_0^{\frac{c-d}{2}} w(\mathrm{T}) d\,\mathrm{T} + g(d) \int_0^{\frac{d-c}{2}} w(\mathrm{T}) d\,\mathrm{T},$$

$$\Delta_{1,\frac{c+d}{2}} = -h(c) \int_0^{\frac{c-d}{2}} w(\mathrm{T}) d\,\mathrm{T} + h(d) \int_0^{\frac{d-c}{2}} w(\mathrm{T}) d\,\mathrm{T}. \tag{2.12}$$

Remark 2.7. For $w(t) = \frac{1}{d-c}$ in Corollary 2.6, Corollary 4.1 of [XI] is recaptured.

Now, in the following section we would present applications to probability density functions (PDF) and cumulative distribution functions (CDF).

III. Applications for PDF and CDF

Throughout this section we consider $w : [c, d] \to [0,1]$. Suppose random variable X is continuous with PDF $g : [c, d] \to \mathbb{R}_+$ and the expectationofX is stated as

$$\mathcal{E}_w(X) = \int_c^d \mathrm{T}\; w(\mathrm{T})\; g(\mathrm{T})\; d\,\mathrm{T}\;. \tag{3.1}$$

In the similar way we state cumulative distribution function (CDF) in thefollowingand denoted by G as

$$G(x) = \int_c^x w(\mathrm{T}) g(\mathrm{T})\; d\,\mathrm{T}, \tag{3.2}$$

$\forall\, x \in [c, d]$.

Further that let another random variable Y be continuous with another PDF $h: [c,d] \rightarrow \mathbb{R}_+$ and the expectation of Y is stated as

$$\mathcal{E}_w(Y) = \int_c^d \mathsf{T}\ w(\mathsf{T})h(\mathsf{T})\, d\,\mathsf{T}\ . \tag{3.3}$$

Again in the similar way we state CDF of H as

$$H(y) = \int_c^y w(\mathsf{T})h(\mathsf{T})\ d\,\mathsf{T}, \tag{3.4}$$

$\forall\, y \in [c,d]$. Then

$$\int_c^d G(x)\ dx = d - \mathcal{E}_w(X),$$

$$G(c) = 0,\ G(d) = 1 \tag{3.5}$$

and

$$\int_c^d H(y)\ dy = d - \mathcal{E}_w(Y),$$

$$H(c) = 0,\ H(d) = 1. \tag{3.6}$$

Theorem 3.1. Let X,Y,G and H be stated as above. Further we assume that the probability density function $w : [c,d] \rightarrow [0,1]$ is differentiable. Then we have

$$\left|\left(\int_0^{\frac{c-d}{2}} w(\mathsf{T})d\,\mathsf{T} + \int_c^d \mathsf{T}\ w'(\mathsf{T})\ G(\mathsf{T})\ d\,\mathsf{T} + \mathcal{E}_w(X) - w(d)\right)\left(\int_0^{\frac{c-d}{2}} w(\mathsf{T})d\,\mathsf{T} + \int_c^d \mathsf{T}\ w'(\mathsf{T})\ H(\mathsf{T})\ d\,\mathsf{T} + \mathcal{E}_w(Y) - w(d)\right)\right| \leq M^{\frac{2}{s}}_{1,\frac{c+d}{2}} \|G'\|_r \|H'\|_r.$$

*Proof.*By choosing g = G and h = H in (2.10)−(2.12) and simplifying with the help of (3.1)−(3.6), we get the required inequality. □

Remark3.2. By putting $w(\mathsf{T}) = \frac{1}{d-c}$in Theorem 3.1 we get following result whichis in fact Proposition 4.1 of [XI].

Corollary3.3. Let X,Y,G and H be stated as above. Then

$$\left|\left(\frac{1}{2} + \frac{\mathcal{E}(X) - d}{d - c}\right)\left(\frac{1}{2} + \frac{\mathcal{E}(Y) - d}{d - c}\right)\right| \leq \frac{1}{4}\left(\frac{d - c}{s + 1}\right)^{\frac{2}{s}} \|g\|_r \|h\|_r$$

holds.

Remark 3.4. If we select G = H in Corollary 3.3, then we obtain

$$\left|\frac{1}{2} + \frac{\mathcal{E}(X) - d}{d - c}\right| \leq \frac{1}{2}\left(\frac{d - c}{s + 1}\right)^{\frac{1}{s}} \|h\|_r.$$

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In literature, this is said to be "Trapezoid inequality" for CDF (see [XVI], p. 34 for$g$=H).

Remark 3.5. In similar manner we can state applications in case of L_∞- norm as well.

IV. Conclusion

Inpresent paper, we established generalization of Čebyšev-type inequalities involving weights for absolutely continuous functions whose first derivatives belong to$L_r(c,d)$ for $r \geq 1$ with a couple of corollaries. We also recaptured some results stated in [VIII], [XI] and [XVII] as our special cases.Moreover, we have given an application for expectation of a continuous random variable and probability density function.

References

I. Asif R. Khan, J. E. Pečarić, M. Praljak, Weighted Montgomery's Identities for Higher Order Differentiable Functions of Two Variables, , Rev. Anal. Numer. Theor. Approx., **42** (1) (2013), 49-71

II. Asif R. Khan, J. E. Pečarić, M. Praljak, Generalized Cebysev and Ky Fans Identities and Inequalities, J. Math. Inequal.,**10** (1) (2016), 185-204.

III. Asif R. Khan, J. E. Pečarić, M. Praljak and S. Varošanec, Genral linear Inequalities and Positivity/Higher Order Convexity, Monographs in inequalities, 12, Element, Zagreb, 2017.

IV. Asif. R. Khan and Faraz Mehmood, Double Weighted Integrals Identities of Montgomery for Differentiable Function of Higher Order, Journal of Mathematics and Statistics, **15** (1) (2019), 112-121.

V. Asif R. Khan and FarazMehmood, Generalized Identities and Inequalities of Čebyšev and Ky Fan for ∇ −convex function, Submitted.

VI. B. G. Pachpatte, On Trapezoid and Grüss like integral inequalities, Tamkang J. Math., 34(4) (2003) 365-369.

VII. B. G. Pachpatte, On Ostrowski-Grüss-Čebyšev type inequalities for functions whose modulus of derivatives are convex, J. Inequal. Pure and Appl. Math., 6(4) (2005). Art. 128.

VIII. B. G. Pachpatte, On Čebyšev type inequalities involving functions whose derivatives belong to Lp spaces, J. Inequal. Pure Appl. Math., 7(2) (2006). Art. 58.

IX. D. S. Mitrinović, J. E. Pečarić and A. M. Fink, Classical and new inequalities in analysis, Kluwer Academic Publishers Group, Dordrecht, 1993.

Copyright reserved © J. Mech. Cont.& Math. Sci.

X. FarazMehmood, On Functions WithNondecreasing Increments, (Unpublished doctoral dissertation), Department of Mathematics, University of Karachi, Karachi, Pakistan, 2019.

XI. FizaZafar, Some generalizations of Ostrowski inequalities and their applications to numerical integration and special means, (Unpublished doctoral dissertation). BahauddinZakariya University, Multan, 2010.

XII. H. P. Heining and L. Maligranda, Čebyšev inequality in function spaces, Real Analysis Exchange, 17 (1991-1992) 211-247.

XIII. J. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial orderings, and statistical applications, Mathematics in science and engineering, vol. 187, Academic Press, 1992.

XIV. P. Cerone, OnČebyšev Functional Bounds, Proceedings of the Conference on Differential and Difference Equations and Applications, Hindawi Publishing Corporation, 267-277.

XV. P. L. Čebyšev, Sur les expressions approximatives des integrales par les auters prises entre les memes limites, Proc. Math. Soc. Charkov, 2 (1882) 93-98.

XVI. S. S. Dragomir, Th. M. Rassias (Editors). Ostrowski Type Inequalities and Applications in Numerical Integration. Kluwer Academic Publishers, Dordrecht/Boston/London 2002.

XVII. Zheng Liu, Generalizations of some new Čebyševtype inequalities, J. Inequal. Pure Appl. Math, 8(1) (2007). Art. 13.

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