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α-ideals in a 0-distributive lattice

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Abstract

In this paper we have studied the α-ideals in a 0-distributive lattice. We have described the α-ideals by two definition and proved that these are equivalent. We have given several characterizations. They have proved that a lattice L is disjunctive if and only if each ideal is an α-ideals. We have also included a prime separation theorem for α-ideals. At the end we have studied the α-ideals in a sectionally quasi-complemented lattice.

Keywords: α-ideals, 0-distributive lattice, separation theorem, quasi-complemented lattice.

I. Introduction

α-ideals have been studied by many authors concluding Cornish [V] in case of distributive lattices with 0. In a non-distributive lattice L with 0, if $I(L)$ is pseudo complemented, then concept of α-ideals is possible. Thus, in particular, we can study the α-ideals for a 0-distributive lattice as a lattice L with 0 is 0-distributive if and only if $I(L)$ is pseudo complemented. A lattice L with 0 is called a 0-distributive lattice if for all $a,b,c \in L$, $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. 0-distributive lattices were first studied by [XII]. Then a series of lattice theorists have studied the topic including [II], [III], [IV], [VIII], [IX], [X] and [XI]. In [VII], Jayaram has given a result on Prime Separation Theorem using α-ideals in a 0-distributive lattice. On the other hand, Noor, Ayub and Islam [I] have generalized the result of [V] for distributive near lattices. In this paper, we would like to discuss the α-ideals of a 0-distributive lattice in a very simple way. Then we generalized several results of [V] in a 0-distributive lattice.

For an ideal J in L, we define $\alpha(J) = \{(x]^* : x \in J\}$ is a filter in $A_0(L)$, where $A_0(L)$ is the set of all annihilator ideals of the form $(x]^*; x \in L$, which is a sublattice of lattice of annihilator ideals of L. We also define $\overleftarrow{\alpha}(F) = \{x \in L : (x]^* \in F\}$ where F is any filter in $A_0(L)$. It is easy to check tha $\overleftarrow{\alpha}(F)$ is an ideal in L.

We start with the following result which is due to [V] Problem 3.1.

Proposition 1.1. Let L be a lattice with 0. The following holds:

a) For an ideal J in L, $\alpha(J) = \{(x]^* : x \in J\}$ is a filter in $A_0(L)$,

b) For a filter F in $A_0(L)$, $\overleftarrow{\alpha}(F) = \{x \in L : (x]^* \in F\}$ is an ideal in L,

c) If J_1, J_2 are ideals in L then $J_1 \subseteq J_2$ implies $\alpha(J_1) \subseteq \alpha(J_2)$ and F_1, F_2 are filter in $A_0(L)$ then $F_1 \subseteq F_2$ implies $\overleftarrow{\alpha}(F_1) \subseteq \overleftarrow{\alpha}(F_2)$,

d) The map $l \to \overleftarrow{\alpha}\alpha(l)\{= \overleftarrow{\alpha}(\alpha(l))\}$ is a closure operation on the lattice of ideals of L.

In a 0-distributibve lattice L, an ideal J_0 is called an α-ideal if $\overleftarrow{\alpha}\alpha(J) = J$. Thus α-ideals are simply the closed elements with respect to the closure operation of proposition 1.1. Thus the following result is an immediate consequence of above result.

Proposition 1.2. The α-ideals of a lattice L with 0 form a complete lattice isomorphic to the lattice of filters ordered by set inclusion of the lattice $A_0(L)$ of annulets of L.

The infimum of a set of α-ideals J_i is $\cap J_i$ is their set theoretic intersection. The supremum is $\overleftarrow{\alpha}\alpha(\vee J)$ where $\vee J$ is the supremum in the lattice of ideals of L.

Proposition 1.3. For an ideal J of a distributive lattice L, the following are equivalent:

a) J is an α-ideal

b) For $x, y \in L$, $(x]^* = (y]^*$ and $x \in J \Rightarrow y \in J$

c) $x \in J$ implies $(x]^{**} \subseteq J$.

Proof. (a) $\Leftrightarrow$ (b) Suppose a) holds, so $\overleftarrow{\alpha}\alpha(J) = J$. Let $(x]^* = (y]^*$ and $x \in J$. Then $(y]^* = (x]^* \in \alpha(J)$. So by the definition, $y \in \overleftarrow{\alpha}\alpha(J) = J$. This implies $(x]^* \in \alpha(J)$. Then by definition of $\alpha(J)$, $(x]^* = (p]^*$ for some $p \in J$. Hence by (b) $x \in J$, and so $\overleftarrow{\alpha}\alpha(J) \subseteq J$. Since by Proposition 1.1, reverse inclusion always holds. Thus $\overleftarrow{\alpha}\alpha(J) = J$ and hence J is an α-ideal.

$(b) \Leftrightarrow (c)$ Suppose (b) holds. Let $x \in J$ and $y \in (x]^{**}$. Then $(x]^* \subseteq (y]^*$. Thus, $(y]^* = (x]^* \vee (y]^* = ((x]^{**} \wedge (y]^{**})^* = (x \wedge y]^{***} = (x \wedge y]^*$. Moreover,

$x \wedge y \in J$ as $x \in J$. Hence applying (b), we have $y \in J$. This implies $(x]^* \subseteq J$ and so (c) holds. Finally suppose (c) holds. Let $x \in J$ and $(x]^* = (y]^*$: $y \in L$. By (c), $(x]^{**} \subseteq J$, and so $y \in (y]^{**} = (x]^{**} \subseteq J$ implies $y \in J$, which is (b).

Thus we can define an α - ideal as follows:
An ideal I in a 0-distributive lattice L is called an α-ideal if for each $x \in L$, $x \in I$ implies $(x]^{**} \subseteq I$.

We know that $A^{\perp} = \{x \in L : x \wedge a = 0 \text{ for all } a \in L\}$ is an ideal if L is 0-distributive and if A is an ideal, then $A^{\perp} = A^*$ is the annihilator ideal.

Theorem 1.4. For any ideal J in a 0-distributive lattice L, the set $I^e = \{x \in L : (a]^* \subseteq (x]^* \text{ for some } a \in J\}$ is the smallest α-ideal containing J and ideal I in L is an α-ideal if and only if $I = I^e$.

Proof. Let $x \in I^e$. Then $(a]^* \subseteq (x]^*$ for some $a \in I$ and so $(x]^{**} \subseteq (a]^{**}$. Suppose $y \in (a]^{**}$. Thus $(y] \subseteq (a]^{**}$ and so $(a]^* \subseteq (y]^*$. This implies $x \in I^e$. Therefore, $(a]^{**} \subseteq I^e$ and so $(x]^{**} \subseteq I^e$. It follows that I^e is an α-ideal. Now suppose $x \in I$, then by definition, $x \in I^e$ and so $I \subseteq I^e$. Suppose K is an α-ideal containing I. Let $x \in I^e$. Then $(a]^* \subseteq (x]^*$ for some $a \in I \subseteq K$. This implies $(x]^{**} \subseteq (a]^{**} \subseteq K$ as K is an α-ideal. Thus $(x] \subseteq K$ and $x \in K$. Hence $I^e \subseteq K$. That is I^e is the smallest α-ideal containing I.

Theorem 1.5. Every annihilator ideal in a 0-distributive lattice L is an α-ideal.

Proof. Let $I = A^*$ be the annihilator ideal of L. Suppose $y \in I = A^*$ Then $y \wedge a = 0$ for all $a \in A$. Then $(y] \wedge (a] = (0]$ and so $(y] \subseteq (a]^*$. Thus $(y]^{**} \subseteq (a]^{***} = (a]^*$ for all $a \in A$. Hence, $(y]^{**} \subseteq \bigcap_{a \in A} (a]^* = A^* = I$ and so I is an α-ideal.

Theorem 1.6. Let L be a 0-distributive nearlattice. A be a meet semilattice of L. Then A^0 is an α-ideal, where $A^0 = \{x \in L : x \wedge a = 0 \text{ for some } a \in A\}$.

Proof. By [6; Theorem 4.2.8], A^0 is an ideal. Now let $x \in A^0$ and $y \in (x]^{**}$. Clearly $x \in A^0$ implies $x \wedge a = 0$ for some $a \in A$. This shows that $y \in A^0$, consequently $(x]^{**} \subseteq A^e$. Hence A^0 is an α-ideal of L.

Theorem 1.7. If a prime ideal P of a 0-distrbutive lattice L is non dense then P is an α-ideal.

Proof. By assumption $P^* \neq (0]$. Hence there exists $x \in P^*$ such that $x \neq 0$. But then $(x]^* \supseteq P^{**}$ gives $(x]^* \supseteq P$ as $P \subseteq P^{**}$. Further if $t \in (x]^*$, then $x \wedge t = 0 \in P$. But as P is a prime ideal, so $t \in P$ (since $P \cap P^* = (0] \Rightarrow x \notin P$). This implies $(x]^* \subseteq P$. Combining both the inclusions, we get $P = (x]^*$. Hence P is an annihilator ideal and so by Theorem 1.5, P is an α-ideal .

Corollary 1.8. Every non-dense prime ideal of a 0-distributive lattice is an annulet.

Theorem 1.9. For an α-ideal I of a 0-distributive lattice L, $I = \{y \in L : (y] \subseteq (x]^{**} \ \mathit{for\ some}\ x \in I\}$.

Proof. Let $a \in I$. Then -ideal lattice, J. C. Varlet [12] introduced the notion of 0-distributive lattices. Then $(a] \subseteq (a]^{**}$ implies that $a \in \{y \in L : (y] \subseteq (x]^{**} \ \mathit{for\ some}\ x \in I\}$.
Conversely, let $a \in \{y \in L : (y] \subseteq (x]^{**} \ \mathit{for\ some}\ x \in I\}$. Then $(a] \subseteq (x]^{**}$ for some $x \in I$. Since I is an α-ideal, so $(x]^{**} \subseteq I$ and so $(a] \subseteq I$. Hence $a \in I$.

Now we include a prime Separation Theorem for α-ideals in a 0-distributive lattice. This result has also been proved in [1]. But we claim that our proof is much better and easier.

Theorem 1.10. Let F be a filter and I be an α-ideal in a 0-distributive lattice L such that $I \cap F = \phi$. Then there exists a prime α-ideal $P \supseteq I$ such that $P \cap F = \phi$.

Proof. Let χ be the collection of all filters containing F and disjoint from I. χ is non-empty as $F \in \chi$. Then by [6; Lemma 3.3.3] there exists a maximal filter Q containing F and disjoint from I. Suppose Q is not prime. Then there exist $f, g \notin Q$ such that $f \vee g$ exists and $f \vee g \in Q$. Then by [6; Lemma 3.3.4], there exist $a \in Q$, $b \in Q$ such that $a \wedge f \in I$ and $b \wedge g \in I$. Thus we have $a \wedge b \wedge f \in I$ and $a \wedge b \wedge g \in I$. Then $(a \wedge b \wedge f] \subseteq (x]^{**}$ and $(a \wedge b \wedge g] \subseteq (y]^{**}$ for some $x, y \in I$. Thus we have $(a \wedge b \wedge f] \wedge (x]^* = (0] = (a \wedge b \wedge g] \wedge (y]^*$. That is $(a \wedge b] \wedge (x]^*$ $(a \wedge b] \wedge (x]^* \wedge (y]^* \wedge (g]$. Since $I(L)$ is 0-distributive, it follows that
$(a \wedge b] \wedge (x]^* \wedge (y]^* \wedge ((f] \vee (g]) = (0]$.

That is $(a \wedge b] \wedge (x \vee y]^* \wedge (f \vee g] = (0]$, $x \vee y \in I$ as $x, y \in I$. Therefore, $(a \wedge b] \wedge (f \vee g] \subseteq (x \vee y]^{**}$, which implies that $a \wedge b \wedge (f \vee g] \in I$. But $a \in Q$, $b \in Q$, $f \vee g \in Q$ imply $a \wedge b \wedge (f \vee g] \in Q$ which is contradiction to $Q \cap I = \phi$. Therefore, Q must be prime. Thus $P = L - Q$ is a prime ideal containing I such that $P \cap F = \phi$.

Let $x \in P$. If $x \in I$, then $(x]^{**} \subseteq I \subseteq P$. Again if $x \in P - I$, then by maximality of Q, there exist $a \in Q$ such that $a \wedge x \in I$. Thus, $(a]^{**} \wedge (x]^{**} \subseteq I \subseteq P$. Since $(a]^{**} \not\subset P$, so $(x]^{**} \subseteq P$, as P is a prime. Therefore P is an α-ideal.

For an α-ideal I, $\overleftarrow{\alpha}\alpha(I) = I$. Also, it is clear that for any filter F of $A_0(L)$, $\overleftarrow{\alpha}\alpha(F) = F$. Moreover, by proposition 1.1, both α and $\overleftarrow{\alpha}$ are isotone. Hence the lattice of α-ideals of L is isomorphic to the lattice of filters.

Corollary 1.11. Let L be a 0-distributive lattice. Then the set of prime α-ideals of L are isomorphic to the set of prime filters of $A_0(L)$.

A 0-distributive lattice L is called disjunctive if for there is an element $x \in L$ such that $a \wedge x = 0$ where $0 \leq a < b$. It is easy to cheek that is L is disjunctive if and only $(a]^* = (b]^*$ implies $a = b$ for any $a, b \in L$.

Proposition. 1.12. In a 0-distributive lattice L the following conditions are equivalent:

(i) each ideal is an α-ideal.
(ii) each prime ideal is an α-ideal.
(iii) L is disjunctive.

Proof. (i) $\Rightarrow$ (ii); Suppose P is any prime ideal of L then by (i) P is an α-ideal, that is $\overleftarrow{\alpha}\alpha(P) = P$. Let I be any ideal of L then we have $I = \cap(P : P \supseteq I)$ implies $\overleftarrow{\alpha}\alpha(I) = \overleftarrow{\alpha}\alpha(\cap(P : P \supseteq I)) = \cap(\overleftarrow{\alpha}\alpha(P) : P \supseteq I) = \cap(P : P \supseteq I)$ implies that $\overleftarrow{\alpha}\alpha(I) = I$. So I is an α-ideal.

(ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (iii); For any $x, y \in L$, $(x]^* = (y]^*$. Since $(x]$ is an α-ideal, so by definition of α-ideal, $y \in (x]$. Therefore, $y \leq x$. Similarly, $x \leq y$. Hence L is disjunctive.

(iii) $\Rightarrow$ (i); Suppose I is any ideal of L. By proposition 1.1, $(x]^* \subseteq \overleftarrow{\alpha}\alpha(I)$. For the reverse inclusion, let $x \in \overleftarrow{\alpha}\alpha(I)$. Then by definition $(x]^* \in \alpha(I)$, and so $(x]^* = (y]^*$ for some $y \in (x]^*$. This implies $x = y$, as L is disjunctive. So $x \in L$ and hence $\overleftarrow{\alpha}\alpha(I) = I$. Therefore I is an α-ideal of L.

Lemma 1.13. A 0-distributive lattice L is relatively complemented if and only every prime filter is an ultra filter (Proper and maximal).

Proof. By Theorem 2.11 in [XII] we have L is relatively complemented if and only if its prime ideals are unordered. Thus the result follows.

A lattice L with 0 is called a quasi-complemented lattice if for each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $((x] \vee (y])^* = (x]^* \cap (y]^* = (0]$.

A 0-distributive lattice L is called quasi-complemented if for each $x \in L$, there exists $x' \in L$ such that $x \wedge x' = 0$ and $((x] \vee (x'])^* = (0]$.

A lattice L with 0 is called sectionally quasi-complemented if each interval $[0, x]$, $x \in L$ is quasi-complemented.

We conclude the paper with the following result.

Theorem 1.14. Let L be a 0-distributive lattice. Then the following conditions are equivalent:

(i) L is sectionally quasi-complemented.
(ii) each prime α-ideal is a maximal prime ideal.
(iii) each α-ideal is an intersection of minimal prime ideals.

Moreover, the above conditions are equivalent to L being quasi-complemented if and only if there is an element $d \in L$ such that $(d]^* = (0]$.

Proof. (i) $\Rightarrow$ (ii); Suppose L is a sectionally quasi-complemented. Then by [6; Theorem 4.3.7], $A_0(L)$ is relatively complemented. Hence its every prime filter is an ultra filter. Then by Corollary 1.11, each prime α-ideal is a minimal prime ideal.

(ii) $\Rightarrow$ (iii); It is not hard to show that each ideal of L is an intersection of prime α-ideals. This shows (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii); This is obvious by the minimality property of prime α-ideals.

(ii) $\Rightarrow$ (i); Suppose (ii) holds. Then by Corollary 1.11, each prime filter of $A_0(L)$ is maximal. Then by Lemma 1.13, $A_0(L)$ is relatively complemented and so by Proposition 2.7 in [5] L is sectionally quasi-complemented.

Conclusion. This paper shows that α -ideals can be studied in non-distributive lattices by the 0-distributive property of a lattice. Following the technique of this paper, one can generalize those results for a 0-distributive near lattice.

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