

ON THE EXACT SOLVABILITY OF SOME POTENTIALS**By****¹Soumya Das, ²Kusumika Kundu. ³P. S. Majumdar****^{1,2}Narula Institute Of Technology, Kolkata – 700109, West Bengal****²A.P.C. College, New Barrackpore, Kolkata – 700131, West Bengal**<https://doi.org/10.26782/jmcms.2017.01.00001>**Abstract**

In the present paper it is shown that shape invariance is not a necessary condition for exact solvability of a potential in Quantum mechanics .Our contention is established by considering Eckart and Morse II potentials.

Key words: *Exact solvability, Shape independence, Eckart Potential, Morse II Potential.*

I. Introduction

There are some well known potentials for which non – relativistic Schrödinger equations in quantum mechanics can be reduced to hyper geometric or confluent hyper geometric functions.

Such potentials are known as exactly soluble potentials [1-3] Gendenshtein [2] showed that these exactly soluble potentials[4] i.e Eckart [6] , Morse[6 , 7] etc show a property known as shape invariance in the context of super symmetric quantum mechanics [8].

In the present work we attempt to show that the solubility of a potential does not necessarily imply shape invariance.

II. Theoretical Analysis

We consider a potential of the form.

$$U(x, \mu) = -\mu \left(\frac{dy}{dx} \right)^2 + U_2(x) \quad (1)$$

Where y is an invertible function of variable x . μ is a parameter which doesn't occur in $U_2(x)$. Now the well known one dimensional Schrödinger equation is

$$-\frac{1}{2} \frac{d^2 \Psi}{dx^2} + U(x) \Psi(x) = E \Psi(x) \quad (2)$$

It can be transformed into another equation in variable y .

$$-\frac{1}{2} \frac{d^2 \chi}{dy^2} + V(y) \chi(y) = \mu \chi(y) \quad (3)$$

Where

$$V(y) = [U_2(x) - E] \left(\frac{dx}{dy}\right)^2 + \frac{1}{4} v'(y) + \frac{1}{8} v^2(y) \quad (4)$$

$$v(y) = -\frac{d^2 x / dy^2}{dx / dy} \quad (5)$$

$$\chi(y) = (dx/dy)^{-\frac{1}{2}} \Psi(x) \quad (6)$$

The eigen value of the hamiltonian

$H(x, \lambda) = -\frac{1}{2} \frac{d^2}{dx^2} + U(x, \lambda)$ is a differentiable function of the parameter λ , since H itself is a differentiable function of λ [9].

Let us assume that the set of bound state eigen values of $H(x, \lambda)$ can be expressed as,

$$E = E_n(\epsilon); \quad n=0,1,2 \quad (7)$$

For an arbitrary pre assigned value of E ,

i.e, $E = \epsilon$ we can solve the set of equations (7) for ϵ . If these eigen values are denoted

by ϵ_n , we get the eigen values of $H(y, \epsilon) = -\frac{1}{2} \left(\frac{d^2}{dy^2}\right) + V(y, \epsilon)$

Where $V(y, \epsilon)$ is the potential of equation (4) with $E = \epsilon$.

So theoretically an elegant method has been developed by which it is possible to get the eigen values and eigen functions of the Hamiltonian $H(y, \epsilon)$ from $H(x, \lambda)$.

In the present article the formalism developed so far will be utilized to find two new solvable potentials for Eckart [5] and Mose II[6,7] potentials.

III. Results and Discussions

Eckart potential which can be expressed as [5]

$$U(x) = \frac{1}{2} C(C-1) \coth^2 x - D \coth x \quad (8)$$

$D > C^2$, $0 \leq x \leq \infty$ has energy

eigen values [4,7]

$$E_n = \frac{1}{2} [C(C-1) - (C+n)^2 - \frac{D^2}{(C+n)^2}] \quad (9)$$

$$n = 0, 1, 2, 3, \dots, n_{max} < G$$

With corresponding eigen functions

$$\Psi_n(x) = (\sin hx)^{C+n} \exp[-Gx] \times P_n^{(a,b)}(\coth x) \quad (10)$$

$$\text{With } G = \frac{D}{A+n} \quad (11)$$

and

$$a = G - C - n \quad (12)$$

$$b = -G - C - n \quad (13)$$

The above potential (equation (8)) can be written in the form of eqn (1)

with $y(x) = \ln(\sin hx)$ ($0 \leq x < \infty$, $-\infty < y < \infty$).

The corresponding derived potential can be obtained as;

$$V(y) = -\alpha\beta \sqrt{(1/2)(1 + \tanh y)} - \frac{3}{32} \tanh^2 y + \frac{1}{4} \left(\alpha + \beta^2 - \frac{1}{4} \right) \tanh y + \frac{1}{4} \left(\alpha + \beta^2 \frac{5}{8} \right) \quad (14)$$

$$(-\infty < y < \infty)$$

This potential has energy eigen values

$$\epsilon_n = -\frac{1}{2} \alpha_n (\alpha_n - 1)$$

α_n are the real solutions of the cubic equation

$$-\alpha - \beta^2 = \alpha_n(\alpha_n - 1) - (\alpha_n + n)^2 - \left[\frac{\alpha\beta}{(\alpha_n + n)}\right]^2 \quad (16)$$

The corresponding eigen functions are

$$\chi_n(y) = \sqrt{z}(1+z)^{-\sigma_n} \exp(-\mu_n y) \times p_n^{(\mu_n v_n)}(z) \quad (17)$$

Where

$$Z = [1 + \exp(-2y)]^{\frac{1}{2}} \quad (18)$$

$$\sigma_n = \frac{\alpha\beta}{(\alpha_n + n)} ; \mu_n = \sigma_n - \alpha_n - n \quad (19)$$

$$v_n = -\sigma_n - \alpha_n - n \quad (20)$$

The orthonormality of χ_n restricts the number of discrete energy levels by requiring that $\mu_n > 0$. The lowest state energy and wave functions are given by $\lambda_0 = -\frac{1}{2} \alpha (\alpha - 1)$ and

$$\chi_0(y) = [1 + \exp(-2y)]^{\frac{1}{4}} \exp[(\alpha - \beta)y] [1 + \sqrt{(1 + \exp(-2y))}]^{-\beta} \quad (21)$$

Constructing the super potential

$S(y) = -\frac{d\chi_0/dy}{\sqrt{2}\chi_0}$ and the super symmetric pair.

$$V_{\pm}(y) = S^2(y) \pm \frac{dS(y)}{dy} \quad \text{one can write}$$

$$V_-(y) = V(y) - \lambda_0 \quad (22)$$

And

$$V_+(y) = V_-(y) + \frac{1}{4} \operatorname{sech}^2 y [-1 + \sqrt{2}\beta(1 + \tanh y)^{-\frac{1}{2}}] \quad (23)$$

In this case $V_{\pm}(y)$ cannot be generated from $V_{\pm}(y)$ by the well known technique of reparamatrisation .

As a result they do not form a shape invariant pair. But the condition does not distract from the fact like the potential $V(y)$ they are exactly solvable.

Now we consider Morse II potential [6,7]

$$U(x, a) = a \operatorname{sech}^2 x + b \operatorname{sech} x \tanh x, \quad (24)$$

$$(-\infty < x < \infty),$$

One can deduce energy eigen values E_n and eigen values $(\chi_n(y))$ for the potential

$$V(y) = \beta \left(\alpha + \frac{1}{2} \right) \sqrt{\frac{1}{2}(\coth y - 1)} - \frac{3}{32} \coth^2 y + \frac{1}{4} \left(\beta^2 - \alpha - \frac{1}{4} \right) \coth y + \frac{1}{4} \left(\beta^2 - \alpha + \frac{5}{8} \right) \quad (25)$$

$$(0 \leq y < \infty)$$

$$\epsilon_n = \frac{1}{2} (\beta_n^2 - \alpha_n^2 - \alpha_n) \quad (26)$$

and

$$\chi_n(y) = [1 - \exp(-2y)]^{\frac{1}{4}} \exp(-\alpha_n y + \beta_n \theta) \times P_n^{(\mu_n, \mu_n^*)}(iz) \quad (27)$$

Where $z = [\exp(2y) - 1]^{1/2} \cdot \alpha_n$ and β_n are the real solutions of the simultaneous equations.

$$(\alpha_n + \frac{1}{2}) \beta_n = \left(\alpha + \frac{1}{2} \right) \beta, \quad (28)$$

$$\beta_n^2 - (2n+1)\alpha_n + n^2 = \beta^2 - \alpha, \quad (29)$$

$$n = 0, 1, 2, 3 \dots$$

$$\mu_n = -\left(\alpha_n + i\beta_n + \frac{1}{2} \right), \theta = \tan^{-1} \left[\frac{1}{2} (\coth y - 1) \right]^{1/2} \quad (30)$$

The wave function $\chi_n(y)$ is normalisable for $n < \alpha_n$

The Jacobi polynomial $P_n^{(\mu_n, \mu_n^*)}$ is real for even n and pure imaginary for odd n .

The ground state energy eigen value and eigen functions are :

$$E_0 = \frac{1}{2} (\beta^2 - \alpha^2 - \alpha) \quad (31)$$

$$\chi_0(y) = [1 - \exp(-2y)]^{1/4} \exp(-\alpha y + \beta \theta). \quad (32)$$

knowing the ground state wave function one can construct the super symmetry pair of potentials

$$V_-(y) = V(y) - \lambda_0 \quad (33)$$

And

$$V_+(y) = V_-(y) + \frac{1}{4} \operatorname{cosech}^2 y [1 - \beta \sqrt{2} (\coth y - 1)^{-\frac{1}{2}}] \quad (34)$$

It can be observed that it is not possible to get $V_{\pm}(y)$ from each other by mere reparameterisation and $V_{\pm}(y)$ lack the property of shape invariance.

In spite of these potential are exactly solvable as the potential $V(y)$ is solvable.

IV. Conclusion

In the present paper two potentials have been constructed which are exactly solvable but their respective super symmetric partner potentials do not have the property of shape invariance. This shows that shape invariance is not a necessary criterion for the solvability of a potential.

It is to be noted that for all the potentials the eigen values E_n are determined by the cubic equation and the condition that the discriminant of the cubic equation be positive (the condition for obtaining only one real root) turns out to be same as the condition for the normalisability of the eigen function.

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