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NUMEROUS EXACT SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS BY TAN–COT METHOD

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Abstract

This theoretical investigation is made in order to get the new exact solitary wave solutions of nonlinear partial differential equations (PDEs). The well-known Tan-Cot Function method is employed to obtain the exact solutions of Joseph–Egri equation (TRLW), Sharma–Tasso–Olver equation (STO), mKdV (modified Korteweg-de Vries) equation with additional first order dispersion term, and KdV (Korteweg-de Vries) equation with additional fifth order dispersion term,. The results which have been found in this theoretical work could be applicable to understand the characteristics and elastic behavior of nonlinear structures including solitons as well as play an important role in wide range of physical applications.

Keywords : Nonlinear PDEs, Exact solutions, Tan-Cot function method, Joseph–Egri equation, Sharma–Tasso–Olver equation, mKdV equation with additional first order dispersion term, KdV equation with additional fifth order dispersion term, soliton solutions.

I. Introduction

Nonlinear partial differential equations (PDEs) play significant role in the study of a wide variety of scientific applications such as plasma physics, solid state physics and fluid dynamics [1,2,3]. Recently, it has become a more attractive topic to find out the exact solutions of nonlinear PDEs directly.

Considering lots of applications of Joseph–Egri equation (TRLW), Sharma–Tasso–Olver equation (STO), modified Korteweg-de Vries (mKdV) equation with additional first order dispersion term, and Korteweg-de Vries (KdV) equation with additional fifth order dispersion term in contemporary problems, several authors have got inspiration and attained the exact soliton solutions of these equations. The term soliton, from the observation that waves like particles retained their shapes and velocities after interactions, was first introduced by Zabusky and Kruskal [4]. In 1834 John Scott Russel [3], a naval architect, published the first observation

about a solitary wave and explored his experiences to the British Association. By seeking more exact solutions is a superior way to comprehend the nonlinear phenomena of the aforementioned equations. A plenty of methods have been proposed to investigate the exact solutions of nonlinear evolution equations such as tanh -sech method [2,5,6], extended tanh method [7,8,9], hyperbolic function method [10,11], Jacobi elliptic method [13], the first Integral method [14,15], sine-cosine method [16,17]

and Tan-Cot method. A number of authors have been motivated from these researches and applied the Tan – Cot method to solve the Joseph – Egri equation (TRLW), Sharma – Tasso – Olver equation (STO), mKdV equation with additional first order dispersion term, KdV equation with additional fifth order dispersion term equation.

II. The Tan-Cot Function Method

Consider the nonlinear PDE in the form [12]

$$P(u,u_t,u_x,u_{xx},u_{xxx},.....)=0 \qquad (2.1)$$

This PDE can be converted to an ordinary differential equation (ODE)

$$Q(f,\frac{df}{d\xi},\frac{d^2f}{d\xi^2},\frac{d^3f}{d\xi^3},.....)=0 \qquad (2.2)$$

Upon using the following transformation $u(x,t)=f(\xi)$, where $\xi=x-ct+d$, c is the speed of the travelling wave and d is a constant.

Equation (2.2) is then integrated as long as all term contains derivatives where we neglect the integration constants. The solutions of Eq. (2.2) can be expressed in the form:

$$f(\xi)=\alpha\tan^{\beta}(\mu\xi), \quad |\xi|\leq\frac{\pi}{2\mu} \qquad (2.3)$$

or in the form $f(\xi)=\alpha\cot^{\beta}(\mu\xi)$, $\quad |\xi|\leq\frac{\pi}{2\mu}$ (2.4)

where α,μ and β are parameters that will be determined, μ and c are the wave number and the wave speed, respectively. The derivatives of Eq. (2.3) are

$$\frac{df}{d\xi}=\alpha\beta\mu[\tan^{\beta-1}(\mu\xi)+\tan^{\beta+1}(\mu\xi)] \qquad (2.5)$$

$$\frac{d^2 f}{d\xi^2} = \alpha\beta\mu^2[(\beta-1)\tan^{\beta-2}(\mu\xi) + 2\beta\tan^{\beta}(\mu\xi) + (\beta+1)\tan^{\beta+2}(\mu\xi)] \tag{2.6}$$

$$\frac{d^3 f}{d\xi^3} = \alpha\beta\mu^3[(\beta-2)(\beta-1)\tan^{\beta-3}(\mu\xi) + (3\beta^2-3\beta+2)\tan^{\beta-1}(\mu\xi) + (3\beta^2+3\beta+2)\tan^{\beta+1}(\mu\xi) + (\beta+2)(\beta+1)\tan^{\beta+3}(\mu\xi)] \tag{2.7}$$

$$\frac{d^4 f}{d\xi^4} = \alpha\beta\mu^4[(\beta-3)(\beta-2)(\beta-1)\tan^{\beta-4}(\mu\xi) + 4(\beta-1)(\beta^2-2\beta+2)\tan^{\beta-2}(\mu\xi) + 2\beta(3\beta^2+5)\tan^{\beta}(\mu\xi) + 4(\beta+1)(\beta^2+2\beta+2)\tan^{\beta+2}(\mu\xi) + (\beta+3)(\beta+2)(\beta+1)\tan^{\beta+4}(\mu\xi)] \tag{2.8}$$

and so on.
Authors found all solutions by considering only in the form $\alpha\tan^{\beta}(\mu\xi)$. For this reason, all derivatives are calculated intentionally only for this form.

Using Eq. (2.3) and its derivatives in the reduced Eq. (2.2) we can obtain a trigonometric equation in different powers of tangent functions. The parameters are then determined by collecting the coefficients of each pair of tangent functions with same exponents, where these coefficients have to vanish. Consequently, this gives a system of algebraic equations in the unknown parameters α, μ and β that will be determined. The solutions proposed in Eq. (2.3) follow immediately.

III. Applications

A. *Joseph – Egri (TRLW) Equation*

Consider the Joseph – Egri (TRLW) equation

$$u_t + u_x + auu_x + u_{xtt} = 0 \tag{3.1.1}$$

We introduce the transformation

$u(x,t) = f(\xi)$, where $\xi = x - ct + d$, c is the speed of the travelling wave and d is a constant.

By using this transformation equation (3.1.1) transform to the ODE:

$$(1-c)\frac{df}{d\xi} + af(\xi)\frac{df}{d\xi} + c^2\frac{d^3 f}{d\xi^3} = 0 \tag{3.1.2}$$

Integrating (3.1.2) with respect to ξ once with zero constant to get

$$(1-c)f(\xi) + \frac{a}{2}(f(\xi))^2 + c^2\frac{d^2 f}{d\xi^2} = 0 \tag{3.1.3}$$

Seeking the solution in equation (2.6)

$$(1-c)\alpha\tan^{\beta}(\mu\xi)+\frac{a}{2}\alpha^2\tan^{2\beta}(\mu\xi)+ c^2\alpha\beta\mu^2[(\beta-1)\tan^{\beta-2}(\mu\xi)+2\beta\tan^{\beta}(\mu\xi) +(\beta+1)\tan^{\beta+2}(\mu\xi)]=0 \tag{3.1.4}$$

Equating the exponents 2β and $\beta-2$ yield

$$2\beta=\beta-2 \quad \text{i.e.,} \quad \beta=-2$$

Replacing β by -2 into equation (3.1.4) we get

$$(1-c)\alpha\tan^{-2}(\mu\xi)+\frac{a}{2}\alpha^2\tan^{-4}(\mu\xi)+6c^2\alpha\mu^2\tan^{-4}(\mu\xi) +8c^2\alpha\mu^2\tan^{-2}(\mu\xi)=0 \tag{3.1.5}$$

Equation (3.1.5) is satisfied only if the following system of algebraic equations holds: $(1-c)\alpha+8c^2\alpha\mu^2=0$ (3.1.6)

$$\frac{a}{2}\alpha^2+6c^2\alpha\mu^2=0 \tag{3.1.7}$$

Solving Eqs. (3.1.6) and (3.1.7), we find

$$\mu=\pm\frac{1}{2c}\sqrt{\frac{c-1}{2}} \;,\; \alpha=\frac{3(1-c)}{2a} \tag{3.1.8}$$

Substituting Eq. (3.1.8) into Eq. (2.3), we obtain the exact solution of the TRLW equation in the form:

$$u(x,t)=\frac{3(1-c)}{2a}\tan^{-2}\left[\pm\frac{1}{2c}\sqrt{\frac{c-1}{2}}(x-ct+d)\right]$$

i.e., $$u(x,t)=\frac{3(1-c)}{2a}\cot^{2}\left[\pm\frac{1}{2c}\sqrt{\frac{c-1}{2}}(x-ct+d)\right] \tag{3.1.9}$$

Evolutional profile of $u(x,t)$ through the expression (3.1.9) is shown in the figure 1 for $c=a=d=2, -10\le x\le 10$ and $-10\le t\le 10$.

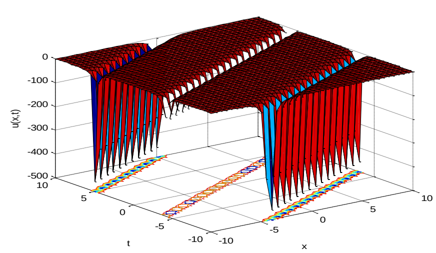


Figure 1: Graphical representation of the travelling wave solution of $u(x,t)$ for $c=a=d=2, -10\le x\le 10$ and $-10\le t\le 10$.

Figure 2 presents the line diagram of $u(x,t)$ for $c=a=d=2, -10\le x\le 10$ and $t=0,6$.

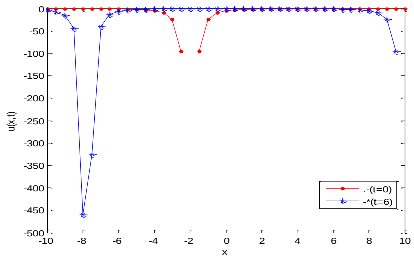


Figure 2: Line diagram of $u(x,t)$ for $c=a=d=2$ and $-10 \le x \le 10$.

B. *Sharma–Tasso–Olver (STO) Equation*

Consider the (1+1) dimensional Sharma – Tasso – Olver (STO) equation

$$u_t + 3u_x{}^2 + 3u^2u_x + 3uu_{xx} + u_{xxx} = 0 \tag{3.2.1}$$

To solve the Eq. (3.2.1) by the tan cot method we introduce the transformation $u(x,t) = f(\xi)$, where $\xi = x - ct + d$, c is the velocity of the travelling wave and d is a constant.

By using this transformation Eq. (3.2.1) becomes

$$-c\frac{df}{d\xi} + 3\left(\frac{df}{d\xi}\right)^2 + 3(f(\xi))^2\frac{df}{d\xi} + 3f(\xi)\frac{d^2f}{d\xi^2} + \frac{d^3f}{d\xi^3} = 0 \tag{3.2.2}$$

Equation (3.2.2) can be re-written as

$$-c\frac{df}{d\xi} + 3\frac{d}{d\xi}\left\{f(\xi)\frac{df}{d\xi}\right\} + \frac{d}{d\xi}\left\{(f(\xi))^3\right\} + \frac{d^3f}{d\xi^3} = 0$$

(3.2.3)

Integrating Eq. (3.2.3) with respect to ξ and ignoring constant of integration, we obtain

$$-cf(\xi) + 3f(\xi)\frac{df}{d\xi} + (f(\xi))^3 + \frac{d^2f}{d\xi^2} = 0 \tag{3.2.4}$$

Seeking the solutions in the Eqs. (2.5) and (2.6)

$$\begin{aligned}&-c\alpha\tan^{\beta}(\mu\xi) + 3\alpha^2\beta\mu\tan^{2\beta-1}(\mu\xi) + 3\alpha^2\beta\mu\tan^{2\beta+1}(\mu\xi)\\&+\alpha^3\tan^{3\beta}(\mu\xi) + \alpha\beta(\beta-1)\mu^2\tan^{\beta-2}(\mu\xi) +\\&2\alpha\beta^2\mu^2\tan^{\beta}(\mu\xi) + \alpha\beta(\beta+1)\mu^2\tan^{\beta+2}(\mu\xi) = 0\end{aligned} \tag{3.2.5}$$

Equating the exponents 3β and $2\beta-1$ yield

$3\beta = 2\beta - 1$ i.e., $\beta = -1$

Replacing β by -1 into Eq. (3.2.5) we get

$$-c\alpha\tan^{-1}(\mu\xi)-3\alpha^2\mu\tan^{-3}(\mu\xi)-3\alpha^2\mu\tan^{-1}(\mu\xi)$$
$$+\alpha^3\tan^{-3}(\mu\xi)+2\alpha\mu^2\tan^{-3}(\mu\xi)+2\alpha\mu^2\tan^{-1}(\mu\xi)=0 \qquad (3.2.6)$$

Equation (3.2.6) is satisfied only if the following system of algebraic equations holds:

$$-c\alpha-3\alpha^2\mu+2\alpha\mu^2=0 \qquad (3.2.7)$$

$$-3\alpha^2\mu+\alpha^3+2\alpha\mu^2=0 \qquad (3.2.8)$$

Solving Eqs. (3.2.7) and (3.2.8) we find the following sets of solutions:

$$\alpha=i\sqrt{c}\ ,\ \ \mu=i\sqrt{c}\ \text{ or }\ \frac{i\sqrt{c}}{2} \qquad (3.2.9)$$

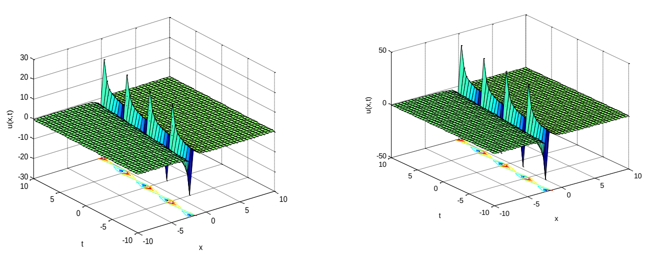
and $\alpha=-i\sqrt{c}\ ,\ \mu=-i\sqrt{c}$ (3.2.10)

Substituting Eqs. (3.2.9) and (3.2.10) into Eq. (2.3), we obtain the exact solution of the STO equation in the form

$$u(x,t)=\sqrt{c}\tanh^{-1}\left[\sqrt{c}(x-ct+d)\right] \qquad (3.2.11)$$

and $u(x,t)=\sqrt{c}\tanh^{-1}\left[\frac{\sqrt{c}}{2}(x-ct+d)\right]$ (3.2.12)

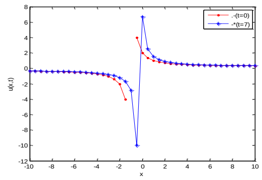
Solution graphs via expression from Eqs. (3.2.11) and (3.2.12) are shown in the figure 3(a) and figure 3(b) respectively for $c=0.1, d=1, -10\le x\le 10$ and $-10\le t\le 10$.

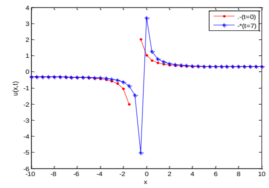


(a) (b)

Figure 3: Graphical representation of the travelling wave solution of $u(x,t)$ for $c=0.1, d=1, -10\le x\le 10$ and $-10\le t\le 10$.

Line diagrams of expression from Eqs. (3.2.11) and (3.2.12) are shown in the figure 4(a) and figure 4(b) respectively for $c=0.1, d=1, -10\le x\le 10$ and $t=0,7$.





(a) **(b)**

Figure 4: Line diagrams of $u(x,t)$ via expression of Eqs. (3.2.11) and (3.2.12) for $c=0.1, d=1$ and $-10 \le x \le 10$.

mKdV equation with additional first order dispersion term

Consider the mKdV equation with additional first order dispersion term

$$u_t - au^2u_x + u_{xxx} + bu_x = 0 \tag{3.3.1}$$

To solve the Eq. (3.3.1) by the tan cot method we introduce the transformation

$u(x,t) = f(\xi)$, where $\xi = x - ct + d$, c is the velocity of the travelling wave and d is a constant.

By using this transformation Eq. (3.3.1) becomes

$$-c\frac{df}{d\xi} - a\left(f(\xi)\right)^2\frac{df}{d\xi} + \frac{d^3f}{d\xi^3} + b\frac{df}{d\xi} = 0 \tag{3.3.2}$$

Integrating Eq. (3.3.2) with respect to ξ and ignoring constant of integration, we obtain

$$(b-c)f(\xi) + \frac{a}{3}\left(f(\xi)\right)^3 + \frac{d^2f}{d\xi^2} = 0 \tag{3.3.3}$$

Seeking the solution in Eq. (2.6)

$$\begin{aligned}&(b-c)\alpha\tan^{\beta}(\mu\xi) + \frac{a}{3}\alpha^3\tan^{3\beta}(\mu\xi) + \alpha\beta(\beta-1)\mu^2\tan^{\beta-2}(\mu\xi)\\&+2\alpha\beta^2\mu^2\tan^{\beta}(\mu\xi) + \alpha\beta(\beta+1)\mu^2\tan^{\beta+2}(\mu\xi) = 0\end{aligned} \tag{3.3.4}$$

Equating the exponents 3β and $\beta-2$ yield

$$3\beta = \beta - 2 \quad \text{i.e.,} \quad \beta = -1$$

Replacing β by -1 into Eq. (3.3.4) we get

$$\begin{aligned}&(b-c)\alpha\tan^{-1}(\mu\xi) + \frac{a}{3}\alpha^3\tan^{-3}(\mu\xi) + 2\alpha\mu^2\tan^{-3}(\mu\xi)\\&+2\alpha\mu^2\tan^{-1}(\mu\xi) = 0\end{aligned} \tag{3.3.5}$$

Equation (3.3.5) is satisfied only if the following system of algebraic equations holds:

$$(b-c)\alpha + 2\alpha\mu^2 = 0 \tag{3.3.6}$$

$$\frac{a}{3}\alpha^3 + 2\alpha\mu^2 = 0 \tag{3.3.7}$$

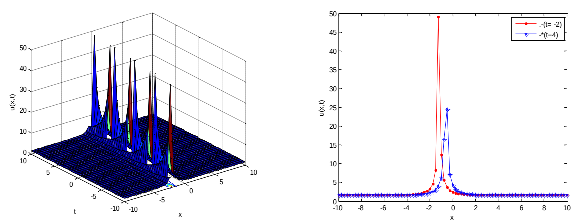
Solving Eq. (3.3.6) and Eq. (3.3.7) we obtain

$$\mu = \pm\sqrt{\frac{(c-b)}{2}}\ ,\ \alpha = \pm\sqrt{\frac{3(b-c)}{a}} \tag{3.3.8}$$

Substituting Eq. (3.3.8) into Eq. (2.3), we obtain the exact soliton solution of the Eq. (3.3.1) equation in the form

$$u(x,t) = \pm i\sqrt{\frac{3(b-c)}{a}}\tanh^{-1}\left[\pm\sqrt{\frac{(b-c)}{2}}(x-ct+d)\right] \tag{3.3.9}$$

Evolutional profile and line diagram of $u(x,t)$ through the expression of Eq. (3.3.9) are shown in the figure 5 (a) and figure 5(b) for $c=0.1, a=b=d=1, -10\le x\le 10$, $-10\le t\le 10$ and $c=0.1, a=b=d=1, -10\le x\le 10$, $t=-2,4$ respectively.



(a) **(b)**

Figure 5: Evolutional profile and line diagram of $u(x,t)$ through the expression of Eq. (3.3.9)

KdV equation with additional fifth order dispersion

We consider the fifth-order KdV equation of the form

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0 \tag{3.4.1}$$

where $u(x,t)$ is differentiable sufficiently.

Equation (3.4.1) is a special case of the standard fifth-order KdV equation

$$u_t + \alpha u^2u_x + \beta u_xu_{xx} + \gamma uu_{xxx} + u_{xxxxx} = 0 \tag{3.4.2}$$

This special case Eq. (3.4.1) is known as the Lax case [1], which is characterized by $\beta = 2\gamma$ and $\alpha = \frac{2}{10}\gamma^2$.

The Eq. (3.4.1) can be written as

$$u_t + 10(u^3)_x + 10(uu_{xx})_x + 5\{(u_x)^2\}_x + u_{xxxxx} = 0\,, \tag{3.4.3}$$

that can be converted to the ODE

$$-cf(\xi) + 10\big(f(\xi)\big)^3 + 10f(\xi)\frac{d^2 f}{d\xi^2} + 5\left(\frac{df}{d\xi}\right)^2 + \frac{d^4 f}{d\xi^4}\,, \tag{3.4.4}$$

upon using the transformation

$u(x,t) = f(\xi)$, where $\xi = x - ct + d$, c is the velocity of the travelling wave, d is a constant and integrating once.

Seeking the solutions in Eqs. (2.5), (2.6) and (2.8)

$$\begin{aligned}
&-c\alpha\tan^{\beta}(\mu\xi) + 10\alpha^3\tan^{3\beta}(\mu\xi) + 10\alpha^2(\beta-1)\beta\mu^2\tan^{2\beta-2}(\mu\xi)\\
&+20\alpha^2\beta^2\mu^2\tan^{2\beta}(\mu\xi) + 10\alpha^2\beta(\beta+1)\mu^2\tan^{2\beta+2}(\mu\xi)\}\\
&+5\alpha^2\beta^2\mu^2\tan^{2\beta-2}(\mu\xi) + 10\alpha^2\beta^2\mu^2\tan^{2\beta}(\mu\xi)\\
&+5\alpha^2\beta^2\mu^2\tan^{2\beta+2}(\mu\xi) + \alpha\beta\mu^4[(\beta-3)(\beta-2)(\beta-1)\tan^{\beta-4}(\mu\xi)\\
&+4(\beta-1)(\beta^2-2\beta+2)\tan^{\beta-2}(\mu\xi) + 2\beta(3\beta^2+5)\tan^{\beta}(\mu\xi) +\\
&4(\beta+1)(\beta^2+2\beta+2)\tan^{\beta+2}(\mu\xi)\\
&+(\beta+3)(\beta+2)(\beta+1)\tan^{\beta+4}(\mu\xi)] = 0
\end{aligned} \tag{3.4.5}$$

Equating the exponents 3β and $\beta-4$ yield

$$3\beta = \beta - 4 \quad \text{i.e.,} \quad \beta = -2$$

Replacing β by -2 into Eq. (3.4.5) we get

$$\begin{aligned}
&-c\alpha\tan^{-2}(\mu\xi) + 40\alpha^2\mu^2\tan^{-2}(\mu\xi) + 136\alpha\mu^4\tan^{-2}(\mu\xi)\\
&+120\alpha^2\mu^2\tan^{-4}(\mu\xi) + 240\alpha\mu^4\tan^{-4}(\mu\xi)\\
&+10\alpha^3\tan^{-6}(\mu\xi) + 80\alpha^2\mu^2\tan^{-6}(\mu\xi)\\
&+120\alpha\mu^4\tan^{-6}(\mu\xi) = 0
\end{aligned} \tag{3.4.6}$$

Setting the coefficients of each $\tan^i(\mu\xi)$ to zero gives the system

$$\begin{aligned}
&-c\alpha + 40\alpha^2\mu^2 + 136\alpha\mu^4 = 0\\
&120\alpha^2\mu^2 + 240\alpha\mu^4 = 0\\
&10\alpha^3 + 80\alpha^2\mu^2 + 120\alpha\mu^4 = 0
\end{aligned} \tag{3.4.7}$$

Solving the system Eq. (3.4.7) leads the following sets of solutions:

$$\alpha = \sqrt{\frac{c}{14}}\;,\;\; \mu = \pm i\,\sqrt[4]{\frac{c}{56}} \tag{3.4.8}$$

and

$$\alpha = -\sqrt{\frac{c}{14}}\;,\;\; \mu = \pm\,\sqrt[4]{\frac{c}{56}} \tag{3.4.9}$$

Substituting Eq. (3.4.7) into Eq. (2.3), we obtain the exact solution of the Eq. (3.4.1) equation in the form

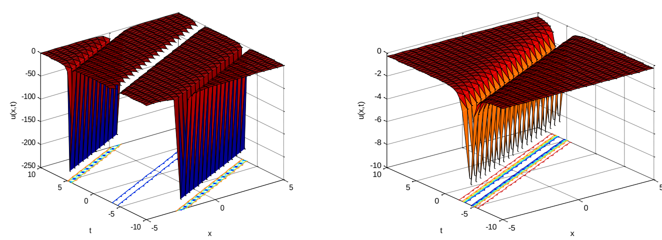
$$u(x,t) = -\sqrt{\frac{c}{14}}\tanh^{-2}\left[\sqrt[4]{\frac{c}{56}}(x-ct+d)\right] \tag{3.4.10}$$

Solution graph via expression of Eq. (3.4.10) is shown in the figure 5.1 for $c=1, -5 \le x \le 5$ and $-10 \le t \le 10$.

Again, Substituting Eq. (3.4.9) into Eq. (2.3), we obtain the exact solution of the Eq. (3.4.1) in the form

$$u(x,t) = -\sqrt{\frac{c}{14}}\tan^{-2}\left[\pm\sqrt[4]{\frac{c}{56}}(x-ct+d)\right] \tag{3.4.11}$$

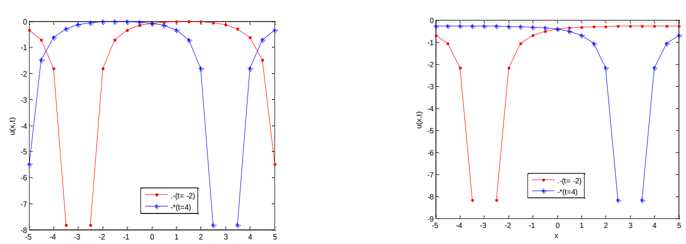
Evaluation profiles via expression of Eqs. (3.4.10) and (3.4.11) are shown in the figure 6(a) and figure 6(b) respectively for $c=d=1, -5 \le x \le 5$ and $-10 \le t \le 10$.



(a) **(b)**

Figure 6: Graphical representation of the travelling wave solution of $u(x,t)$ for $c=d=1, -5 \le x \le 5$ and $-10 \le t \le 10$.

Line diagrams of expression of Eqs. (3.4.10) and (3.4.11) are shown in the figure 7(a) and figure 7(b) respectively for $c=d=1, -5 \le x \le 5$ and $t=-2,4$.



(a) **(b)**

Figure 7: Line diagram of $u(x,t)$ via expression of Eqs. (3.4.10) and (3.4.11) for $c=d=1$ and $-5 \le x \le 5$.

IV. Conclusions

In our present attempt, the Tan-Cot Function method has been successfully implemented to find the exact solutions of different nonlinear PDEs including solitons, solitary waves and travelling waves. The investigation presented here has been an advanced nonlinear wave mode analysis and this investigation predicts unique findings on nonlinear wave solutions.

As a consequence, it is stressed that this direct and effective algebraic method can be extended to solve the problems of nonlinear structures which arising in the theory of solitons and other nonlinear science areas.

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