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LAMINAR CONVECTION OVER A VERTICAL PLATE WITH CONVECTIVE BOUNDARY CONDITION

BY

Asish Mitra

College of Engineering & Management, Kolaghat. East Midnapur, India-721171

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Abstract

In the present numerical study, laminar convection over a vertical plate with convective boundary condition is presented. It is found that the similarity solution is possible if the convective heat transfer associated with the hot fluid on the left side of the plate is proportional to $x^{1/2}$, and the thermal expansion coefficient β is proportional to x^{-1}. The numerical solutions thus obtained are analyzed for a range of values of the embedded parameters and for representative Prandtl numbers of 0.72, 1, 3 and 7.1. The results of the present simulation are then compared with the reports published in literature and find a good agreement.

Keywords: Convective Boundary Condition, Laminar Convection, Matlab, Numerical Simulation, Vertical Plate.

NOMENCLATURE

Bi_x parameter in Eq. (14), dimensionless
c constant in Eq. (15)
c_p specific heat capacity, J/Kg.K
f function defined in Eq. (6)
g gravitational acceleration, 9.81 m/s^2
Gr_x Grashof number based on x, dimensionless
h_f heat transfer coefficient, $W/m^2.K$
k thermal conductivity, W/m.K
m constant in Eq. (15)
Pr Prandtl number, dimensionless
Re_x Reynolds Number, dimensionles
T temperature, K
T_f hot fluid temperature, K
T_∞ free streams temperature, K
U_∞ free stream velocity, m/s
u_1, u_2 initial values in Eq. (20)

u velocity component in x, m/s
v velocity component in y, m/s
x coordinate from the leading edge, m
y coordinate normal to plate, m
z_1, z_2, z_3, z_4, z_5 variables, Eq. (17)

Greek Symbols

α thermal diffusivity, m^2/s
θ function defined in Eq. (10), dimensionless
β coefficient of thermal expansion, 1/K
μ dynamic viscosity, $N.s/m^2$
ν kinematic viscosity, m^2/s
η similarity variables, Eq. (7)
ψ stream function, m^2/s
ρ density, kg/m^3
ρ_∞ free stream density, kg/m^3

I. Introduction

There have been a lot of studies on natural convection from a vertical plate with convective boundary condition due to its relevance to a variety of industrial applications and naturally occurring processes such as solar collectors, pipes, ducts, electronic packages, walls and windows etc. The earliest analytical investigation was a similarity analysis of the boundary layer equations Blasius, H [1]. The problem is also analyzed by several researchers from different perspectives [2-10].

In the present numerical investigation, a simple accurate numerical simulation of laminar free-convection flow and heat transfer over a vertical plate with constant heat flux is developed. The paper is organized as follows: Mathematical model of the problem, its solution procedure, development of code in Matlab, interpretation of the results, comparison with previous works.

II. MATHEMATICAL MODEL

A steady, laminar, two-dimensional, no dissipation fluid flows over a vertical plate. On the right side of the plate, a stream of cold fluid at temperature T_∞ moving up with a uniform velocity U_∞ whereas the left surface of the plate is heated by convection from a hot fluid at temperature T_f. We assume the fluid to be Newtonian with constant properties, including density, with one exception: the density difference $\rho - \rho_\infty$ is to be considered since it is this density difference between the inside and the outside of the boundary layer that gives rise to buoyancy force and sustains flow. (This is known as the *Boussinesq approximation.*). The upward direction along the plate is *x*, and the direction normal to surface is *y*, as shown in figure 1.

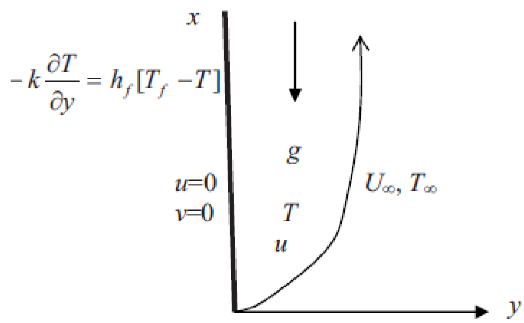

Fig. 1. Physical Model and its coordinate system

The equations governing the flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \beta g(T - T_\infty) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial u}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

The boundary conditions on the solution are:

At y = 0: u = v = 0, $-k\frac{\partial T}{\partial y} = h_f[T_f - T(x,0)]$

For large y: u→ U_∞, T→T_∞ (4)

The continuity equation (1) is automatically satisfied through introduction of the stream function:

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{5}$$

A similarity solution is possible if

$$\psi = U_\infty\sqrt{\frac{\nu x}{U_\infty}}f(\eta) \tag{6}$$

where

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} = \frac{y}{x}\sqrt{\mathrm{Re}_x} \tag{7}$$

Then the velocity components can be written as

$$u = U_\infty f'(\eta) \tag{8}$$

$$v = \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}(\eta f' - f) \tag{9}$$

Now, with the dimensionless temperature

$$\theta = \frac{T - T_\infty}{T_f - T_\infty} \tag{10}$$

the partial differential Eqs. (2) and (3) are transformed to ordinary differential equations (with a prime denoting differentiation with respect to η)

$$f''' + \frac{1}{2}ff'' + Gr_x\theta = 0 \tag{11}$$

$$\theta'' + \frac{1}{2}\Pr f\theta' = 0 \tag{12}$$

Eqs. (11) and (12) constitute a pair of simultaneous nonlinear ordinary differential equations for the velocity and temperature functions, f' and θ. They must be solved subject to the following boundary conditions:

At y = 0: u = 0 i.e., at η = 0: f′ = 0

At y = 0: v = 0 i.e., at η = 0: f = 0

At y = 0: $-k\frac{\partial T}{\partial y} = h_f[T_f - T(x,0)]$

i.e., at η = 0: $\theta'(0) = -Bi_x[1 - \theta(0)]$

For large y: u→ 0 i.e., for large η: f′ = 1

For large y: T→ T_∞ i.e., for large η: θ = 0 (13)

$$\Pr = \frac{c_p\mu}{k} = \frac{\nu}{\alpha}, \quad Bi_x = \frac{h_f}{k}\sqrt{\frac{\nu x}{U_\infty}}, \quad Gr_x = \frac{\nu x g(T_f - T_\infty)}{U_\infty^2} \tag{14}$$

Pr, Gr_x and Bi_x are Prandtl number, Grashof number and local convective heat transfer parameter respectively.

In order to get a similar solution of the momentum and energy equations, the parameters, Gr_x and Bi_x , defined in Eq. (14), must be constants and not functions of x. This condition can be met if the heat transfer coefficient h_f is proportional to $x^{-1/2}$ and the thermal expansion coefficient β is proportional to x^{-1}. Accordingly

$$h_f = cx^{-1/2}, \quad \beta = mx^{-1} \tag{15}$$

where c and m are constants. Substituting Eq. (15) into Eq. (14), we get

$$Bi = \frac{c}{k}\sqrt{\frac{\nu}{U_\infty}}, \quad Gr = \frac{\nu m g(T_f - T_\infty)}{U_\infty^2} \tag{16}$$

With Bi and Gr defined by Eq. (16), the solutions of Eqs. (11) and (12) yield the similarity solutions.

III. SOLUTION PROCEDURE

Eqs. (11) and (12) are coupled and must be solved simultaneously, which is always the case in convection problems. No analytic solution is known, so numerical integration is necessary. There are two unknown initial values at the wall. $f''(0)$ and $\theta(0)$. One must find the proper values of $f''(0)$ and $\theta(0)$ so that integration gives $f'(\infty)=1$ and $\theta(\infty)=0$. Pr, Gr and Bi are parameters.

III.i. Reduction of Equations to First-order System

This is done easily by defining new variables:

$$\begin{aligned} z_1 &= f \\ z_2 &= z_1' = f' \\ z_3 &= z_2' = z_1'' = f'' \\ z_4 &= \theta \\ z_5 &= \theta' \end{aligned} \tag{17}$$

Therefore from Eqs. (11) and (12), we get the following set of differential equations

$$\begin{aligned} z_1' &= f' \\ z_2' &= z_1'' = f'' \\ z_3' &= z_2'' = z_1''' = f''' = -\frac{1}{2}ff'' - Gr\theta = -\frac{1}{2}z_1 z_3 - Gr z_4 \quad z_4' = z_5 \\ z_5' &= -\frac{1}{2}\Pr f\theta' = -\frac{1}{2}\Pr z_1 z_5 \end{aligned} \tag{18}$$

with the following boundary conditions:

$$z_1(0) = f(0) = 0$$
$$z_2(0) = z_1'(0) = f'(0) = 0$$
$$z_2(\infty) = z_1'(\infty) = f'(\infty) = 1$$
$$z_4(\infty) = \theta(\infty) = 0$$
$$z_5(0) = z_4'(0) = \theta'(0) = -Bi[1-\theta(0)] = -Bi[1-z_4(0)] \tag{19}$$

Eq. (11) is third-order and is replaced by three first-order equations, whereas Eq. (12) is second-order and is replaced with two first-order equations.

III.ii. Conversion to Initial Value Problems

To solve Eq. (18), let the two unknown initial values $f''(0)$ and $\theta(0)$ be u_1 and u_2 respectively, the set of initial conditions is then:

$$z_1(0) = f(0) = 0$$
$$z_2(0) = z_1'(0) = f'(0) = 0$$
$$z_3(0) = z_2'(0) = z_1''(0) = f''(0) = u_1$$
$$z_4(0) = \theta(0) = u_2$$
$$z_5(0) = z_4'(0) = \theta'(0) = -Bi[1-\theta(0)] = -Bi[1-z_4(0)] = -Bi[1-u_2] \tag{20}$$

If Eqs. (18) are solved with adaptive Runge-Kutta method using the initial conditions in Eq. (20) for a particular value of Bi, the computed boundary values at $\eta = \infty$ depend on the choice of u_1 and u_2. Let this dependence be

$$z_2(\infty) = z_1'(\infty) = f'(\infty) = f_1(u_1)$$
$$z_4(\infty) = \theta(\infty) = f_2(u_2) \tag{21}$$

The correct choice of u_1 and u_2 yields the given boundary conditions at $\eta = \infty$; that is, it satisfies the equations

$$f_1(u_1) = 1$$
$$f_2(u_2) = 0 \tag{22}$$

These are simultaneous nonlinear equations that can be solved by the Newton-Raphson method. A value of 10 is fine for infinity, further integration will change nothing.

III.iii. Program Details

This section describes a set of Matlab routines for the solution of Eqs. (18) along with the boundary conditions (20). They are listed in Table 1.

Table 1. A set of Matlab routines used sequentially to solve Eqs. (18).

Matlab code	Brief Description
deqs.m	Defines the differential Eqs. (18).
incond.m	Describes initial values for integration, u_1 and u_2 are guessed values, Eq. (20)
runKut5.m	Integrates the initial value problem (18) using adaptive Runge-Kutta method.
residual.m	Provides boundary residuals and approximate solutions.
newtonraphson.m	Provides correct values u_1 and u_2 using approximate solutions from residual.m
runKut5.m	Again integrates the initial value problem (18) using correct values of u_1 and u_2.

The output of the code runKut5.m gives the tabulated values of $f,\ f',\ f'',\ \theta,\ \theta'$ as functions of η for various values of Pr, Gr, and Bi numbers. The set of discrete values of the parameters, Pr, Gr and Bi used in the codes are given in the Table 2.

Table 2. Values of input parameters

Input parameters	Values
Pr	0.72, 1, 3, 7.1
Gr	0.1, 0.5, 1, 1.2
Bi	0.05, 0.1, 0.2, 0.4, 0.5, 0.6, 0.8, 1, 5, 10, 20

IV. INTERPRETATION OF COMPUTATIONAL RESULTS

Physical quantities are related to the dimensionless functions f and θ through Eqs. (6) to (10). f and θ are now known.

IV.i. Comparison with previous numerical experiments

The numerical results obtained from the abovementioned codes are validated by comparing with previously published results [7] for Pr = 0.72 and Gr = 0 in Table 3 and are found in excellent agreement.

Table 3. Comparison with previous results [7] for Pr = 0.72 and Gr = 0.

Bi	$\theta(0)$ Aziz [7]	$-\theta'(0)$ present	$\theta'(0)$ Aziz [7]	$-\theta'(0)$ Present
0.05	0.1447	0.1447	0.0428	0.0428
0.01	0.2528	0.2528	0.0747	0.0747
0.2	0.4035	0.4035	0.1193	0.1193
0.4	0.5750	0.5750	0.1700	0.1700
0.6	0.6699	0.6699	0.1981	0.1981
0.8	0.7302	0.7302	0.2159	0.2159
1	0.7718	0.7718	0.2282	0.2282
5	0.9441	0.9442	0.2791	0.2791
10	0.9713	0.9713	0.2871	0.2871
20	0.9854	0.9854	0.2913	0.2913

IV.ii. Some initial values for different parameters

Some accurate initial values from this computation are listed in Table 4. The values of the skin-friction coefficient and the local Nusselt number are represented by $f''(0)$ and $-\theta'(0)$ respectively. Table 4 indicates that the skin-friction and the heat transfer rate at the plate surface increases with the increase in local Grashof number and convective surface heat transfer parameter. On the other hand, an increase in the Prandtl number decreases the skin-friction but increases the rate of heat transfer at the plate surface.

Table 4. Some initial values for different parameters

Pr	Bi	Gr	$f''(0)$	$\theta(0)$	$-\theta'(0)$
0.72	0.1	0.1	0.3688	0.2492	0.0751
		0.5	0.4970	0.2386	0.0761
		1.0	0.6320	0.2296	0.0770
		1.2	0.6810	0.2267	0.0773
Pr	Gr	Bi	$f''(0)$	$\theta(0)$	$-\theta'(0)$
0.72	0.1	0.1	0.3688	0.2492	0.0751
		0.5	0.4207	0.6182	0.1909
		1.0	0.4404	0.7625	0.2375
		10.0	0.4679	0.9694	0.3056
Bi	Gr	Pr	$f''(0)$	$\theta(0)$	$-\theta'(0)$
0.1	0.1	0.72	0.3688	0.2492	0.0751
		1	0.3632	0.2286	0.0771
		3	0.3494	0.1695	0.0830
		7.1	0.3427	0.1328	0.0867

IV.iii. Effect of Bi on Dimensionless velocity (f')

Variations of dimensionless velocity *f'* with η for various values of Bi are shown graphically in figure 2 for Pr = 0.72 and Gr = 0.1.

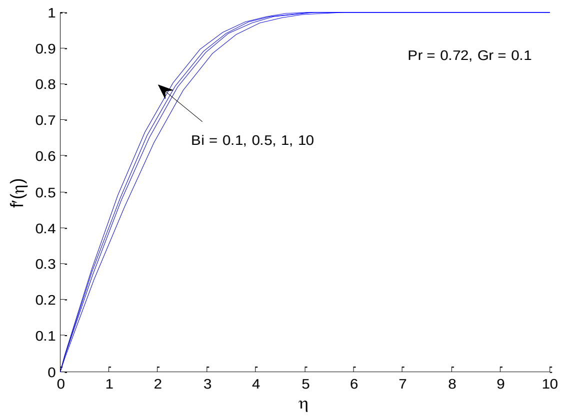

Fig. 2. Dimensionless velocity distributions for various values of Bi

IV.iv. Effect of Gr on Dimensionless velocity (f')

Variations of dimensionless velocity *f'* with η for various values of Gr are shown graphically in figure 3 for Pr = 0.72 and Bi = 0.1. From figures 2 and 3, it is seen see that the effect of local Grashof number (Gr) on velocity profiles is more articulate than the effect of convection parameter (Bi).

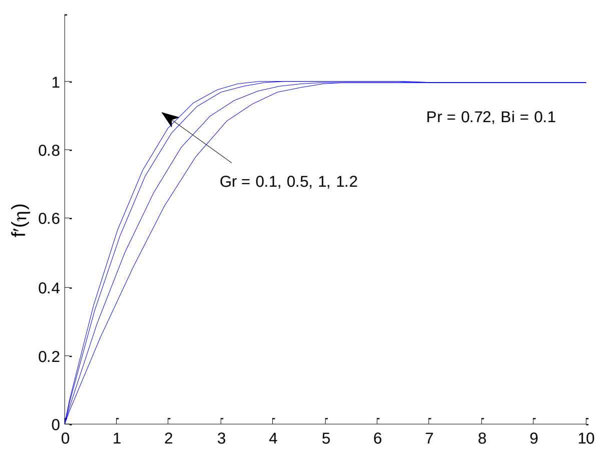

Fig. 3. Dimensionless velocity distributions for various values of Gr.

IV.v. Effect of Bi on Temperature profiles (θ)

Variations of temperature distributions θ with η for various values of Bi are shown graphically in figure 4 for Pr = 0.72 and Gr = 0.1.

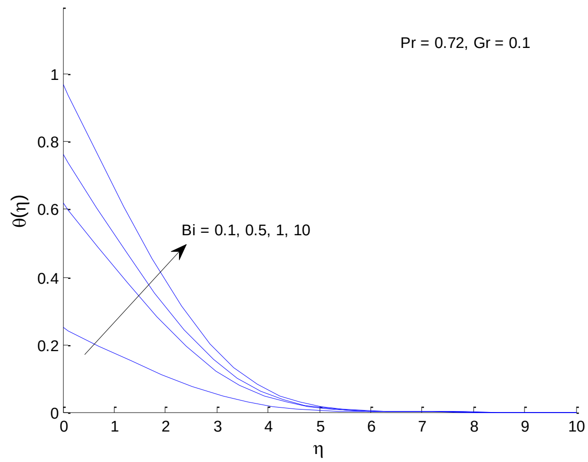

Fig. 4. Dimensionless temperature distributions for various values of Bi.

IV.vi. Effect of Gr on Temperature profiles (θ)

Variations of temperature distributions θ with η for values of Gr are shown graphically in figure 5 for Pr = 0.72 and Bi = 0.1.

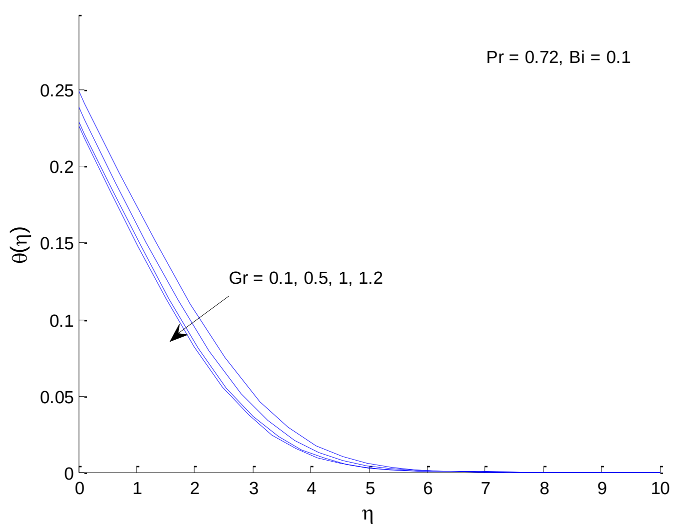

Fig. 5. Dimensionless temperature distributions for various values of Gr.

IV.vii. Effect of Pr on Temperature profiles (θ)

Finally, variations of temperature distributions θ with η for values of Pr are shown graphically in figure 6 for Gr = 0.1 and Bi = 0.1.

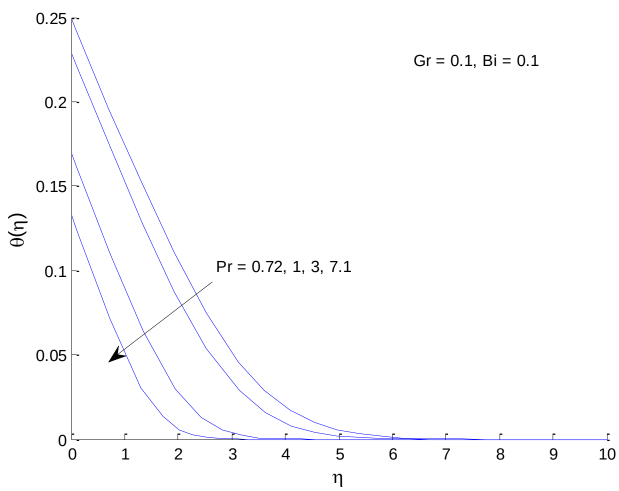

Fig. 6. Dimensionless temperature distributions for various values of Pr.

From figures 4 to 6, it is seen that thermal boundary layer thickness increases with an increase in Bi, but decreases with an increase in Gr and Pr.

V. Conclusions

In the present numerical simulation, laminar convection over a vertical plate with convective boundary condition is presented. A similarity solution is possible if the convective heat transfer associated with the hot fluid on the left side of the plate is proportional to $x^{1/2}$, and the thermal expansion coefficient β is proportional to x^{-1}. Details of the solution procedure of the nonlinear coupled partial differential equations of flow are discussed. The computer codes are developed for this numerical analysis in Matlab environment. Numerical solutions are computed for a range of values of the embedded parameters and for representative Prandtl numbers of 0.72, 1, 3 and 7.1using these codes. The computed results are in good agreement with reports published in literatures.

References

1) Blasius, H., “Grenzschichten in Flussigkeiten mit kleiner reibung,” *Z. Math Phys.*, vol. 56, pp. 1–37, 1908.
2) Weyl, H., “On the Differential Equations of the Simplest Boundary Layer Problem,” *Ann. Math.*, vol. 43, pp. 381–407, 1942.
3) Magyari, E., “The Moving Plate Thermometer,” *Int. J. Therm. Sci.*, **47**, pp. 14361441, 2008.
4) Cortell, R., “Numerical Solutions of the Classical Blasius Flat-Plate Problem,” *Appl. Math. Comput.*, vol. 170, pp. 706–710, 2005.
5) He, J. H., “A Simple Perturbation Approach to Blasius Equation,” *Appl. Math. Comput.*, vol. 140, pp. 217–222, 2003.
6) Bataller, R. C., “Radiation Effects for the Blasius and Sakiadis Flows With a Convective Surface Boundary Condition,” *Appl. Math. Comput.*, vol. 206, pp. 832–840, 2008.
7) Aziz, A., “A Similarity Solution for Laminar Thermal Boundary Layer Over a Flat Plate With a Convective Surface Boundary Condition,” *Commun. Nonlinear Sci. Numer. Simul.*, vol. 14, pp. 1064–1068, 2009.
8) Makinde, O. D., and Sibanda, P., “Magnetohydrodynamic Mixed Convective Flow and Heat and Mass Transfer Past a Vertical Plate in a Porous Medium With Constant Wall Suction,” *ASME J. Heat Transfer*, vol. 130, pp. 112602, 2008.
9) Makinde, O. D., “Analysis of Non-Newtonian Reactive Flow in a Cylindrical Pipe,” *ASME J. Appl. Mech.*, vol. 76, pp. 034502, 2009.
10) Cortell, R., “Similarity Solutions for Flow and Heat Transfer of a Quiescent Fluid Over a Nonlinearly Stretching Surface,” *J. Mater. Process. Technol.*, pp. 176–183, 2008.