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An Extension of the Krylov-Bogoliubov-Mitropolskii (KBM) Method for Third Order Critically Damped Nonlinear Systems

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Abstract.

Krylov-Bogoliubov-Mitropolskii (KBM) method has been extended and applied to certain third order non-oscillatory nonlinear systems characterizing critically damped systems. For different set of initial conditions as well as for different eigenvalues the solutions obtained by the extended KBM method show good coincidence with those obtained by the numerical method. The method is illustrated by an example.

সংক্ষিপ্তসার

উচ্চতর ক্রাইলভ - বোগোলিউভব - মিট্রোপোলস্কি পদ্ধতিটি সম্প্রসারিত করা হয়েছে এবং কোন তৃতীয় ক্রমের অ-দোদুল্যমান অ-রৈখিক তন্ত্রকে ক্রান্তিক অবমন্দিত তন্ত্র-বর্গের বৈশিষ্ট্য রূপে ধরে ইহাতে প্রয়োগ করা হয়েছে। সম্প্রসারিত KBM পদ্ধতির দ্বারা প্রারম্ভিক শর্তযুক্ত বিভিন্ন সেটের জন্য এবং বিভিন্ন আইগেন-মানের জন্য যে সমাধান নির্ণয় করা হয়েছে, আবার সাংখ্যমানের সাহায্যে যে সমাধান পাওয়া যায় তা একই। এই পদ্ধতিটি উদাহরণ সহযোগে বুঝানো হয়েছে।

1. Introduction.

The asymptotic method of Krylov-Bogoliubov-Mitropolskii (KBM) [2, 4] is particularly convenient and is a widely used tool to obtain approximate solutions of weakly nonlinear systems. Originally, the method was developed for obtaining the periodic solutions of second order nonlinear systems with small nonlinearities. The method was then extended by Popov [9] to damped oscillatory nonlinear systems. Owing to physical importance, Popov's results were rediscovered by Mendelson [5] and Bojadziev [3]. Murty *et al.* [6, 7] extended the KBM method to nonlinear over-damped systems. Murty [8] has presented a unified KBM method for solving a second order nonlinear system which covers the undamped, damped and over-damped cases. Sattar [11] has investigated an asymptotic solution of a second order critically damped nonlinear system. Shamsul [14] has developed a new asymptotic method for obtaining approximate solutions of second order over-damped and critically damped nonlinear systems. Shamsul and Sattar [13] have extended Bogoliubov's asymptotic method to third order critically damped nonlinear systems. Recently, Rokibul *et al.* [10] have developed a new technique for solving third order critically damped nonlinear systems.

In the present paper, the KBM method has again been extended and asymptotic solutions for certain third order critically damped nonlinear systems are investigated. For different set of initial conditions as well as for different eigenvales, the solutions found in this paper show good coincidence with numerical solutions.

The method

Consider a third order weakly nonlinear differential system

$$\dddot{x} + k_1\ddot{x} + k_2\dot{x} + k_3 x = -\varepsilon f(x, \dot{x}, \ddot{x}) \qquad (1)$$

where over dots are used to denote first, second and third derivatives of x with respect to t; k_1, k_2, k_3 are constants, ε is the small parameter and $f(x, \dot{x}, \ddot{x})$ is the given nonlinear function. As the equation is third order, so, we shall get three real negative eigenvalues, where two of the eigenvalues are equal, because the system is critically damped. Suppose the eigenvalues are $-\lambda, -\lambda, -\mu$.

When $\varepsilon = 0$, the equation (1) becomes linear and the solution of the corresponding linear equation is

$$x(t, 0) = (a_0 + b_0 t)e^{-\lambda t} + c_0 e^{-\mu t} \qquad (2)$$

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where a_0, b_0, c_0 are constants of integration.

When $\varepsilon \neq 0$, following [17] a solution of the equation (1) is sought in the form

$$x(t,\varepsilon) = (a + b\,t)e^{-\lambda t} + c\, e^{-\mu t} + \varepsilon\, u_1(a,b,c,t) + \cdots \tag{3}$$

where a, b, c are functions of time t, and satisfy the first order differential equation

$$\begin{aligned} \dot{a}(t) &= \varepsilon\, A_1(a,b,c,t) + \cdots \\ \dot{b}(t) &= \varepsilon\, B_1(a,b,c,t) + \cdots \\ \dot{c}(t) &= \varepsilon\, C_1(a,b,c,t) + \cdots \end{aligned} \tag{4}$$

We only consider a first few terms in the series expansion of (3) and (4), we evaluate the functions u_i and A_i, B_i, C_i, $i = 1, 2, \cdots, n$ such that a, b, c appearing in (3) and (4) satisfy the given differential equation (1) with an accuracy of order ε^{n+1}. In order to determine these unknown functions it is customary in the KBM method that the correction terms, u_i, $i = 1, 2, \cdots, n$ must exclude terms (known as secular terms), which make them large. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first [8].

Now differentiating the equation (3), three times with respect to t, substituting the value of x and the derivatives $\dot{x}, \ddot{x}, \dddot{x}$ in the original equation (1), utilizing the relations presented in (4) and finally equating the coefficients of ε, we obtain

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$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial A_1}{\partial t}+t\frac{\partial B_1}{\partial t}+2B_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2 C_1 + \left(\frac{\partial}{\partial t}+\lambda\right)^2\left(\frac{\partial}{\partial t}+\mu\right)u_1 = -f^{(0)}(a,b,c,t) \quad (5)$$

where $f^{(0)}(a,b,c,t) = f(x_0, \dot{x}_0, \ddot{x}_0)$ and $x_0 = (a+bt)e^{-\lambda t} + c\,e^{-\mu t}$

Now, we expand $f^{(0)}$ in the Taylor's series (see also [7] for details) of the form

$$f^{(0)} = \sum_{i,j=0}^{\infty} F_0(a,b,c)e^{-(i\lambda+j\mu)t} + t\sum_{i,j=0}^{\infty} F_1(a,b,c)e^{-(i\lambda+j\mu)t} + t^2\sum_{i,j=0}^{\infty} F_2(a,b,c)e^{-(i\lambda+j\mu)t} + t^3\sum_{i,j=0}^{\infty} F_3(a,b,c)e^{-(i\lambda+j\mu)t} + \cdots \quad (6)$$

Therefore, using (6) into (5), we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial A_1}{\partial t}+t\frac{\partial B_1}{\partial t}+2B_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2 C_1 + \left(\frac{\partial}{\partial t}+\lambda\right)^2\left(\frac{\partial}{\partial t}+\mu\right)u_1$$
$$= -\sum_{i,j=0}^{\infty} F_0(a,b,c)e^{-(i\lambda+j\mu)t} - t\sum_{i,j=0}^{\infty} F_1(a,b,c)e^{-(i\lambda+j\mu)t} - t^2\sum_{i,j=0}^{\infty} F_2(a,b,c)e^{-(i\lambda+j\mu)t} - t^3\sum_{i,j=0}^{\infty} F_3(a,b,c)e^{-(i\lambda+j\mu)t} + \cdots \quad (7)$$

According to the KBM method u_1 does not contain the fundamental terms (the solution presented in the equation (2) is called generating solution and its terms are called fundamental terms) of $f^{(0)}$ (see also [6, 11, 13-15] for details). Therefore equation (7) can be separated in the following way:

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial B_1}{\partial t} = -\sum_{i,j=0}^{\infty} F_1(a,b,c)\,e^{-(i\lambda+j\mu)t} \quad (8)$$

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$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial A_1}{\partial t}+2B_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2 C_1=-\sum_{i,j=0}^{\infty}F_0(a,b,c)e^{-(i\lambda+j\mu)t} \tag{9}$$

and

$$\left(\frac{\partial}{\partial t}+\lambda\right)^2\left(\frac{\partial}{\partial t}+\mu\right)u_1=-t^2\sum_{i,j=0}^{\infty}F_2(a,b,c)e^{-(i\lambda+j\mu)t}-t^3\sum_{i,j=0}^{\infty}F_3(a,b,c)e^{-(i\lambda+j\mu)t}-\dots \tag{10}$$

Solving equation (8), we obtain

$$B_1=-\sum_{i,j=0}^{\infty}\frac{F_1(a,b,c)e^{-((i-1)\lambda+j\mu)t}}{(i\lambda+(j-1)\mu)((i-1)\lambda+j\mu)} \tag{11}$$

Substituting the value of B_1 from (11) into equation (9), we obtain

$$\begin{aligned}&e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial A_1}{\partial t}+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2 C_1\\&=-\sum_{i,j=0}^{\infty}F_1(a,b,c)e^{-(i\lambda+j\mu)t}-2\sum_{i,j=0}^{\infty}\frac{F_1(a,b,c)e^{-(i\lambda+j\mu)t}}{((i-1)\lambda+j\mu)}\end{aligned} \tag{12}$$

Now we have only one equation (12) for obtaining the unknown functions A_1 and C_1. So, for determining these unknown functions, we need to impose some restrictions (see also [16, 18] for details). In this paper, we have imposed the restriction that the term $e^{-(i\lambda+j\mu)t}$ balance with A_1 if $i \geq j$ and the term $e^{-(i\lambda+j\mu)t}$ balance with C_1 if $j > i$. This is an important restriction, since under this restriction the coefficients of A_1 and C_1 do not become large (the principle of the KBM method is, the coefficients of A_1, B_1 and C_1 must be small) and the solution is also useful in the case of more critically damped (three of the eigenvalues are equal) systems. On the other hand, if any other relations exist among the eigenvalues, our solution gives desire results in those cases also. This restriction is not used in the previous papers [1, 11-16, 18]. Thus, we shall be

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able to separate the equation (12) into two equations, one for A_1 and the other is for C_1 and solving them we shall get the unknown functions A_1 and C_1. Equation (10) is a non-homogeneous linear equation, so it can be solve for u_1 by well known operator method.

Since $\dot{a}, \dot{b}, \dot{c}$ are proportional to the small parameter ε, so they are slowly varying functions of time t. Therefore, it is plausible to replace a, b, c by their respective values obtained in the linear case (*i. e.* the values of a, b, c obtained when $\varepsilon = 0$) in the right hand side of (4). This replacement was first made by Murty *et al.* [6, 7] to solve similar type of nonlinear equations. Thus substituting the values of A_1, B_1 and C_1 into the equation (4) and integrating, we obtain

$$\left.\begin{aligned} a &= a_0 + \varepsilon \int_0^t A_1(a_0, b_0, c_0, t)\, dt \\ b &= b_0 + \varepsilon \int_0^t B_1(a_0, b_0, c_0, t)\, dt \\ c &= c_0 + \varepsilon \int_0^t C_1(a_0, b_0, c_0, t)\, dt \end{aligned}\right\} \tag{13}$$

Substituting the values of a, b, c and u_1 in the equation (3), we shall get the complete solution of (1).

Thus the determination of the first order approximate solution is completed.

Example

As an example of the above method, we have considered a Duffing equation type third order weakly nonlinear system

$$\dddot{x} + k_1 \ddot{x} + k_2 \dot{x} + k_3 x = -\varepsilon x^3 \tag{14}$$

Here $f = x^3$, and $x_0 = (a+bt)\,e^{-\lambda t} + c\,e^{-\mu t}$

Therefore,
$$\begin{aligned} f^{(0)} &= a^3 e^{-3\lambda t} + 3a^2 c e^{-(2\lambda+\mu)t} + 3ac^2 e^{-(\lambda+2\mu)t} + c^3 e^{-3\mu t} \\ &+ t\left(3a^2 b e^{-3\lambda t} + 6abc e^{-(2\lambda+\mu)t} + 3bc^2 e^{-(\lambda+2\mu)t}\right) \\ &+ t^2\left(3ab^2 e^{-3\lambda t} + 3b^2 c e^{-(2\lambda+\mu)t}\right) + t^3 b^3 e^{-3\lambda t} \end{aligned} \tag{15}$$

Comparing equation (6) and (15), we obtain

$$\sum_{i,j=0}^{\infty} F_0(a,b,c) e^{-(i\lambda+j\mu)t} = a^3 e^{-3\lambda t} + 3a^2 c e^{-(2\lambda+\mu)t} + 3ac^2 e^{-(\lambda+2\mu)t} + c^3 e^{-3\mu t},$$

$$\sum_{i,j=0}^{\infty} F_1(a,b,c) e^{-(i\lambda+j\mu)t} = 3a^2 b e^{-3\lambda t} + 6abc e^{-(2\lambda+\mu)t} + 3bc^2 e^{-(\lambda+2\mu)t},$$

$$\sum_{i,j=0}^{\infty} F_2(a,b,c) e^{-(i\lambda+j\mu)t} = 3ab^2 e^{-3\lambda t} + 3b^2 c e^{-(2\lambda+\mu)t} \tag{16}$$

and

$$\sum_{i,j=0}^{\infty} F_2(a,b,c) e^{-(i\lambda+j\mu)t} = b^3 e^{-3\lambda t}.$$

Therefore, equations (8)-(10) respectively become

$$e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\frac{\partial B_1}{\partial t} = -\left(3a^2 b e^{-3\lambda t} + 6abc e^{-(2\lambda+\mu)t} + 3bc^2 e^{-(\lambda+2\mu)t}\right) \tag{17}$$

$$\begin{aligned} &e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\left(\frac{\partial A_1}{\partial t} + 2B_1\right) + e^{-\mu t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)^2 C_1 \\ &= -\left(a^3 e^{-3\lambda t} + 3a^2 c e^{-(2\lambda+\mu)t} + 3ac^2 e^{-(\lambda+2\mu)t} + c^3 e^{-3\mu t}\right) \end{aligned} \tag{18}$$

and

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$$\left(\frac{\partial}{\partial t}+\lambda\right)^2\left(\frac{\partial}{\partial t}+\mu\right)u_1=-t^2 3ab^2e^{-3\lambda t}-t^2 3b^2ce^{-(2\lambda+\mu)t}-b^3t^3e^{-3\lambda t} \quad (19)$$

The solution of the equation (17) is

$$B_1=a^2bl_1e^{-2\lambda t}+abcl_2e^{-(\lambda-\mu)t}+bc^2l_3e^{-2\mu t} \quad (20)$$

where $l_1=\dfrac{-3}{2\lambda(3\lambda-\mu)}$, $l_2=\dfrac{-3}{\lambda(\lambda+\mu)}$, $l_3=\dfrac{-3}{2\mu(\lambda+\mu)}$

Putting the value of B_1 from equation (20) into the equation (18), we obtain

$$\begin{aligned}&e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial A_1}{\partial t}+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2C_1\\&=2(3\lambda-\mu)a^2bl_1e^{-3\lambda t}+2\lambda abcl_2e^{-(2\lambda+\mu)t}+2(\lambda+\mu)bc^2l_3e^{-(\lambda+2\mu)t}\\&-\left(a^3e^{-3\lambda t}+3a^2ce^{-(2\lambda+\mu)t}-3ac^2e^{-(\lambda+2\mu)t}+c^3e^{-3\mu t}\right)\end{aligned} \quad (21)$$

To separate the equation (21) for determining the unknown functions A_1 and C_1, in this paper we have imposed the restriction that the terms $e^{-(i\lambda+j\mu)t}$ balance with A_1 if $i\geq j$ and with C_1 if $j>i$. Under this restriction, we obtain

$$\begin{aligned}&e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial A_1}{\partial t}=2(3\lambda-\mu)a^2bl_1e^{-3\lambda t}\\&+2\lambda abcl_2e^{-(2\lambda+\mu)t}-a^3e^{-3\lambda t}-3a^2ce^{-(2\lambda+\mu)t}\end{aligned} \quad (22)$$

and

$$e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2C_1=2(\lambda+\mu)bc^2l_3e^{-(\lambda+2\mu)t}-3a^2ce^{-(\lambda+2\mu)t}-c^3e^{-3\mu t} \quad (23)$$

The particular solutions of (22)-(23) respectively become

$$A_1=m_1a^2be^{-2\lambda t}+m_2abc_2e^{-(\lambda+\mu)t}+m_3a^3e^{-2\lambda t}+m_4ca^2e^{-(2\lambda+\mu)t} \quad (24)$$

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$$C_1 = n_1 b c^2 e^{-(\lambda+\mu)t} + n_2 a^2 c e^{-(\lambda+\mu)t} + n_3 c^3 e^{-2\mu t} \tag{25}$$

where

$$m_1 = \frac{-3}{2\lambda^2(3\lambda-\mu)}, \qquad m_2 = \frac{-6}{2\lambda(\lambda+\mu)^2},$$

$$m_3 = \frac{-1}{2\lambda(3\lambda-\mu)}, \qquad m_4 = \frac{-3}{2\lambda(\lambda+\mu)},$$

$$n_1 = \frac{1}{4\mu^3}, \quad n_2 = \frac{-3}{4\mu^2}, \quad n_3 = \frac{-1}{(\lambda-3\mu)^2}$$

The solution the equation (19) is

$$\begin{aligned} u_1 &= ab^2\left(r_1 t^2 + r_2 t + r_3\right)e^{-3\lambda t} + b^2 c\left(r_4 t^2 + r_5 t + r_6\right)e^{-(\mu+2\lambda)t} \\ &\quad + b^3\left(r_7 t^3 + r_8 t^2 + r_9 t + r_{10}\right)e^{-3\lambda t} \end{aligned} \tag{26}$$

where

$$r_1 = \frac{-3}{4\lambda^2(\mu-3\lambda)}, \qquad r_2 = \frac{-6(\mu-2\lambda)}{4\lambda^3(\mu-3\lambda)^2},$$

$$r_3 = \frac{3}{2\lambda^2(\mu-3\lambda)}\left\{\frac{3}{2\lambda^2} + \frac{2}{\lambda(\mu-3\lambda)} + \frac{2}{(\mu-3\lambda)^2}\right\},$$

$$r_4 = \frac{3}{2\lambda(\lambda+\mu)^2}, \qquad r_5 = \frac{3(\mu+5\lambda)}{2\lambda^2(\lambda+\mu)^3},$$

$$r_6 = \frac{3}{2\lambda(\lambda+\mu)^2}\left\{\frac{6}{(\mu+\lambda)^2} + \frac{2}{\lambda(\mu+\lambda)} + \frac{1}{2\lambda^2}\right\},$$

$$r_7 = -\frac{1}{4\lambda^2(\mu-3\lambda)}, \qquad r_8 = -\frac{1}{4\lambda^2(\mu-3\lambda)}\left\{\frac{3}{\lambda} - \frac{3}{(\mu-3\lambda)}\right\},$$

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$$r_9=-\frac{1}{4\lambda^2(\mu-3\lambda)}\left\{\frac{9}{2\lambda^2}-\frac{6}{\lambda(\mu-3\lambda)}+\frac{6}{(\mu-3\lambda)^2}\right\},$$

$$r_{10}=-\frac{1}{4\lambda^2(\mu-3\lambda)}\left\{\frac{3}{\lambda^3}-\frac{9}{2\lambda^2(\mu-3\lambda)}+\frac{6}{\lambda(\mu-3\lambda)^2}-\frac{6}{(\mu-3\lambda)^2}\right\}.$$

Substituting the values of A_1, B_1, C_1 from the equation (24), (20) and (25) respectively, into the equation (13) and integrating, we obtain

$$a=a_0+\varepsilon\left\{\frac{m_1a_0^2b_0\left(1-e^{-2\lambda t}\right)}{2\lambda}+\frac{m_2a_0b_0c_0\left(1-e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)}\right. \tag{27}$$
$$\left.-\frac{m_4a_0^3\left(1-e^{-2\lambda t}\right)}{2\lambda}+\frac{m_5c_0a_0^2\left(1-e^{-(2\lambda+\mu)t}\right)}{(2\lambda+\mu)}\right\}$$

$$b=b_0+\varepsilon\left\{\frac{l_1a_0^2b_0\left(1-e^{-2\lambda t}\right)}{2\lambda}+\frac{l_2a_0b_0c_0\left(1-e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)}+\frac{l_3b_0c_0^2\left(1-e^{-2\mu t}\right)}{2\mu}\right\} \tag{28}$$

$$c=c_0+\varepsilon\left\{+\frac{n_1b_0c_0^2\left(1-e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)}+\frac{n_2a_0^2c_0\left(1-e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)}+\frac{n_3c_0^3\left(1-e^{-2\mu t}\right)}{2\mu}\right\} \tag{29}$$

Therefore, we obtain the first order approximate solution of the equation (14) as

$$x(t,\varepsilon)=(a+bt)e^{-\lambda t}+ce^{-\mu t}+\varepsilon u_1 \tag{30}$$

where a, b, c are given by the equation (27)-(29) and u_1 given by (26).

Results and Discussion.

By means of the extended KBM method an approximate solutions of a third order critically damped nonlinear system has been found in this paper. It is usual to compare the perturbation solutions, obtained by a certain perturbation method, to the numerical solutions to test the accuracy of the

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approximate solutions. With regard to such a comparison concerning the presented KBM method of this paper, we refer the work of Murty *et al.* [6]. In the present paper, we have compared the solutions obtained by the equation (30) to those obtained by fourth order Runge-Kutta method for different set of initial conditions as well as for different eigenvalues.

First of all $x(t, \varepsilon)$ has been computed by the equation (30) in which a, b, c are calculated by equations (27)-(29) and u_1 is calculated by the equation (26) together with initial conditions $a_0 = 0.8$, $b_0 = 0.1$, $c_0 = 0.1$ [or $x(0) = 0.900004$, $\dot{x}(0) = -2.472107$, $\ddot{x}(0) = 7.150955$] for $k_1 = 7.2$, $k_2 = 15.81$, $k_3 = 9.61$; *i. e.* for $\lambda = 3.1$ and $\mu = 1$ when $\varepsilon = 0.1$. The solutions for various values of t are presented in the second column of Table I. The corresponding numerical solutions (designated by x^*) have been computed by fourth order Runge-Kutta method and are given in the third column of the Table I. Percentage errors have also been calculated and are given in the fourth column of the Table I.

Table I

T	x	x^*	Error%
0.0	0.900004	0.900004	0.0000
0.5	0.242944	0.242266	-0.2799
1.0	0.078903	0.078598	-0.3881
1.5	0.032447	0.032333	-0.3526
2.0	0.016222	0.016178	-0.2720
2.5	0.009066	0.009046	-0.2211
3.0	0.005326	0.005316	-0.1881
3.5	0.003192	0.003186	-0.1883
4.0	0.001927	0.001924	-0.1559
4.5	0.001167	0.001165	-0.1717
5.0	0.000708	0.000706	-0.2833

Initial conditions are $a_0 = 0.8$, $b_0 = 0.1$, $c_0 = 0.1$, $\lambda = 3.1$, $\mu = 1$ and $\varepsilon = 0.1$

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x is computed by (30)

x^* is computed by *Runge-Kutta* method

Secondly, we have computed $x(t, \varepsilon)$ by (30) for another set of initial conditions $a_0 = 0.4$, $b_0 = 0.4$, $c_0 = 0.2$ [or $x(0) = 0.600008$, $\dot{x}(0) = -1.799403$, $\ddot{x}(0) = 6.197403$] for $k_1 = 11$, $k_2 = 35$, $k_3 = 25$; *i. e.* for $\lambda = 5$ and $\mu = 1$ when $\varepsilon = 0.1$. The solutions for various values of t are presented in the second column of Table II. The corresponding numerical solutions (designated by x^*) have been computed by fourth order Runge-Kutta method and are given in the third column of the Table II. Percentage errors have also been calculated and are given in the fourth column of the Table II.

Table II

t	X	X^*	Error%
0.0	0.600008	0.600008	0.0000
0.5	0.170661	0.170096	-0.3322
1.0	0.079045	0.078838	-0.2626
1.5	0.045230	0.045150	-0.1772
2.0	0.027153	0.027111	-0.1549
2.5	0.016441	0.016415	-0.1584
3.0	0.009970	0.009953	-0.1708
3.5	0.006047	0.006036	-0.1822
4.0	0.003667	0.003660	-0.1913
4.5	0.002224	0.002220	-0.1802
5.0	0.001349	0.001346	0.2229

Initial conditions are $a_0 = 0.4$, $b_0 = 0.4$, $c_0 = 0.2$, $\lambda = 5$, $\mu = 1$ and $\varepsilon = 0.1$

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x is computed by (30)

x^* is computed by *Runge-Kutta* method

Finally, we have computed $x(t, \varepsilon)$ by (30) for another set of initial conditions $a_0 = 0.6$, $b_0 = 0.4$, $c_0 = 0.1$ [or $x(0) = 0.700441$, $\dot{x}(0) = -1.005450$, $\ddot{x}(0) = 1.233213$] for $k_1 = 6$, $k_2 = 12$, $k_3 = 8$; *i. e.* for $\lambda = \mu = 2$ when $\varepsilon = 0.1$. The solutions for various values of t are presented in the second column of Table **III**. The corresponding numerical solutions (designated by x^*) have been computed by fourth order Runge-Kutta method and are given in the third column of the Table **III**. Percentage errors have also been calculated and are given in the fourth column of the Table **III**.

Table III

t	X	x^*	Error%
0.0	0.700441	0.700441	0.0000
0.5	0.330913	0.330721	-0.0581
1.0	0.148715	0.148614	-0.0680
1.5	0.064641	0.064575	-0.1022
2.0	0.027435	0.027392	-0.1570
2.5	0.011437	0.011411	-0.2279
3.0	0.004702	0.004688	-0.2987
3.5	0.001912	0.001904	-0.4202
4.0	0.000770	0.000767	-0.3911
4.5	0.000308	0.000306	-0.6536
5.0	0.000122	0.000122	0.0000

Initial conditions are $a_0 = 0.6$, $b_0 = 0.4$, $c_0 = 0.1$, $\lambda = \mu = 2$ and $\varepsilon = 0.1$

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x is computed by (30)

x^* is computed by *Runge-Kutta* method

Conclusion.

In this paper, a formula is developed for solving third order critically damped nonlinear systems with small nonlinearities. This is an extension of the KBM method. This extension is important, because the formula equation (30) gives not only the desired results when the system is critically damped but also gives the desired results when the system undergoes more critical damping (when three of the eighenvalues are equal). The results presented in Table III are obtained for $\lambda = \mu = 2$. In this case the system undergoes more critical damping. From Table III, we see that the results obtained by (30) show good coincidence with numerical results in the case of more critical damping also.

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