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TRANSIENT MOTION OF A REINER-RIVLIN FLUID BETWEEN TWO CONCENTRIC POROUS CIRCULAR CYLINDERS IN PRESENCE OF RADIAL MAGNETIC FIELD

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Abstract

This paper is concerned with the motion of a non-Newtonian fluid of Reiner-Rivlin type through an annulus with porous walls in presence of radial magnetic field. Here, the inner cylinder rotates about its axis with a transient angular velocity while the outer one is kept fixed.

Keywords : *Reiner*-Rivlin fluid, Circular cylinder, Radial magnetic field, transient angular velocity, Hankel functions

I. Introduction

The unsteady motion of a viscous conducting liquid between two porous concentric circular cylinders acted on by a radial magnetic field was studied by Mahapatra [1]. The motion of an ordinary viscous fluid between two co-axial porous circular cylinders when the inner cylinder is oscillating was investigated by Khamrui [2]. In this paper, an attempt has been made to investigate the flow of a Reiner-Rivlin fluid through two porous co-axial circular cylinders acted on by a radial magnetic field. The inner cylinder rotates with a transient angular velocity and the outer one is at rest. The approximate solutions for two extreme cases have been derived and in solving the problem Hankel functions have been used.
The well-known constitutive relation of Reiner-Rivlin fluid is given by

$$\tau_{ij} = -p\delta_{ij} + \mu e_{ij} + \mu_c e_{ik} e_{kj} \quad (1)$$

where μ is the coefficient of viscosity, μ_c is the coefficient of cross-viscosity, τ_{ij} is the stress tensor,e_{ij} is the rate of strain-tensor and $\delta_{ij} = 1, i = j \; and \; \delta_{ij} = 0, i \neq j$. Here μ and μ_c are assumed to be constants.

II. Formulation and Solutions of the Problem

Let us consider the radii of the inner and outer cylinders be 'a' and 'b' respectively. Choosing the cylindrical polar co-ordinates (r, θ, z) with z-axis as the common axis of the cylinders, the velocity components can be taken as $(u, v, 0)$. It is assumed that all the physical quantities are independent of θ and z.

Neglecting the effects due to the induced electric and magnetic fields, the equations of motion can be written as

$$\left.\begin{array}{c} \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}-\frac{v^2}{r}\right)=\frac{\partial \tau_{rr}}{\partial r}+\frac{\partial \tau_{zr}}{\partial z}+\frac{\tau_{rr}-\tau_{\theta\theta}}{r} \\ and \\ \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{uv}{r}+\frac{\sigma B_0^2}{\rho}.\frac{v}{r^2}\right)=\frac{\partial \tau_{r\theta}}{\partial r}+\frac{\partial \tau_{\theta z}}{\partial z}+\frac{2\tau_{r\theta}}{r} \end{array}\right\} \tag{2}$$

where B_0 is the magnetic induction vector and σ is the conductivity of the fluid. The equation of continuity is

$$\frac{\partial u}{\partial r}+\frac{u}{r}=0 \tag{3}$$

Now, $e_{rr}=2\frac{\partial u}{\partial r}$, $e_{\theta\theta}=\frac{2u}{r}$, $e_{rz}=e_{\theta z}=e_{zz}=0$, $e_{r\theta}=\frac{\partial v}{\partial r}-\frac{v}{r}$.

Hence, we have on using (1),

$$\tau_{rr}=-p+2\mu\frac{\partial u}{\partial r}+\mu_c\left\{4\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2\right\}$$

$$\tau_{\theta\theta}=-p+2\mu\frac{u}{r}+\mu_c\left\{\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2+4\frac{u^2}{r^2}\right\}$$

$$\tau_{rz}=\tau_{\theta z}=\tau_{zz}=0$$

and $\quad \tau_{r\theta}=\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)+\mu_c\,.\,2\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)$.

Hence from (2), with the help of (3), we get

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}-\frac{v^2}{r}\right)=-\frac{\partial p}{\partial r}+2\mu\frac{\partial^2 u}{\partial r^2}+\mu_c\frac{\partial}{\partial r}\left\{4\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2\right\}$$

$$+\frac{1}{r}\left[2\mu\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right)+4\mu_c\left\{\left(\frac{\partial u}{\partial r}\right)^2-\frac{u^2}{r^2}\right\}\right] \tag{4}$$

and

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{uv}{r}+\frac{\sigma B_0^2}{\rho}.\frac{v}{r^2}\right)=\mu\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right) \tag{5}$$

It is assumed that the rate of suction at the inner cylinder is equal to the rate of injection at the outer cylinder. So, we have on using (3)

$$u=\frac{au_a}{r}=\frac{bu_b}{r} \tag{6}$$

where u_a and u_b are the radial velocities at the inner and outer walls respectively.

Substituting for u in (5) and considering u_a to be constant, we get

$$\rho\left\{\frac{\partial v}{\partial t}+\frac{au_a}{r}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+\frac{\sigma B_0^2}{\rho}.\frac{v}{r^2}\right\}=\mu\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right) \qquad (7)$$

Again, substituting $v=r\omega$ in (7), we obtain

$$r\frac{\partial\omega}{\partial t}+\frac{au_a}{r}\left(r\frac{\partial\omega}{\partial r}+2\omega\right)=\nu\left[\frac{\partial}{\partial r}\left(r\frac{\partial\omega}{\partial r}\right)+2\frac{\partial\omega}{\partial r}\right]-\frac{\sigma B_0^2}{\rho}.\frac{\omega}{r} \qquad (8)$$

where ν is the kinematic coefficient of viscosity and ω is the angular velocity of the inner cylinder.

Further, putting $\omega=\varphi(r).e^{-nt}$ in (8), we get

$$r^2\frac{d^2\varphi}{dr^2}+(3-r)r\frac{d\varphi}{dr}+\left(\frac{n}{\nu}r^2-2R-M^2\right)\varphi=0 \qquad (9)$$

where $M=B_0\sqrt{\frac{\sigma}{\mu}}$ and $R=\frac{au_a}{\nu}$ is the cross-flow Reynolds number in Newtonian fluid.

If the inner cylinder rotates with a transient angular velocity Ωe^{-nt}, the boundery conditions are

$$\left.\begin{array}{c}\varphi(r)=\Omega\ \text{ when r = a}\\ \text{and}\\ \varphi(r)=0\ \text{ when r = b}\end{array}\right\} \qquad (10)$$

Now, substituting $\varphi(r)=r^{\frac{R}{2}-1}.F(r)$ in (9), we have

$$r^2\frac{d^2F}{dr^2}+r\frac{dF}{dr}+(k^2r^2-m^2)F=0 \qquad (11)$$

which is Bessel's equation of m-th order, where $m=\sqrt{\left(1+\frac{R}{2}\right)^2+M^2}$

and $k=\sqrt{\frac{n}{\nu}}$.

For practical need, the solution of equation (11) can be taken as

$$F(r)=CH_m^{(1)}(kr)+DH_m^{(2)}(kr)$$

where $H_m^{(1)}(kr)$ and $H_m^{(2)}(kr)$ are Hankel functions of the first kind and second kind respectively and C, D are constants.

Hence we obtain

$$\varphi(r)=r^{\frac{R}{2}-1}\left[CH_m^{(1)}(kr)+DH_m^{(2)}(kr)\right].$$

Applying the boundary conditions (10), we obtain

$$C = \frac{\Omega a^{1-\frac{R}{2}} H_m^{(2)}(kb)}{H_m^{(1)}(ka)H_m^{(2)}(kb) - H_m^{(1)}(kb)H_m^{(2)}(ka)} \quad \text{and} \quad D = \frac{-\Omega a^{1-\frac{R}{2}} H_m^{(1)}(kb)}{H_m^{(1)}(ka)H_m^{(2)}(kb) - H_m^{(1)}(kb)H_m^{(2)}(ka)}.$$

Hence we obtain the azimuthal velocity as

$$v = \Omega r^{\frac{R}{2}} a^{1-\frac{R}{2}} e^{-nt} \left[\frac{H_m^{(1)}(kr)H_m^{(2)}(kb) - H_m^{(1)}(kb)H_m^{(2)}(kr)}{H_m^{(1)}(ka)H_m^{(2)}(kb) - H_m^{(1)}(kb)H_m^{(2)}(ka)} \right] \tag{12}$$

Using the relation (6), equation (4) becomes

$$\rho\left(-\frac{a^2u_a^2}{r^3} - \frac{v^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu_c \frac{\partial}{\partial r}\left\{\frac{4a^2u_a^2}{r^4} + \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2\right\}$$

which on integration gives

$$p = p_0 + \rho \int \frac{v^2}{r} dr - \frac{a^2u_a^2\rho}{2r^2} + \mu_c\left\{\frac{4a^2u_a^2}{r^4} + \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2\right\}$$

$$= p_0 + p_I + p_{II} + p_{III} \ \ (say) \tag{13}$$

which shows that the pressure is made up of the components p_I due to the rotatory motion, p_{II} due to the porosity of the cylinders and p_{III} due to the cross-viscosity. It is also evident from the solution that the velocity is unaffected by the cross-viscosity while the pressure is affected by it.

III. Flow for Low Frequencies

We have the expressions (Watson [3], p. 73) for $H_m^1(z)$ and $H_m^2(z)$ in terms of $J_m(z)$ and $J_{-m}(z)$ as

$$H_m^1(z) = \frac{e^{-\frac{m}{\pi i}} J_m(z) - J_{-m}(z)}{-i\, sinm.\,\pi} \quad \text{and} \quad H_m^2(z) = \frac{e^{\frac{m}{\pi i}} J_m(z) - J_{-m}(z)}{i\, sinm.\,\pi}.$$

Hence we get

$$\frac{H_m^{(1)}(kr)H_m^{(2)}(kb) - H_m^{(1)}(kb)H_m^{(2)}(kr)}{H_m^{(1)}(ka)H_m^{(2)}(kb) - H_m^{(1)}(kb)H_m^{(2)}(ka)} = \frac{J_m(kr)J_{-m}(kb) - J_m(kb)J_{-m}(kr)}{J_m(ka)J_{-m}(kb) - J_m(kb)J_{-m}(ka)} \tag{14}$$

Again, we have (Watson [3], p.40)

$$J_m(z) = \left(\frac{z}{2}\right)^m . \left[\frac{1}{\Gamma(m+1)} - \frac{z^2}{4\Gamma(m+2)} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots..\right]$$

$$= \left(\frac{z}{2}\right)^m . (a_0 - a_1 z^2 + \ldots\ldots\ldots\ldots\ldots..)$$

and

$$J_{-m}(z) = \left(\frac{z}{2}\right)^{-m} . \left[\frac{1}{\Gamma(-m+1)} - \frac{z^2}{4\Gamma(-m+2)} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots..\right]$$

$$= \left(\frac{z}{2}\right)^{-m} . (b_0 - b_1 z^2 + \ldots\ldots\ldots\ldots\ldots..),$$

where
$$\left.\begin{array}{ll} a_0 = \frac{1}{\Gamma(m+1)}\ , & a_1 = \frac{1}{4\Gamma(m+2)}\ , \\ b_0 = \frac{1}{\Gamma(-m+1)}\ , & b_1 = \frac{1}{4\Gamma(-m+2)} \end{array}\right\} \tag{15}$$

The absolute values of *kr*, *ka* and *kb* are small for low frequencies and so avoiding terms involving powers of *k* higher than two, we get

$$J_m(kr)J_{-m}(kb) - J_m(kb)J_{-m}(kr) =$$

$$= a_0 b_0 \left[\left(\frac{r}{b}\right)^m - \left(\frac{b}{r}\right)^m\right]$$

$$- k^2 \left[(a_0 b_1 b^2 + a_1 b_0 r^2)\left(\frac{r}{b}\right)^m - (a_0 b_1 r^2 + a_1 b_0 b^2)\left(\frac{b}{r}\right)^m\right]$$

and

$$J_m(ka)J_{-m}(kb) - J_m(kb)J_{-m}(ka) =$$

$$= a_0 b_0 \left[\left(\frac{a}{b}\right)^m - \left(\frac{b}{a}\right)^m\right]$$

$$- k^2 \left[(a_0 b_1 b^2 + a_1 b_0 a^2)\left(\frac{a}{b}\right)^m - (a_0 b_1 a^2 + a_1 b_0 b^2)\left(\frac{b}{a}\right)^m\right].$$

Putting in (14), we have

$$\frac{H_m^{(1)}(kr)H_m^{(2)}(kb) - H_m^{(1)}(kb)H_m^{(2)}(kr)}{H_m^{(1)}(ka)H_m^{(2)}(kb) - H_m^{(1)}(kb)H_m^{(2)}(ka)} =$$

$$= \frac{a_0 b_0\left[\left(\frac{r}{b}\right)^m - \left(\frac{b}{r}\right)^m\right] - k^2\left[(a_0 b_1 b^2 + a_1 b_0 r^2)\left(\frac{r}{b}\right)^m - (a_0 b_1 r^2 + a_1 b_0 b^2)\left(\frac{b}{r}\right)^m\right]}{a_0 b_0\left[\left(\frac{a}{b}\right)^m - \left(\frac{b}{a}\right)^m\right] - k^2\left[(a_0 b_1 b^2 + a_1 b_0 a^2)\left(\frac{a}{b}\right)^m - (a_0 b_1 a^2 + a_1 b_0 b^2)\left(\frac{b}{a}\right)^m\right]} .$$

Now, substituting $= \sqrt{\frac{n}{\nu}}$, we finally obtain the velocity from (12) as

$$v = \Omega r^{\frac{R}{2}} a^{1-\frac{R}{2}} e^{-nt} . \left\{ \frac{\left[\left(\frac{r}{b}\right)^m - \left(\frac{b}{r}\right)^m\right] - \frac{n}{\nu a_0 b_0}\left[(a_0 b_1 b^2 + a_1 b_0 r^2)\left(\frac{r}{b}\right)^m - (a_0 b_1 r^2 + a_1 b_0 b^2)\left(\frac{b}{r}\right)^m\right]}{\left[\left(\frac{a}{b}\right)^m - \left(\frac{b}{a}\right)^m\right] - \frac{n}{\nu a_0 b_0}\left[(a_0 b_1 b^2 + a_1 b_0 a^2)\left(\frac{a}{b}\right)^m - (a_0 b_1 a^2 + a_1 b_0 b^2)\left(\frac{b}{a}\right)^m\right]} \right\} \qquad (16)$$

IV. Flow for High Frequencies

The value of $|k|$ is very large for high frequencies and so, we use the asymptotic formulas (Sommerfeld [4], p.100)

$$H_m^1(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i\left\{z - \frac{1}{2}\pi\left(m+\frac{1}{2}\right)\right\}} \quad and \quad H_m^2(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-i\left\{z - \frac{1}{2}\pi\left(m+\frac{1}{2}\right)\right\}} .$$

Hence we get

$$H_m^{(1)}(kr)H_m^{(2)}(kb) - H_m^{(1)}(kb)H_m^{(2)}(kr) = -\frac{2}{\pi k\sqrt{rb}}\left\{e^{ik(b-r)} - e^{-ik(b-r)}\right\}$$
$$\approx -\frac{2}{\pi k\sqrt{rb}} . e^{ik(b-r)} ,$$

$since\ b > r$

and

$$H_m^{(1)}(ka)H_m^{(2)}(kb) - H_m^{(1)}(kb)H_m^{(2)}(ka) = -\frac{2}{\pi k\sqrt{ab}}\left\{e^{ik(b-a)} - e^{-ik(b-a)}\right\}$$
$$\approx -\frac{2}{\pi k\sqrt{ab}} . e^{ik(b-a)} , \ since\ b > a .$$

Substituting in (12) and simplifying, we finally obtain the velocity for high frequencies as

$$v = \Omega r^{\frac{1}{2}(R-1)} a^{\frac{1}{2}(3-R)} e^{-nt} Re\left[e^{-ik(r-a)}\right]$$
$$= \Omega r^{\frac{1}{2}(R-1)} a^{\frac{1}{2}(3-R)} e^{-nt} \cos k(r-a) \qquad (17)$$

where $k = \sqrt{\frac{n}{\nu}}$.

It is clear from the solution that the velocity is not affected by the magnetic field for high frequencies.

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