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ON THE SOLVABILITY OF A CLASS OF NONLINEAR FUNCTIONAL EQUATIONS

BY
D. C. Sanyal
(Retired) Professor of Mathematics, University of Kalyani
email: dcs_klyuniv@yahoo.com

Abstract

It is proposed to introduce some simple criteria regarding the existence of unique solutions of a class of nonlinear functional equations in supermetric and metric spaces followed by suitable examples. The results obtained may be of much useful to many physical problems arising nonlinear equations.

Key words: Supermetric space, metric space, functional equation, Hammerstein equation.

1. Introduction

Iterative processes for the solutions of equations of various types are of common use in diversified fields. But the method gets little reflection when dealing with abstract spaces, e.g. supermetric space, metric space, Hilbert space etc. Approximate iterative solution of a class of functional equations and its uniqueness in supermetric space has been given by Sen (1971), Sen and Mukherjee (1983, 1988) and others. On the other hand, the process got no light when the supermetric space is replaced by a generalised space, viz. a metric space.

With this end in view, we intend to introduce some simple criteria regarding the existence of solutions of a class of nonlinear functional equations by the method of approximate iterative process in supermetric and metric spaces followed by suitable examples. It is expected that the results will be fruitful to those dealing with physical problems in which nonlinear equations arise frequently.

2. Some general concepts

A space R is a set of elements f, g, ... In the applications, these elements may be real or complex numbers, vectors, matrices, functions of one or more variables etc. Ordinarily the spaces that occur in the applications are linear and hence we shall consider linear spaces only.

A linear space is characterized by the following properties:

(1) An operation is defined, which we call addition and this follows the rules of ordinary addition; if f and g are elements of R, then $f + g \in R$ and $\exists$ a null element $\theta \in R$ s.t. $\theta + f = f, \forall f \in R$.

(2) A multiplication is defined between the elements f and scalars c of a field F. This multiplication obeys the rule of ordinary vector algebra, i.e. if $f \in R$ and $c \in F$, then $cf \in R$. The field F is usually the field of rational, real or complex numbers.

Now we consider **transformations** (**operations, mapping**) T which associate certain elements *f* of the original space R uniquely with elements h of a linear space R^*, called the image space. It will frequently be the case that R = R^*. In the case when R^* is a number space, T associates each *f* with a number and T is called a real functional.

An operator T is called **linear** when T is defined $\forall f \in R$ and when

$$T(c_1 f_1 + c_2 f_2) = c_1 T f_1 + c_2 T f_2,$$

holds for all elements $f_1, f_2 \in R$ and $c_1, c_2 \in F$. In all other cases T is called a nonlinear operator.

3. Solutions of equations

Let the unknown quantity u be an element of a given linear space R. T and S are given linear or nonlinear operators. Three types of equations can be distinguished:

(i) $Tu = u$, (1)

(ii) $Su = \theta$, (2)

(iii) $Tu = \lambda u$ (3)

where the image elements are in the same space R, θ belongs to the image space, $u \neq \theta$ and $\lambda \in F$.

Equation (2) is the most general one since it contains (1) and usually (3) as special cases. Let I be the identity operator which maps each element into itself. Then equation (1) is seen to be of the form (2) by putting $S = T - I$.

The eigenvalue problem (3) consists in finding those values of λ, the eigenvalue, for which there exist vectors u, different form θ and which satisfy equation (3). Here T is frequently assumed to be linear. We assume further that it is possible to satisfy a normalisation condition

$$Gu = 1 \tag{4}$$

where G is a given functional with a value different form unity when operating on the null element, $G\theta \neq 1$.

Example 1 : Consider the eigenvalue problem

$$y''(x) = \lambda y(x),\ y(0) = y(\pi) = 0.$$

For the functional G, we may choose, for example, Gu = u(1) or Gu = $[\int_0^x u^2 dx]^{\frac{1}{2}}$. In order to show that (3) is indeed equivalent to (2), we consider pairs v, a or in vector notation $\binom{v}{a}$ with $v \in R$, $a \in F$, these pairs being elements of a new space R_1. A new transformation T_1 for the elements $\binom{v}{a}$ is defined by

$$T_1\binom{v}{a} = \binom{Tv - av}{Gv - 1} \tag{5}$$

Let the null element be $\theta_1 = \binom{\theta}{0}$ of the space R_1. Then

$$T_1\binom{v}{a} = \theta_1 \tag{6}$$

which is equivalent to (3) and has the form (2).

Conversely, we find: if under the transformation (2) the original and image elements lie in the same space R, equation (2) assumes the form (1) if we put S = T – I. If the original space R and the image space R^* are different, the existence of a linear one-to-one mapping L or R onto R^* is sufficient to transform (2) into (1) by means of T = $L^{-1}(L - S) = I - L^{-1}S$. It is often of advantage for numerical purposes to choose L in such a manner that L – S is nearly constant in the neighbourhood of a solution. Equation (1) can be seen as a special case of (3).

4. Supermetric Space

A metric linear space R is supermetric (Collatz, 1966) when the distance ρ satisfies the relation

$$\rho\,(f_1\,,\,f_2 + f_3) = \rho\,(f_1 + f_2\,,\,f_3), \text{ for any three elements } f_1, f_2, f_3 \in R \tag{7}$$

Example 2: The space R of all complex numbers z with the distance

$$\rho\,(z_1,\ z_2) = \frac{|z_1 - z_2|}{1 + |z_1 - z_2|}$$

is supermetric. For, if $z_1,\ z_2\ ,\ z \in R$, then

$$\rho\,(z_1,\ z_2 + z_3) = \frac{|z_1 - (z_2 + z_3)|}{1 + |z_1 - (z_2 + z_3)|}$$

$$= \frac{|(z_1 - z_2) - z_3|}{1 + |(z_1 - z_2) - z_3)|}$$

$$= \rho(z_1 - z_2,\ z_3).$$

Example3 : In complex variable theory the chordal distance $\chi\ (f,\ g)$ of two complex numbers f, g is defined by

$$\chi\,(f, g) = \frac{|f - g|}{[(1 + |f|^2)\,(1 + g^2)\,]^{\frac{1}{2}}}\ ,$$

f and g may be infinite and it is always true that $\chi\ (f, g) < 1$.

The chordal distance is an example of a space which is metric, but not supermetric, since for $f_1 = f_2 = f_3$ the distances

$$\rho\ (1,\ 2) = \chi\,(1, 2) = \frac{1}{\sqrt{10}} \quad \text{and} \quad \rho\ (0,\ 1) = \chi\,(0, 1) = \frac{1}{\sqrt{2}}$$

are different.

The following inequality holds in any supermetric space:

$$\rho\ (f_1 + f_2,\ f_3 + f_4) \leq \rho\ (f_1\ ,\ f_3) + \rho\ (f_2,\ f_4) \ \text{ for } f_i \in R, \tag{8}$$

since the L.H.S. is equal to

$$\rho\ (f_1,\ f_3 + f_4 - f_2) \leq \rho\ (f_1\ ,\ f_3) + \rho\ (\ f_3,\ f_3 + f_4 - f_2\ )$$

$$= \rho\ (f_1\ ,\ f_3) + \rho\ (\ f_2,\ f_4).$$

In the supermetric space, the norm of an element is defined as

$$\|f\| = \rho\ (f, \theta).$$

This norm satisfies the relations

$$\|f\| = 0 \ \text{ iff } \ f = \theta,$$

$$\|f + g\| \leq \|f\| + \|g\|\ ,$$

$$|\ \|f\| - \|f\|\ | \leq \|f - g\|.$$

If the norm satisfies a homogeneity condition, the space becomes a normed space. Supermetric spaces which occur in the applications often are normed spaces. The concept of a supermetric space is used to demonstrate where in the applications the concept of distance suffices and where a norm with the homogeneity property is necessary.

Condition number of a linear, bounded operator

Let T be a linear bounded operator and inverse (linear) operator T^{-1} exists and is bounded also. Let T and T^{-1} be defined in normed space. Then

$$\kappa(T) = \|T\|\|T^{-1}\|$$

is called the condition number of the operator T.

Theorem 3.1: (Collatz, 1966)Let T satisfies the assumptions in the definition above and let v be an approximation to a solution u of the operator equation Tu = r with r $\neq \theta$. Using the condition number $\kappa(T)$ and the defect d = Tv – r of the approximation v, we can find bounds for the 'relative error' $\rho(u, v)\|u\|$:

$$\frac{1}{\kappa(T)} \frac{\rho(Tu, r)}{\|r\|} \leq \frac{\rho(u, v)}{\|u\|} \leq \kappa(T) \frac{\rho(Tv, r)}{\|r\|}. \qquad (9)$$

Moreover, if bounds for $\|u\|$ are known, the norm $\rho(u, v)$ of the error f = v – u can also be bounded from above and below.

From (8), there follows immediately.

Theorem 3.2: (Collatz, 1966) The assumptions of the previous theorem imply:

The condition number $\kappa(T)$ of the operator T is $\kappa(T) \geq 1$.For $\kappa(T) = 1$, we have

$$\frac{\rho(Tu, r)}{\|r\|} = \frac{\rho(u, v)}{\|u\|} \text{ (for r} \neq \theta). \qquad (10)$$

Error estimated for an iteration process

Let T, T_1, and T_2 be linear, bounded (and hence continuous) operators with a common domain D $\subseteq R$ and range in R; R itself is assumed to be linear and supermetric. Let T = $T_1 + T_2$. In the operator equation

$$X = Tx + r, r \in R \qquad (11)$$

Theorem 3.3: (Collatz, 1966) Under the assumptions about the operator T, we use for the operator equation (11) the following general iterative method:

$$x_{n+1} = T_1\, x_{n+1} + T_2\, x_n + r, \; (n = 0, 1, 2, 3, \dots) \qquad (12)$$

Starting with an element $x_0 \in D$; if there is a solution of (11) and if $\|T\| \le \kappa < 1$, then the following error estimate holds:

$$\rho(x_{n+1}, x) \le \frac{\|T_2\|}{1 - \|T\|}\rho(x_{n+1}, x_n).$$

5. Solvability of nonlinear functional equations in supermatric space

In solving the physical problems, the presence of more than one solution sometimes creates difficulty to find a true solution of the problem and thus there is lack of agreement of the solution with experimental results. Sen (1971) and Sen and Mukherjee (1983, 1988) devised some methods to overcome such difficulties.

In the following, we proceed to reduce the constraints occurred in the above works and improve the results to a great extent.

Consider a nonlinear complete supermetric space R and f be a number of R. Let A is a nonlinear mapping R into R. Our problem is to solve the nonlinear equation of the type

$$u = Au + f, \; f \in R. \qquad (13)$$

We consider the iterate of the form

$$u_{n+1} = A\, u_n + f, \; (n = 0, 1, 2, \dots\dots\dots\dots\dots\dots) \qquad (14)$$

where u_0 is prechosen.

Theorem 4.1: Let the following conditions be fulfilled:

L is the bounded linear operator mapping R into R such that

(i) $\rho(Au, Av) \le \rho(Lu, Lv), \forall\, u, v \in R$

(ii) $\rho(L^m u, L^m v) \le \alpha\rho(u, v), \forall\, u, v \in R$ and $0 < \alpha < 1$ for fixed m.

(iii)f belongs to the range of $I - A$.

Then the sequence of iterates $\{x_n\}$ defined by (14) converges uniquely to the solution u^* of (13)

Proof. Let u^* be the solution of (13). Then $u^* = Au^* + f$.

The space being supermetric, we have

$$\rho(u_{n+1}, u^*) = \rho(Au_m + f, u^*)$$
$$= \rho(Au_m + f, Au^* + f)$$
$$= \rho(Au_m , Au^*) \text{ (}\because \text{ the space is supermetric)}$$
$$\leq \rho(Lu_m , Lu^*) \text{ (by (i))}$$
$$\leq \alpha\rho(u_m, u^*)$$
$$\leq \alpha^m \rho(u_0 , u^*)$$
$$\to 0 \text{ as m} \to \infty \ (\because 0 < \alpha < 1).$$

To prove the uniqueness, we suppose that v^* is another solution of (13). Then

$$\rho(u^*, v^*) = \rho(Au^* + f, Av^* + f)$$
$$= \rho(Au^*, Av^*)$$
$$\leq \rho(Lu^*, Lv^*) \text{ (by (i))}$$
$$\leq \alpha\rho(u^*, v^*)$$
. . . .
$$\leq \alpha^m \rho(u^*, v^*)$$
$$\to 0 \text{ as m} \to \infty \ (\because 0 < \alpha < 1).$$

Hence $u^* = v^*$.

Thus if *f* belongs to the range of *I* – *A*, the sequence of iterates $\{u_n\}$ converges uniquely to the solution u^* of (13).

Example: Let us consider Hammerstein equation

$$u(x) = 1 + \int_0^1 |x - t| \left[u(t) - \frac{1}{2} u^2(t)\right] dt \qquad (15)$$

in C(0, 1). Using the theory of monotonically decomposable operator, (Collatz , 1966) proved the existence of a unique solution u(x) of (15) provided

$$\frac{1}{2(x - x^2)} \leq u(x) \leq 2(1 - x - x^2).$$

Let $R = \left[\frac{1}{2(x - x^2)} \leq u(x) \leq 2(1 - x - x^2)\right]$.

Here $Au = \int_0^1 |x - t| \left[u(t) - \frac{1}{2}u^2(t)\right] dt$.

We choose $Lu = \int_0^1 |x-t| u(t)dt$ and

$$\rho\,(u, v) = \|u - v\| = \max_{0 \le x \le 1} |u(x) - v(x)|, \forall\, u(x), v(x) \in C(0,1).$$

It is also supposed that the metric in R is induced by the metric in C(0, 1) and is complete in R w. r. t. the induced metric so that

$$Au - Av = \int_0^1 |x-t| \left[u(t) - \tfrac{1}{2}u^2(t)\right] dt - \int_0^1 |x-t| \left[v(t) - \tfrac{1}{2}v^2(t)\right] dt$$

$$= \int_0^1 |x-t|\, [u(t) - v(t)]dt - \tfrac{1}{2}[u(\xi) + v(\xi)] \int_0^1 |x-t|\, u(t)dt$$

$$- \frac{1}{2}[u(\eta) + v(\eta)] \int_0^1 |x-t|\, v(t)dt$$

where $0 < \xi < 1$, $0 < \eta < 1$.

Now

$$\max_{0 \le x \le 1} \left|1 - \tfrac{u(x) + v(x)}{2}\right| \le 1, \forall\, u(x), v(x) \in R,$$

$$\rho(Lu,\ Lv) = \max_{0 \le x \le 1} |u(\bar{\xi}) - v(\bar{\eta})| \int_0^1 |x-t|\, dt,\ 0 < \bar{\xi} < 1,\ 0 < \bar{\eta} < 1,$$

$$\int_0^1 |x-t| v(t)dt \le \frac{v(\bar{\eta})}{u(\bar{\xi}) - v(\bar{\eta})} \rho(Lu,\ Lv)$$

Neglecting the quantities of second and higher orders in ξ and η, we get

$$|Au - Av| \le \left[1 - \tfrac{1}{2}\{u(\xi) + v(\eta)\} + v(\bar{\eta})\right] \rho(Lu,\ Lv)$$

Also $\left[1 - \tfrac{1}{2}\{u(\xi) + v(\eta)\} + v(\bar{\eta})\right] = 1 - 2\,(\xi - \xi^2) + 2\,(1 - \bar{\eta} + \bar{\eta}^2) \approx 1,$

so that $\rho(Au,\ Av) \le \rho(Lu,\ Lv), \forall\, u, v \in R$

and by Schwartz inequality, we have

$$\rho(L^2 u,\ L^2 v) \le \tfrac{1}{3}\rho(u,\ v).$$

Hence by the use of the above theorem, the sequence $\{u_n\}$ defined by

$$u_{n+1} = 1 + \int_0^1 |x-t| \left[u_n(t) - \tfrac{1}{2}\, {u_n}^2(t)\right] dt, \text{ (n = 0, 1, 2, ...)}$$

converges to the unique solution of the equation (15) in R.

6. Solvability of nonlinear functional equations in metric space

In this section, we propose to consider the solvability of a class of nonlinear functional equations of the type $Au = Pu$ (16)

in a complete metric space R, where u $\in R$, A and P are two nonlinear onto and into self-mapping of R. To do this, we construct the iterative sequence $\{u_n\}$ such that

$$\mathrm{A}u_{i+1} = Pu_i, \text{ (i = 0, 1, 2, ...)} \tag{17}$$

u_0 being prechosen.

We also consider the existence and uniqueness of the solution of the equations of the form

$$A^n u = P^m u, u \in R, \tag{18}$$

m and n being positive integers.

To prove this, we use Kannan's theorem (1968) for the existence of a fixed point of an operator as given below:

Theorem 5.1(Kannan, 1968)**:** T is a map of complete metric space X into itself and

$$\rho(Tx, Ty) \le \nu[\rho(x, Tx) + \rho(y, Ty)], \tag{19}$$

where x, y $\in X$ and $0 < \nu < \frac{1}{2}$. Then T has a unique fixed point.

Theorem 5.2: Let the following conditions are fulfilled for all u, v $\in R$:

(i) $\rho(Au, Av) \ge \alpha\rho(u, v)$, $\alpha > 1$,

(ii) $\rho(Pu, Pv) \le \beta(u, v)$,

(iii) $0 < 3\beta < \alpha$.

Then the sequence of iterates $\{u_n\}$ defined by (17) converges to the unique solution of the equation

$$Au = Pu, u \in R.$$

The error estimate is given by

$$\rho(u_i, u^*) \le \nu \left(\frac{\nu}{1-\nu}\right)^{i-1} \rho(u_0, A^{-1}Pu_0), \tag{20}$$

where $\nu = \frac{\beta}{\alpha - \beta}$.

Proof: The existence of A^{-1}, its boundedness and continuity follow from (i). Thus the sequence $\{u_n\}$, where

$$u_n = A^{-1}Pu_{n-1}, \text{ (n = 1, 2, 3, ...)} \tag{21}$$

u_0 being prechosen, is well defined.

Now

$$\rho(u, \ v) \leq \frac{1}{\alpha}\rho(Au, \ Av) \text{ (by (i))}$$

or, $\rho(A^{-1}Pu, \ \ A^{-1}Pv) \leq \frac{1}{\alpha}\rho(Au, \ Av) = \frac{1}{\alpha}\rho(Pu, \ Pv)$ (by equation (16))

$\leq \frac{\beta}{\alpha}\rho(u, \ v)$ (by (ii)) (22)

$\leq \frac{\beta}{\alpha}[\rho(u, \ A^{-1}Pu) + \rho(A^{-1}Pu, v)]$

$\leq \frac{\beta}{\alpha}[\rho(u, \ A^{-1}Pu) + \rho(A^{-1}Pu, A^{-1}Pv) + \rho(A^{-1}Pv, v)]$

or, $\left(1 - \frac{\beta}{\alpha}\right) \rho(A^{-1}Pu, \ A^{-1}Pv) \leq \frac{\beta}{\alpha}[\rho(u, \ A^{-1}Pu) + \rho(A^{-1}Pv, v)]$

i.e. $\rho(A^{-1}Pu, \ A^{-1}Pv) \leq \nu\,[\rho(u, \ A^{-1}Pu) + \rho(A^{-1}Pv, v)]$, (23)

since $0 < 3\beta < \alpha$, so $0 < \nu < \frac{1}{2}$. Thus by Theorem 5.1, $A^{-1}P$ will have a unique fixed point u^* (say)

To find the error estimate, we have

$\rho(u_i, u^*) = \rho(A^{-1}Pu_{i-1}, A^{-1}Pu^*)$ (by (21))

$\leq \nu\,[\rho(u_{i-1}, \ A^{-1}Pu_{i-1}) + \rho(u^*, A^{-1}Pu^*)]$ (by (23))

$= \nu\,[\rho(u_{i-1}, \ A^{-1}Pu_{i-1}) + \rho(u^*, u^*)]$

$= \nu\,\rho(u_{i-1}, \ A^{-1}Pu_{i-1})$

$= \nu\,\rho(A^{-1}Pu_{i-2}, \ A^{-1}Pu_{i-1})$ (by (21))

$= \nu\,\frac{\beta}{\alpha}\rho(u_{i-2}, \ u_{i-1})$ (by (22))

i.e $\rho(u_i, u^*) \leq \nu\left(\frac{\nu}{1-\nu}\right)\rho(u_{i-2}, \ u_{i-1})$

$= \nu\left(\frac{\nu}{1-\nu}\right)\rho(A^{-1}Pu_{i-3}, \ \ A^{-1}Pu_{i-2})$ (by (21))

$\leq \nu\left(\frac{\nu}{1-\nu}\right)^2 \rho(u_{i-3}, \ u_{i-2})$

....

$\leq \nu\left(\frac{\nu}{1-\nu}\right)^{i-1} \rho(u_0, \ u_1)$

$\therefore \rho(u_i, u^*) \leq \nu\left(\frac{\nu}{1-\nu}\right)^{i-1} \rho(u_0, \ \ A^{-1}Pu_0).$ (24)

Since $0 < \nu < \frac{1}{2}$, so $0 < \frac{\nu}{1-\nu} < 1$ and hence $\rho(u_i, u^*) \to$ as $i \to \infty$. Thus the sequence converges to the unique solution u^* and (24) gives the required error estimate.

Theorem 5.3: Let the following conditions be fulfilled in a linear metric space R for all u, v $\in R$:

(i) $\rho\,(Au,\ Av) \geq\ \alpha\rho(u,v)$, $\alpha > 1$,

(ii) $\rho\,(Pu,\ Pv) \leq\ \beta(u,v)$,

(iii) A and P commute,

(iv) $0 < 3\beta < \alpha$.

Then the sequence $\{u_n\}$ defined by

$$A^n u_{i+1} =\ P^m u_i,\ (i = 0, 1, 2, \ldots\ldots), \tag{25}$$

where u_0 being prechosen, n and m are positive and $n \geq m$ converges to the unique solution u^* of the equation

$$A^n u =\ P^m u. \tag{26}$$

The error estimate is given by

$$\rho(u_i, u^*) \leq \frac{1}{\alpha^{i-i}}\left(\frac{\nu}{1-\nu}\right)^m \rho(u_0,\ \ A^{-1}Pu_0). \tag{27}$$

Proof : The existence of A^{-1}, its boundedness and continuity follow from (i). Then the sequence $\{u_n\}$ expressed by (25) gives

$$A^{n-1}u_{i+1} = A^{-1}P^m u_i$$

so that we obtain similarly

$$u_{i+1} = (A^{-1})^n P^m u_i \ \text{ i.e. } \ u_{i+1} = (A^n)^{-1}P^m u_i. \tag{28}$$

Since A^{-1} exists and A commutes with P, so $A^{-1}P = PA^{-1}$ and , therefore, A^{-1} commutes with P. Hence

$$(A^n)^{-1}P^m = (A^{-1})^n P^m = (A^{-1})^p (A^{-1})^m P^m = (A^{-1})^p (A^{-1}P)^m$$

so that

$$(A^n)^{-1}P^m = \begin{cases} (A^{-1})^p (A^{-1}P)^m & p \geq 1 \\ (A^{-1}P)^m & p = 0 \end{cases} \tag{29}$$

Hence

$u_{i+1} = (A^{-1})^p (A^{-1}P)^m u_i$ for p ≥ 1

Now proceeding along the same lines as in Theorem 5.2 it follows that $A^{-1}P$ has a unique fixed point u^* (say), i.e. $u^* = (A^{-1}P)u^*$. We now show that u^* is also the unique fixed point of $(A^{-1}P)^m$. If possible suppose v^* is another fixed point of $(A^{-1}P)^m$ and $u^* \neq v^*$. Then

$$\rho(u^*, v^*) = \rho((A^{-1}P)^m u^*, (A^{-1}P)^m v^*)$$

$$\leq \nu \; [\rho(u^*, (A^{-1}P)^m u^*) + \rho(v^*, (A^{-1}P)^m v^*)] \quad \text{(by Theorem 5.1)}$$

$$= \nu \, [\rho(u^*, v^*) + \rho((A^{-1}P)^m v^*, v^*)]$$

i.e. $\rho(u^*, v^*) \leq \frac{\nu}{1-\nu} \rho((A^{-1}P)^m v^*, (A^{-1}P)^{m-1} v^*)$

$$\leq \left(\frac{\nu}{1-\nu}\right) . \left(\frac{\nu}{1-\nu}\right) \rho((A^{-1}P)^{m-1} v^*, (A^{-1}P)^{m-2} v^*)$$

....

$$\leq \left(\frac{\nu}{1-\nu}\right)^m \rho((A^{-1}P)v^*, v^*)$$

$$= \left(\frac{\nu}{1-\nu}\right)^m \rho(v^*, (A^{-1}P)v^*)$$

$$= \left(\frac{\nu}{1-\nu}\right)^m \rho((A^{-1}P)^m v^*, (A^{-1}P)^{m+1} v^*)$$

$$\leq \left(\frac{\nu}{1-\nu}\right) \left(\frac{\nu}{1-\nu}\right)^m \rho((A^{-1}P)^{m-1} v^*, (A^{-1}P)^m v^*)$$

....

$$\leq \left(\frac{\nu}{1-\nu}\right) \left(\frac{\nu}{1-\nu}\right)^{2m-1} \rho(v^*, (A^{-1}P)v^*)$$

....

$$\leq \left(\frac{\nu}{1-\nu}\right)^{im} \rho(v^*, (A^{-1}P)v^*), \; (i = 1, 2, \ldots)$$

$$\to 0 \text{ as } i \to \infty, \text{ since } 0 < \frac{\nu}{1-\nu} < 1.$$

Hence $u^* = v^*$.

Since A is an onto mapping, A^{-1} exists, is continuous and is also an onto mapping. Furthermore, it follow from (i) that $(A^{-1})^p$ is a contraction mapping and hence has a unique fixed point in R. Also $(A^{-1})^p$ and $(A^{-1}P)^m$ commute and each of them has a unique fixed point, so $(A^{-1})^p (A^{-1}P)^m$ has a unique fixed point u^* (say).

$$\rho(u_i, u^*) = \rho(\ (A^{-1})^p (A^{-1}P)^m u_{i-1},\ \ (A^{-1})^p (A^{-1}P)^m u^*)$$
$$\leq \frac{1}{\alpha} \rho(u_{i-1},\ u^*)$$
$$= \frac{1}{\alpha}\ \rho(\ (A^{-1})^p (A^{-1}P)^m u_{i-2},\ \ (A^{-1})^p (A^{-1}P)^m u^*)$$
$$\leq \frac{1}{\alpha^2} \rho(u_{i-2},\ u^*)$$
$$\cdots\ \ \cdots\ \ \cdots\ \ \cdots$$
$$\leq \frac{1}{\alpha^{i-2}} \rho(u_2,\ u^*)$$
$$= \frac{1}{\alpha^{i-2}}\ \rho(\ (A^{-1})^p (A^{-1}P)^m u_1,\ \ (A^{-1})^p (A^{-1}P)^m u^*)$$
$$\leq \frac{1}{\alpha^{i-1}}\ \rho((A^{-1}P)^m u_1,\ (A^{-1}P)^m u^*)$$
$$\leq \frac{1}{\alpha^{i-1}} \left(\frac{\nu}{1-\nu}\right)\ \rho((A^{-1}P)^{m-1} u_1,\ (A^{-1}P)^{m-1} u^*)$$
$$\cdots\ \ \cdots\ \ \cdots\ \ \cdots\ \ \cdots$$
$$\leq \frac{1}{\alpha^{i-1}} \left(\frac{\nu}{1-\nu}\right)^{m-1}\ \rho\big((A^{-1}P)u_1,\ (A^{-1}P)u^*\big)$$
$$\leq \frac{1}{\alpha^{i-1}} \left(\frac{\nu}{1-\nu}\right)^{m}\ \rho(u_1,\ u^*)$$
$$= \frac{1}{\alpha^{i-1}} \left(\frac{\nu}{1-\nu}\right)^{m}\ \rho\big((A^{-1}P)u_0,\ u^*\big)$$
$$\to 0 \text{ as } i \to \infty$$

This shows that $u_i \to u^*$ as $i \to \infty$ and the error estimate is

$$\rho(u_i, u^*) \leq \frac{1}{\alpha^{i-1}} \left(\frac{\nu}{1-\nu}\right)^{m}\ \rho(A^{-1}Pu_0,\ u^*).$$

Theorem 5.4 : Let R be a linear metric space and the following conditions be fulfilled for all u, v $\in R$:

(i) $\rho\ (Au, Av) \geq\ \alpha\rho(u,v),\ \alpha > 0,$

(ii) $\rho\big((A^{-n}P^m)^{\lambda}u\ ,\ \theta\big) \leq k\rho(u,\theta)$, for all positive integers λ,

(iii) $A^{-n}P^m$ is continuous at its fixed point,

(iv) A and P commute,

(v) P is compact

(vi) $\rho((A^n)^{\nu}u\ ,(P^m)^{\nu}u\ )\ \geq\ \rho(A^n u, P^m u)$ for all finite positive integers ν.

Then the sequence $\{u_n\}$ defined by

$$A^n u_{i+1} = P^m u_i,\ (i = 0, 1, 2, . \ . \ . \ .) \qquad (30)$$

where u_0 being prechosen and n and m (n < m) are positive integers will converge to the unique solution u^* of the equation

$$A^n u = P^m u. \qquad (31)$$

Proof: The sequence $\{u_n\}$ expressed by

$$u_i = A^{-n}P^m u_{i-1} = (A^{-n}P^m)^2 u_{i-2} = . \ . \ . \ . = (A^{-n}P^m)^i u_0, \text{ (i = 1, 2, . . .)}$$

is well defined. Let us denote $A^{-n}P^m$ by Q. Since R is a linear metric space, so $\theta \in R$.

Now by condition (ii)

$$\rho(u_i, \theta) = \rho\big((A^{-n}P^m)^i u_0, \theta\big) \ \leq k\ \rho(u_0, \theta)$$

which implies that $\{u_i\}$ is bounded.

Since P is compact and $\{u_i\}$ is bounded, $\{Pu_i\}$ is sequentially compact and hence bounded. Thus $\{P(Pu_i)\}$ is again compact so that P^2 is compact. In general P^m is compact with m positive integer. Also $\{u_i\}$ and A^{-1} being bounded, $\{(A^n)^{-1}u_i\}$ is also bounded. Moreover, P^m is compact and A^{-1} commutes with P^m and so $\{(A^n)^{-1}P^m u_i\}$ is compact. Thus the sequence $\{u_i\}$ defined by u_i = $A^{-n}P^m u_{i-1}$ = Qu_{i-1}, (i = 1, 2, ...) contains a convergent subsequence $\{u_{i_p}\}$, (say).

Let $u_{i_p} \to u^*$ as p $\to \infty$ and since $u_{i_p} = Q^k u_{i_{p-1}}$ for some integer k, so $Q^k u_{i_{p-1}} \to u^*$ as p $\to \infty$ for finite k.

Again, since $(A^n)^{-1}$ and P^m commute, so

$$Q^k u_{i_{p-1}} = (A^{-n}P^m)^k u_{i_{p-1}} = (A^{-n})^k u_{i_p}$$

and A^{-n} being continuous

$$\lim_{p \to \infty} (A^{-n})^k u_{i_p} = (A^{-n})^k u^* = u^*.$$

Hence

$$\lim_{p \to \infty} (A^{-n}P^m)^k u_{i_{p-1}} = \lim_{p \to \infty} (A^{-n})^k u_{i_p} = u^*$$

and so $Q^k u^* = u^*$.

Thus u^* is a solution of $(A^n)^k$u = $(P^m)^k$u and, therefore, by virtue of the condition (vi), u^* is also a solution of $A^n u = P^m u$.

Now Q being continuous at its fixed point,

$$\lim_{p \to \infty} u_{i_{p+1}} = \lim_{p \to \infty} Qu_{i_p} = Qu^* = u^*.$$

Hence the sequence $\{u_i\}$ converges to u^*, a solution of the equation (31).

The uniqueness of the solution follows in the usual way.

Example : Let $u(x) \in C(0,1)$;

$Au = u^2(x) + 2(x + 15)u(x) - 1.5$, $D(A) = \{u(x): 0.05 \leq u(x) \leq 1.5\}$;

$P(u) = 7 \int_0^1 |x - t| \left[u(t) - \frac{1}{8} u^2(t)\right] dt$, $D(P) = \{u(x): 0.06 \leq u(x) \leq 0.13\}$.

We propose to test the solvability of the integral equation *Au = Pu.*

Solution : We have

$$\begin{aligned} Pu &= 7 \int_0^1 |x - t| \left[u(t) - \tfrac{1}{8} u^2(t)\right] dt \\ &\geq 7 \int_0^1 |x - t| \left[(0.06) - \tfrac{(0.13)^2}{8}\right] dt \\ &= 0.203(1 - 2x + 2x^2) \end{aligned}$$

and since $\min_{0 \leq x \leq 1} (1 - 2x + 2x^2) = 0.5$,

so $Pu \geq 0.102$ for $u \in D(P)$.

Again,

$$\begin{aligned} Pu &= 7 \int_0^1 |x - t| \left[u(t) - \tfrac{1}{8} u^2(t)\right] dt \\ &\leq 7 \int_0^1 |x - t| \left[(0.13) - \tfrac{(0.06)^2}{8}\right] dt \\ &= 0.455(1 - 2x + 2x^2) \end{aligned}$$

and since $\max_{0 \leq x \leq 1} (1 - 2x + 2x^2) = 1.5$,

so $Pu \leq 0.682$ for $u \in D(P)$.

Thus $0.102 \leq Pu \leq 0.682, \forall u \in D(P)$ and hence $D(A) \supseteq D(P)$.

We introduce in D(A) the $L_2(0,1)$ norm (i.e. $\|u\|^2 = \int_0^1 u^2 dx$ for all $u \in D(A)$) and *D(A)* is complete w.r.t. the above norm. Now *D(A)* is a subspace of $L_2(0,1)$ and so we introduce the scalar product

$$(u, v) = \int_0^1 uv\, dx, \forall u, v \in D(A)$$

and $\|u\|^2 = (u, v)$.

On the choice of the metric $\rho(u,v) = \|u-v\|$, $\forall\, u,v \in D(A)$, *D(A)* becomes a complete metric space.

Since A is continuous, so *D(A)* is closed.

Now for all u $\in D(A)$

(Au – Av, u – v) = 2

$$\int_0^1 (x+15)(u-v)^2dx + \int_0^1 (u+v)(u-v)^2dx$$

$$\geq 30 \int_0^1 (u-v)^2dx + \int_0^1 (u+v)(u-v)^2dx$$

$$= 30 \int_0^1 (u-v)^2dx + [u(\xi) + v(\xi)] \int_0^1 (u-v)^2dx,$$

$0 < \xi < 1$

$$\geq (30 + 2\ 0.05)\|u-v\|^2$$

$$= 30.1\|u-v\|^2$$

Thus we have $\alpha = 30.1$.

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