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Solution of Linear System of the First Order Delay Differential Inequalities

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Abstract

In this paper, we will present the existence of necessary and sufficient conditions for linear systems of the first order delay deferential inequalities and equations to have oscillatory, eventually negative solutions and has ultimately positive solutions. Also, some illustrative examples of each case are given.

Keywords: Delay, Differential, System, Eventually, Positive, Negative, Oscillatory, Equation, Inequality, Bounded, Solution.

I. Introduction

Systems of DDEs now occupy a place of central importance in all areas of science and particularly in the biological sciences (e.g., population dynamics and epidemiology). Interest in such systems often arises when more realistic distributed assumptions replace traditional pointwise modelling assumptions, for example, when the birth rate of predators is affected by prior levels of predators or prey rather than by only the current levels in a predator-prey model. The manner in which the properties of systems of delay differential equations differ from those of systems of ordinary differential equations has been and remains an active area of research; and for typical examples of such studies for a description of several common models.In (1978), KITAMURA and KUSANO, studied the two-dimensional differential system with deviating argument. In (1987), Gopalsamy, found sufficient conditions that achieves particular oscillation, eventually negative and eventually positive of system DDES and system delay differential inequalities. In (2000), Cahlon and Schmidt, have given necessary and sufficient conditions for the asymptotic stability of the zero solution of the system of linear with two constant delay differential equations.

In this paper, we found sufficient conditions, in general, satisfies oscillation, eventually negative and eventually positive of system DDES and system delay differential inequalities. Also take examples of matrices.

II. Existence of the Solution of Linear System of the First Order Delay Differential Inequalities

Consider the following system first order delay differential inequalities:

$$\frac{dy_i(t)}{dt} + \sum_{j=1}^{n} a_{ij}\, y_j(t) + \sum_{j=1}^{n} p_{ij}\, y_j\left(t-\tau_{ij}\right) \leq 0; i = 1,2,\dots,n \qquad \dots(a)$$

$$\frac{dy_i(t)}{dt} + \sum_{j=1}^{n} a_{ij}\, y_j(t) + \sum_{j=1}^{n} p_{ij}\, y_j\left(t-\tau_{ij}\right) \geq 0; i = 1,2,\dots,n \qquad \dots(b)$$

and the DDE:

$$\frac{dy_i(t)}{dt} + \sum_{j=1}^{n} a_{ij}\, y_j(t) + \sum_{j=1}^{n} p_{ij}\, y_j\left(t-\tau_{ij}\right) = 0; i = 1,2,\dots,n \qquad \dots(c)$$

wherea_{ij} , p_{ij} and τ_{ij}arerealconstants.

Wegive sufficient conditions under which:

(a) Inequality has eventually negative solutions.

(b) Inequality has eventually positive solutions.

(c) Equation has oscillatory solutions only.

Remark (1):

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \vdots \\ \frac{dy_n}{dt} \end{bmatrix} + \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} +$$

$$\begin{bmatrix} p_{11} & \dots & p_{1n} \\ p_{21} & \dots & p_{2n} \\ \vdots & & \vdots \\ p_{n1} & \dots & p_{nn} \end{bmatrix} \begin{bmatrix} y_1(t-\tau_{11}) & \dots & y_n(t-\tau_{1n}) \\ y_1(t-\tau_{21}) & \dots & y_n(t-\tau_{2n}) \\ \vdots & & \vdots \\ y_n(t-\tau_{n1}) & \dots & y_n(t-\tau_{nn}) \end{bmatrix} \leq 0 \qquad \dots \qquad (1)$$

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Remark (2):

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \vdots \\ \frac{dy_n}{dt} \end{bmatrix} + \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} p_{11} & \dots & p_{1n} \\ p_{21} & \dots & p_{2n} \\ \vdots & & \vdots \\ p_{n1} & \dots & p_{nn} \end{bmatrix} \begin{bmatrix} y_1(t-\tau_{11}) & \dots & y_n(t-\tau_{1n}) \\ y_1(t-\tau_{21}) & \dots & y_n(t-\tau_{2n}) \\ \vdots & & \vdots \\ y_n(t-\tau_{n1}) & \dots & y_n(t-\tau_{nn}) \end{bmatrix} \geq 0 \qquad (2)$$

Remark (3):

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \vdots \\ \frac{dy_n}{dt} \end{bmatrix} + \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} p_{11} & \dots & p_{1n} \\ p_{21} & \dots & p_{2n} \\ \vdots & & \vdots \\ p_{n1} & \dots & p_{nn} \end{bmatrix} \begin{bmatrix} y_1(t-\tau_{11}) & \dots & y_n(t-\tau_{1n}) \\ y_1(t-\tau_{21}) & \dots & y_n(t-\tau_{2n}) \\ \vdots & & \vdots \\ y_n(t-\tau_{n1}) & \dots & y_n(t-\tau_{nn}) \end{bmatrix} = 0 \qquad (3)$$

III Main Theorems and Results

If the assumptions

a_{ij}, p_{ij} and τ_{ij};$i,\ j = 1,2,\dots,n$ are real constants (4)

such that:

(i) a_{ii}, p_{ii} and τ_{ii} are positive constants; $i = 1,2,\dots,n$.

(ii) $\tau_{ij} \geq 0;\ i,j = 1,2,\dots,n$.

$$e\left[\min_{1\leq i\leq n}\{\tau_{ii}\}\right]\left[\min_{1\leq i\leq n}\left(p_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}|p_{ji}|\right)\right] > e^{-\left[\min_{1\leq i\leq n}\{\tau_{ii}\}\right]\left[\min_{1\leq i\leq n}\left(a_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ji}|\right)\right]} \qquad (5)$$

are hold. Then, the system (1) has eventually negative boundedsolutionson$[0,\infty)$.

We will prove that the existence of an eventually positive solution leadsto a contradiction with condition. Let us then suppose this systemhas eventually positive bounded solution $u(t) = \{u_1(t), u_2(t), \ldots, u_n(t)\}$on $[0,\infty)$. There exists $t_1 > 0$, such that:

$$u_i(t) > 0, \text{ for } t \geq t^*; i = 1,2,\ldots,n \tag{6}$$

and

$$\frac{du_i(t)}{dt} \leq -a_{ii}u_i(t) - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}\, u_j(t) - p_{ii}u_i(t-\tau_{ii}) - \sum_{\substack{j=1 \\ j\neq i}}^{n} p_{ij}\, u_j(t-\tau_{ij}) \tag{7}$$

It follows from the bounded and eventually positive solutions$u_1, u_1 \ldots, u_n$ converges as t ⟶∞. We let:

$$\lim_{t\to\infty} u_i(t) = c_i \geq 0, \; i = 1,2,\ldots,n \tag{8}$$

We claim that $c_i = 0, i = 1,2,\ldots,n$; and if not, then the non-oscillation of$u_1, u_1 \ldots, u_n$ and the eventually positive of $u_1, u_1, \ldots, u_n$shows that the convergences in inequality (6) is monotonic in t eventually and hence there exists $t_2 > t_1 + \tau$ $(\tau = \max(\tau_{ij};\ i, j = 1,2,\ldots,n))$, such that:

$$\left.\begin{aligned} u_i(t) < c_i + \varepsilon, \text{ for } t > t_2 \\ u_i(t) > c_i - \varepsilon,\; i = 1,2,\ldots,n \end{aligned}\right\} \tag{9}$$

Whereεis any arbitrary positive number. We have from inequality:

$$\frac{d}{dt}\left(\sum_{i=1}^{n} u_i(t)\right) \leq -\sum_{i=1}^{n} a_{ii}u_i(t) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\, u_j(t) - \sum_{i=1}^{n} p_{ii}u_i(t-\tau_{ii}) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} |p_{ij}|\, u_j(t-\tau_{ij}),\, t > t_2 + \tau \tag{10}$$

$$\leq -\sum_{i=1}^{n}\left\{a_{ii}(c_i - \varepsilon) - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\,(c_i + \varepsilon)\right\} - \sum_{i=1}^{n}\left\{p_{ii}(c_i - \varepsilon) - \right.$$

j=1j≠inpij(ci+ε) (11)

$$\leq -\sum_{i=1}^{n}\left(a_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ji}|\right)c_i + \varepsilon\left[\sum_{\substack{j=1\\ j\neq i}}^{n}\left(a_{ii} + \sum_{j=1}^{n}|a_{ji}|\right)\right]$$

$$-\sum_{i=1}^{n}\left(p_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}|p_{ji}|\right)c_i + \varepsilon\left[\sum_{\substack{j=1\\ j\neq i}}^{n}\left(p_{ii} + \sum_{j=1}^{n}|p_{ji}|\right)\right] -$$

$$q\sum_{i=1}^{n} c_i + Q\varepsilon \qquad (12)$$

$$\leq -m\sum_{i=1}^{n} c_i + M\varepsilon$$

where:

$$m = \min_{1\leq i\leq n}\left(a_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ji}|\right), M = \sum_{i=1}^{n} a_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ji}| \qquad (13)$$

and

$$q = \min_{1\leq i\leq n}\left(p_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}|p_{ji}|\right), Q = \sum_{i=1}^{n} p_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n}|p_{ji}| \qquad (14)$$

The assumption (5) implies that $m, q > 0$. Now if $\sum_{i=1}^{n} c_i > 0$, then choosing ε small enough then one can show that there exist positive numbers m, q, such that:

$$\frac{d}{dt}\left(\sum_{i=1}^{n} u_i(t)\right) \leq -m - q, \text{ for } t > t_2 + \tau \qquad (15)$$

This leads to:

$$\frac{d}{dt}\left(\sum_{i=1}^{n} u_i(t)\right) \leq -m(t - t_2) + \sum_{i=1}^{n} u_i(t_2) - q(t - t_2 - \tau) + \sum_{i=1}^{n} u_i(t_2 - \tau),$$
$$\text{for } t > t_2 + \tau \qquad (16)$$

Implying that $\sum_{i=1}^{n} u_i(t)$may become negative for large enough t; but this is impossible.Thus, we have $\sum_{i=1}^{n} c_i = 0$ and hence$c_i = 0, i = 1,2,\dots,n$.Hence:

$$\lim_{n\to\infty} u_i(t) = 0, i = 1,2,\dots,n \qquad (17)$$

and the convergence in (17) is monotonic in t eventually due to the nonoscillatory nature of $u(t) = \{u_1(t), u_2(t), \dots, u_n(t)\}$. It follows from (10) that:

$$\frac{d}{dt}\left(\sum_{i=1}^{n} u_i(t)\right) \le -\sum_{i=1}^{n} a_{ii}\, u_i(t) - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji}|\, u_i(t) - \sum_{i=1}^{n} p_{ii}\, u_i(t-\tau_{ii}) - \sum_{\substack{j=1 \\ j\neq i}}^{n} |p_{ji}|\, u_i(t-\tau_{ji}) \tag{18}$$

whichis upon using:

$$t_{ii} \ge \tau_{ji} \to t-\tau_{ii} \le t-\tau_{ji} \to u_i(t-\tau_{ii}) \ge u_i(t-\tau_{ji}), i, j = 1,2, \dots, n \tag{19}$$

leads to:

$$\frac{d}{dt}\left(\sum_{i=1}^{n} u_i(t)\right) \le -\sum_{i=1}^{n}\left(a_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji}|\right) u_i(t) - \sum_{i=1}^{n}\left(p_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |p_{ji}|\right) u_i(t - \tau ii, t > t2+\tau \tag{20}$$

$$\le -m\sum_{i=1}^{n} u_i(t) - q\sum_{i=1}^{n} u_i(t-\tau_{ii}), t > t_2+\tau \tag{21}$$

$$\le -m\sum_{i=1}^{n} u_i(t) - q\sum_{i=1}^{n} u_i(t-\sigma) \tag{22}$$

where:

$$m = \min_{1\le i\le n}\left(a_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji}|\right), q = \min_{1\le i\le n}\left(p_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |p_{ji}|\right), \sigma = \min\{\tau_{11}, \tau_{22}, \dots, \tau_{nn}\} \tag{23}$$

Note that $\sigma > 0$ is due to (5) and we have used the eventual monotonic convergence of $u_i(t-\tau_{ii})$ to zero as $t \to \infty (i = 1,2, \dots, n)$ in the derivation of (22) from (21).

Now, if we let:

$$y(t) = \sum_{i=1}^{n} u_i(t); \text{ fort} > t_2 + \tau$$

Then, we have from (22):

$$\frac{dy(t)}{dt} \le -my(t) - qy(t-\sigma); \text{fort} > t_2 + \tau \tag{24}$$

andy is any positive solution of the scalar delay differential inequality (24) in which the constantsm, q and σ satisfy:

$$eq\sigma > e^{-m\sigma} \tag{25}$$

as a consequence of (5) and inequality(23).

It is well known that the scalar delay differential inequality (24) cannot have the eventually positive solution (Ladas and Stavroulakis[9]) when (25) holds.This contradiction shows that system(1)has ultimately negative bounded solutionsand this completes the proof.

Assume that (4) and (5) hold. Then the system of inequalities (2) has eventually positive bounded solutions. The conclusion follows from the result of the above theorem (3.1) since an eventually negative bounded solution of system (2) is an eventually positive bounded solution of system (1).

Assume that (4) and (5) hold. Then all bounded solutions of system (3) are oscillatory. The assertion is a consequence of the fact that system (3) cannot havenon-oscillatory bounded solutions, which are eventually positive or eventually negative.

IV. Applications

Here, we introduce some applied examples to investigate sufficient conditions that satisfies oscillation, eventually negative and eventually positive of system DDES and system delay differential inequalities.

Example(1):

Consider the system:

$$\frac{dy_1(t)}{dt} + 7y_1(t) + 9y_1(t-1) - 2y_2(t) - y_2(t-2) - y_3(t) + 2y_3(t-3) \leq 0$$

$$\frac{dy_2(t)}{dt} + 2y_1(t) - 4y_1(t-4) + 9y_2(t) + 6y_2(t-6) + 3y_3(t) - 3y_3(t-1) \leq 0$$

$$\frac{dy_3(t)}{dt} - 3y_1(t) + 4y_1(t-2) + 4y_2(t) - 2y_2(t-7) + 6y_3(t) + 7y_3(t-8) \leq 0$$

where the system have positive solution:

$$y_1(t) = e^{-2t}, y_2(t) = e^{-t} \text{and} y_3(t) = e^{-3t}$$

$$a = \begin{bmatrix} 7 & -2 & -1 \\ 2 & 9 & 3 \\ -3 & 4 & 6 \end{bmatrix}, p = \begin{bmatrix} 9 & -1 & 2 \\ -4 & 6 & -3 \\ 4 & -2 & 7 \end{bmatrix}, \tau = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 1 \\ 2 & 7 & 8 \end{bmatrix}$$

$$m = \min_{1\le i\le n}\left(a_{ii} - \sum_{\substack{j=1 \\ j\ne i}}^{n} |a_{ji}| \right)$$

$$= \min_{1\le i\le 3}\big(a_{11} - (|a_{21}| + |a_{31}|, a_{22} - (|a_{12}| + |a_{32}|), a_{33} - (|a_{13}| + |a_{23}|)\big)$$

$$= \min_{1\le i\le 3}(7-(2+3), 9-(2+4), 6-(1+3))$$

$$= \min_{1\le i\le 3}(2,3,2) = 2$$

$$q = \min_{1\le i\le n}\left(p_{ii} - \sum_{\substack{j=1 \\ j\ne i}}^{n} |p_{ji}| \right)$$

$$= \min_{1\le i\le 3}\big(p_{11} - (|p_{21}| + |p_{31}|, p_{22} - (|p_{12}| + |p_{32}|), p_{33} - (|p_{13}| + |p_{23}|)\big)$$

$$= \min_{1\le i\le 3}(9-(4+4), 6-(1+2), 7-(2+3))$$

$$= \min_{1\le i\le 3}(1,3,2) = 1$$

$$\sigma = \min\{\tau_{11}, \tau_{22}, \tau_{33}\}$$

$$= \min\{1,6,8\} = 1$$

$$eq\sigma > e^{-m\sigma}$$

$$e > e^{-2}$$

2.718>0.135

and hence the condition is hold.

Example(2):

Consider the system:

$$\frac{dy_1(t)}{dt} + 7y_1(t) + 9y_1(t-1) - 2y_2(t) - y_2(t-2) - y_3(t) + 2y_3(t-3) \ge 0$$

$$\frac{dy_2(t)}{dt} + 2y_1(t) - 4y_1(t-4) + 9y_2(t) + 6y_2(t-6) + 3y_3(t) - 3y_3(t-1) \geq 0$$

$$\frac{dy_3(t)}{dt} - 3y_1(t) + 4y_1(t-2) + 4y_2(t) - 2y_2(t-7) + 6y_3(t) + 7y_3(t-8) \geq 0$$

where the system have negative solution:

$y_1(t) = -e^{-2t}$, $y_2(t) = -\,e^{-t}$ and$y_3(t) = -e^{-3t}$

$$a = \begin{bmatrix} 7 & -2 & -1 \\ 2 & 9 & 3 \\ -3 & 4 & 6 \end{bmatrix}, p = \begin{bmatrix} 9 & -1 & 2 \\ -4 & 6 & -3 \\ 4 & -2 & 7 \end{bmatrix}, \tau = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 1 \\ 2 & 7 & 8 \end{bmatrix}$$

$$m = \min_{1\leq i\leq n}\left(a_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji}| \right)$$

$$= \min_{1\leq i\leq 3}\big(a_{11} - (|a_{21}| + |a_{31}|, a_{22} - (|a_{12}| + |a_{32}|), a_{33} - (|a_{13}| + |a_{23}|)\big)$$

$$= \min_{1\leq i\leq 3}(7 - (2+3), 9 - (2+4), 6 - (1+3))$$

$$= \min_{1\leq i\leq 3}(2,3,2) = 2$$

$$q = \min_{1\leq i\leq n}\left(p_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |p_{ji}| \right)$$

$$= \min_{1\leq i\leq 3}\big(p_{11} - (|p_{21}| + |p_{31}|, p_{22} - (|p_{12}| + |p_{32}|), p_{33} - (|p_{13}| + |p_{23}|)\big)$$

$$= \min_{1\leq i\leq 3}(9 - (4+4), 6 - (1+2), 7 - (2+3))$$

$$= \min_{1\leq i\leq 3}(1,3,2) = 1$$

$$\sigma = \min\{\tau_{11}, \tau_{22}, \tau_{33}\}$$

$$= \min\{1,6,8\} = 1$$

$$eq\sigma > e^{-m\sigma}$$

$$e > e^{-2}$$

2.718>0.135

and hence the condition is hold.

Example(3):

Consider the system:

$$\frac{dy_1(t)}{dt} + 5y_1(t) + 4y_1(t-2) - y_2(t) + y_2(t-1) - 2y_3(t) - 2y_3(t-4) \leq 0$$

$$\frac{dy_2(t)}{dt} + y_1(t) - 3y_1(t-3) + 4y_2(t) + 6y_2(t-6) + 6y_3(t) - y_3(t-6) \leq 0$$

$$\frac{dy_3(t)}{dt} + 7y_1(t) + 6y_1(t-7) + 2y_2(t) - 4y_2(t-1) + 3y_3(t) + 9y_3(t-10) \leq 0$$

where the system have negative solution:

$y_1(t) = -e^{-5t}$, $y_2(t) = -e^{-2t}$and $y_3(t) = -e^{-t}$

$$a = \begin{bmatrix} 5 & -1 & -2 \\ 1 & 4 & 6 \\ 7 & 2 & 3 \end{bmatrix}, p = \begin{bmatrix} 4 & 1 & -2 \\ -3 & 6 & -1 \\ 6 & -4 & 9 \end{bmatrix}, \tau = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 5 & 6 \\ 7 & 1 & 10 \end{bmatrix}$$

$$m = \min_{1\leq i\leq n} \left(a_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji}| \right)$$

$$= \min_{1\leq i\leq 3}\big(a_{11} - (|a_{21}| + |a_{31}|, a_{22} - (|a_{12}| + |a_{32}|), a_{33} - (|a_{13}| + |a_{23}|)\big)$$

$$= \min_{1\leq i\leq 3}(5 - (1+7), 4 - (1+2), 3 - (2+6))$$

$$= \min_{1\leq i\leq 3}(-3,1,-5) = -5$$

$$q = \min_{1 \le i \le n} \left(p_{ii} - \sum_{\substack{j=1 \\ j \ne i}}^{n} |p_{ji}| \right)$$

$$= \min_{1 \le i \le 3} \big(p_{11} - (|p_{21}| + |p_{31}|, p_{22} - (|p_{12}| + |p_{32}|), p_{33} - (|p_{13}| + |p_{23}|)\big)$$

$$= \min_{1 \le i \le 3} (4 - (3 + 6), 6 - (1 + 4), 9 - (2 + 1))$$

$$= \min_{1 \le i \le 3} (-5,1,6) = -5$$

$$\sigma = \min\{\tau_{11}, \tau_{22}, \tau_{33}\}$$

$$= \min\{2,5,9\} = 2$$

$$eq\sigma > e^{-m\sigma}$$

$$e(-5)(2) > e^{-(-5)(2)}$$

$$-27.1828 > 22026.46$$

and hence the condition is hold.

Example(4):

Consider the system:

$$\frac{dy_1(t)}{dt} + 5y_1(t) + 4y_1(t-2) - y_2(t) + y_2(t-1) - 2y_3(t) - 2y_3(t-4) \ge 0$$

$$\frac{dy_2(t)}{dt} + y_1(t) - 3y_1(t-3) + 4y_2(t) + 6y_2(t-6) + 6y_3(t) - y_3(t-6) \ge 0$$

$$\frac{dy_3(t)}{dt} + 7y_1(t) + 6y_1(t-7) + 2y_2(t) - 4y_2(t-1) + 3y_3(t) + 9y_3(t-10) \ge 0$$

where the system have positive solution:

$y_1(t) = e^{-5t}$, $y_2(t) = e^{-2t}$ and$y_3(t) = e^{-t}$

$$a = \begin{bmatrix} 5 & -1 & -2 \\ 1 & 4 & 6 \\ 7 & 2 & 3 \end{bmatrix}, p = \begin{bmatrix} 4 & 1 & -2 \\ -3 & 6 & -1 \\ 6 & -4 & 9 \end{bmatrix}, \tau = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 5 & 6 \\ 7 & 1 & 10 \end{bmatrix}$$

$$m = \min_{1 \le i \le n} \left(a_{ii} - \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ji}| \right)$$

$$= \min_{1 \le i \le 3} \big(a_{11} - (|a_{21}| + |a_{31}|, a_{22} - (|a_{12}| + |a_{32}|), a_{33} - (|a_{13}| + |a_{23}|)\big)$$

$$= \min_{1 \le i \le 3} (5 - (1 + 7), 4 - (1 + 2), 3 - (2 + 6))$$

$$= \min_{1 \le i \le 3} (-3, 1, -5) = -5$$

$$q = \min_{1 \le i \le n} \left(p_{ii} - \sum_{\substack{j=1 \\ j \ne i}}^{n} |p_{ji}| \right)$$

$$= \min_{1 \le i \le 3} \big(p_{11} - (|p_{21}| + |p_{31}|, p_{22} - (|p_{12}| + |p_{32}|), p_{33} - (|p_{13}| + |p_{23}|)\big)$$

$$= \min_{1 \le i \le 3} (4 - (3 + 6), 6 - (1 + 4), 9 - (2 + 1))$$

$$= \min_{1 \le i \le 3} (-5, 1, 6) = -5$$

$$\sigma = \min\{\tau_{11}, \tau_{22}, \tau_{33}\}$$

$$= \min\{2, 5, 9\} = 2$$

$$eq\sigma > e^{-m\sigma}$$

$$e(-5)(2) > e^{-(-5)(2)}$$

$$-27.1828 > 22026.46$$

Therefore, the condition is not hold.

<u>Example(5):</u>

Consider the system

$$\frac{dy_1(t)}{dt} + 2y_1\left(t - \frac{\pi}{2}\right) + y_2(t - 2\pi) = 0$$

$$\frac{dy_2(t)}{dt} + y_1(t - \pi) + 2y_2\left(t - \frac{\pi}{2}\right) = 0$$

where the system have oscillatory solution:

$$y_1(t) = \sin(t) \text{and} y_2(t) = \cos(t)$$

$$a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, p = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tau = \begin{bmatrix} \frac{\pi}{2} & 2\pi \\ \pi & \frac{\pi}{2} \end{bmatrix}$$

$$m = \min_{1 \le i \le n} \left(a_{ii} - \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ji}| \right)$$

$$= \min_{1 \le i \le 3} (a_{11} - |a_{21}|, a_{22} - |a_{12}|)$$

$$= \min_{1 \le i \le 3} (0,0,0) = 0$$

$$q = \min_{1 \le i \le n} \left(p_{ii} - \sum_{\substack{j=1 \\ j \ne i}}^{n} |p_{ji}| \right)$$

$$= \min_{1 \le i \le 3} (p_{11} - |p_{21}|, p_{22} - |p_{12}|)$$

$$= \min_{1 \le i \le 3} (2 - (1), 2 - (1))$$

$$= \min_{1 \le i \le 3} (1,1) = 1$$

$$\sigma = \min\{\tau_{11}, \tau_{22}\}$$

$$\sigma = \min\left\{\frac{\pi}{2}, \frac{\pi}{2}\right\} = \frac{\pi}{2}$$

$$eq\sigma > e^{-m\sigma}$$

$$e\frac{\pi}{2} > e^0$$

$$244.645 > 1$$

Therefore, the condition is hold.

Example(6)

Consider the system:

$$\frac{dy_1(t)}{dt} + 2e^2 y_1(t-2) + e^3 y_2(t-3) = 0$$

$$\frac{dy_2(t)}{dt} + 3e^4 y_1(t-4) + 2e\, y_2(t-1) = 0$$

where the system have nonoscillatory solution:

$$y_1(t) = e^{-t} \text{and} y_2(t) = -e^{-t}$$

$$a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, p = \begin{bmatrix} 2e^2 & e^3 \\ 3e^4 & 2e \end{bmatrix}, \tau = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$m = \min_{1\le i\le n} \left(a_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji}| \right)$$

$$= \min_{1\le i\le 3} (a_{11} - |a_{21}|, a_{22} - |a_{12}|)$$

$$= \min_{1\le i\le 3} (0,0,0) = 0$$

$$q = \min_{1\le i\le n} \left(p_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |p_{ji}| \right)$$

$$= \min_{1\le i\le 3} (p_{11} - |p_{21}|, p_{22} - |p_{12}|)$$

$$= \min_{1\le i\le 3} (2e^2 - (3e^4), 2e - (e^3))$$

$$= \min_{1\le i\le 3} (-149.016, -14.6489) = -149.016$$

$$\sigma = \min\{\tau_{11}, \tau_{22}\}$$

$$= \min\{2,1\} = 1$$

$$eq\sigma > e^{-m\sigma}$$

$$e(-149.016)(1) > e^{-0}$$

$$-405.067 > 1$$

Thus, the condition is not hold.

V. Conclusions

In this paper, we find the conditions satisfy of oscillatory, has no eventually positive solutions and has no ultimately negative solutions of the linear system first order delay deferential equation and inequalities by solution system of delay differential equation and inequity. We prove the system of delay differential inequality have non-oscillatory eventually positive by imposingit possesses oscillatory eventually positive and leads to a contradiction. In the same way, we prove the system of delay differential has non-oscillatory eventually negative. Using the conditions and definitionswe were able to be exampled from matrices.

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