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ON FINITELY GENERATED N-IDEALS WHICH FORM RELATIVELY STONE LATTICES.

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Abstract:

Set of all finitely generated n-ideals of L *is a lattice, denoted by* $F_n(L)$. *In this paper the author has characterized those* $F_n(L)$ *which form relatively Stone lattices. It has been shown that* $F_n(L)$ *is relatively Stone if and only if* $P \vee Q = L$ *for any two incomparable prime n-ideals* P *and* Q *of* L.

Keyword : *lattice, finitely generated ideals, stone lattices.*

¢hj§aÑ pj¡l *(Bengali version of the Abstract)*

L-Hl pp£j i¡-h pªø pLm n-BC¢Xu¡-ml (n-ideal) -pVÚ HL¢V mÉ¡¢Vp (Lattice) k¡q¡-L Fn(L) à¡l¡ ¢Q¢q²a Ll¡ quz HC f-œ fËhåL¡l -pC ph Fn(L) k¡q¡l¡ B-f¢rL -ØV¡e mÉ¡¢Vp (Stone Lattice) NWe L-l-R a¡-cl Q¢lœNa °h¢nø fËc¡e L-l-Rz HV¡ -cM¡-e¡ q-u-R -k Fn(L) B-f¢rL -ØV¡e qu L-Hl -k -L¡e c¤"¢V Aa¥me -j±¢mL n-BC¢Xu¡m P Hhw Q Hl SeÉ k¢c Hhw -Lhmj¡œ k¢c PVQ=L quz

1. Introduction.

Relative annihilators in lattices and semilattices have been studied by many authors including Mandelker [5] and Varlet [8]. Also Cornish [2] has used the annihilators in studying relative normal lattices. Then [6] has introduced the notion of relative annihilators around a fixed element $n \in L$, called relative n-annihilators.

For $a, b \in L$, $<a, b> = \{x \in L : x \wedge a \le b\}$ is known as annihilator of a relative to b or simply a relative annihilator. It is very easy to see that in presence of distributivity, $<a, b>$ is an ideal of L.

Again for $a, b \in L$ we define $<a, b>_d = \{x : x \vee a \ge b\}$, which is called a dual annihilator of a relative to b, or simply a relative dual annihilator. In presence of distributivity of L, $<a, b>_d$ is a dual ideal (filter).

For a fixed element n of a lattice L, a convex sub lattice containing n is called an n-ideal. The idea of n-ideals is a kind of generalization of both ideals and filters of a lattice. The set of all n-ideals of a lattice L is denoted by $I_n(L)$, which is an algebraic lattice under set inclusion. Moreover, $\{n\}$ and L are respectively the smallest and the largest elements of $I_n(L)$.

For any two n-ideals I and J of L, it is easy to check that
$I \wedge J = I \cap J = \{x \in L / x = m(i, n, j) \text{ for some } i \in I,\ j \in J\}$ where
$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and
$I \vee J = \{x \in L / i_1 \wedge j_1 \le x \le i_2 \vee j_2 \text{ for some } i_1, i_2 \in I \text{ and } j_1, j_2 \in J\}$
The n-ideal generated by a finite numbers of elements is called a finitely generated n-ideal. The set of all finitely generated n-ideals is denoted by $F_n(L)$. n-ideal generated by $a_1, a_2, ..., a_m$ is denoted by $<a_1, a_2, ..., a_m>_n$, which is the interval $[a_1 \wedge a_2 \wedge ... \wedge a_m \wedge n,\ a_1 \vee a_2 \vee ... \vee a_m \vee n]$.

Thus, the members of $F_n(L)$ are simply the intervals $[a,b]$ such that $a \leq n \leq b$. A neat description of finitely generated n-ideals can be found in [7]. By [4] and [7], we know that $F_n(L)$ is a lattice and for $[a,b]$, $[a_1, b_1] \in F_n(L)$,

$[a,b] \cap [a_1, b_1] = [a \vee a_1, b \wedge b_1]$ and $[a,b] \vee [a_1, b_1] = [a \wedge a_1, b \vee b_1]$.

The n-ideal generated by a single element a is called principal n-ideal, denoted by $< a >_n$. Clearly, $< a >_n = [a \wedge n, a \vee n]$. By [4] we also know that $< a >_n \cap < b >_n = < m(a, n, b) >_n$, when L is distributive. Let L be a lattice with 0 and 1. An element $a^* \in L$ is called a pseudocomplement of $a \in L$, if $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$. L is called pseudo complemented if its every element has a pseudocomplement. For $a, b \in L$, and a fixed element $n \in L$ we define $< a, b >^n = \{x \in L : m(a, n, x) \in < b >_n\}$ $= \{x \in L : b \wedge n \leq m(a, n, x) \leq b \vee n\}$. By [6], $< a, b >^n$ is called the annihilator of a relative to b around the element n or simply a relative n-annihilator. It is easy to see that for all $a, b \in L$, $< a, b >^n$ is always a convex subset containing n. In presence of distributivity, it is easy to prove that $< a, b >^n$ is an n-ideal.

For two n-ideals A and B of a lattice L, $< A, B > = \{x \in L : m(a, n, x) \in B$ for all $a \in A\}$. In presence of distributivity, clearly $< A, B >$ is an n-ideal. Moreover, we can easily show that $< a, b >^n = << a >_n, < b >_n >$.

A distributive lattice L is called a Stone lattice if it is pseudocomplemented and $x^* \vee x^{**} = 1$, for each $x \in L$. Also recall that a lattice L is relatively pseudocomplernented if its every interval $[a,b]$ $(a, b \in L, \quad a < b)$ is pseudocomplemented. A distributive lattice L is called a relatively Stone lattice if its every interval $[a, b]$ is a Stone lattice. In this paper we include some results on relative n-annihilators of a lattice.

Then we use them in generalizing some results on relatively Stone lattices in terms of n-ideals.

We start the paper with the following result which is due to [4, 7].

Proposition 1.1: *For a lattice* L *with* $n \in L$, $F_n(L) \cong (n]^d \times [n)$.

Following result due to [1] will be needed in proving our main results.

Proposition 1.2: *Let* I *and* J *be two n-ideals of a distributive lattice. Then for any* $x \in I \vee J$, $x \vee n = i_1 \vee j_1$ *and* $x \wedge n = i_2 \wedge j_2$ *for some* $i_1, i_2 \in I$ *and* $j_1, j_2 \in J$ *with* $i_1, j_1 \geq n$ *and* $i_2, j_2 \leq n$.

Now we give a generalization of [2, Lemma-3.6].

Theorem 1.3: *Let* L *be a distributive lattice. Then the following hold:*

(i) $<<x>_n, J> = <<x>_n, <y>_n>$;

(ii) $<<x>_n, J> = \bigvee_{y \in J} <<x>_n, <y>_n>$, *where the supremum of n-ideals* $<<x>_n, <y>_n>$ *is taken in the lattice of n-ideals of* L, *for any* $x \in L$ *and any n-ideal* J.

Proof: (i) L.H.S $\subseteq$ R.H.S is obvious. Let $t \in$R.H.S, then $t \in <<y>_n, <x>_n>$. This implies $m(y, n, t) \in <x>_n$. That is $<m(y, n, t)>_n \subseteq <x>_n$ and so $(<y>_n \cap <t>_n) \vee (<x>_n \cap <t>_n) \subseteq <x>_n$. That is, $<t>_n \cap [<x>_n \vee <y>_n] \subseteq <x>_n$ which implies $t \in << x>_n \vee <y>_n, <x>_n>$. Thus, $t \in$ L.H.S and so (i) holds.

(ii) R.H.S $\subseteq$ L.H.S is obvious. Let $t \in$ L.H.S, then $m(x, n, t) \in J$ that is $m(x, n, t) = j$ for some $j \in J$. This implies $t \in << x>_n, <j>_n>$. Thus $t \in$ R.H.S and so (ii) holds.

Lemma 1.4: *Let L be a distributive lattice with* $n \in L$. *Suppose* $a, b, c \in L$.

(i) *If* $a, b, c \geq n$, *then*

$<< m(a, n, b)>_n, <c>_n> = << a>_n, <c>_n> \vee << b>_n, <c>_n>$ *is equivalent to*

$<a \wedge b, c> = <a, c> \vee <b, c>$;

(ii) *If* $a, b, c \leq n$

then $<< m(a, n, b)>_n, <c>_n> = << a>_n, <c>_n> \vee << b>_n, <c>_n>$

is equivalent to $<a \vee b, c>_d = <a, c>_d \vee <b, c>_d$.

Proof: (i): Suppose $a, b, c \geq n$ and

$<< a>_n \cap <b>_n, <c>_n> = << a>_n, <c>_n> \vee << b>_n, <c>_n>$. Let $x \in <a \wedge b, c>$. Then

$x \wedge a \wedge b \leq c$, $<x>_n \cap <a \wedge b>_n = <x>_n \cap [n, a \wedge b] = [n, (x \vee n) \wedge (a \wedge b)]$

$= [n, (x \wedge a \wedge b) \vee n] \subseteq [n, c]$. Hence

$x \in << a \wedge b>_n, <c>_n> = << m(a, n, b)>_n, <c>_n> = << a>_n, <c>_n> \vee << b>_n, <c>_n>$.

Thus $x \leq p \vee q$, where $p \in << a>_n, <c>_n>$, $q \in << b>_n, <c>_n>$. Then

$<p>_n \cap <a>_n \subseteq <c>_n$. That is, $[p \wedge n, p \vee n] \cap [n, a] \subseteq [n, c]$. Thus,

$[n, (p \vee n) \wedge a] \subseteq [n, c]$ which implies $p \wedge a \leq c$, and so $p \in <a, c>$. Similarly,

$q \in \langle b, c \rangle$ and so $x \in \langle a, c \rangle \vee \langle b, c \rangle$. Hence $\langle a \wedge b, c \rangle \subseteq \langle a, c \rangle \vee \langle b, c \rangle$. But $\langle a, c \rangle \vee \langle b, c \rangle \subseteq \langle a \wedge b, c \rangle$ is obvious, Therefore, $\langle a \wedge b, c \rangle = \langle a, c \rangle \vee \langle b, c \rangle$.

Conversely, suppose $\langle a \wedge b, c \rangle = \langle a, c \rangle \vee \langle b, c \rangle$. Let $x \in \langle \langle m(a, n, b) \rangle_n, \langle c \rangle_n \rangle$. Then

$\langle x \rangle_n \cap \langle m(a, n, b) \rangle_n = [x \wedge n, x \vee n] \cap [n, a \wedge b] \subseteq [n, c]$. That is $[n, (x \vee n) \wedge (a \wedge b)] \subseteq [n, c]$. Thus, $[n, (x \wedge a \wedge b) \vee n] \subseteq [n, c]$ which implies $x \wedge a \wedge b \leq c$, and so $x \in \langle a \wedge b, c \rangle = \langle a, c \rangle \vee \langle b, c \rangle$. This implies $x = r \vee s$, where $r \in \langle a, c \rangle$ and $s \in \langle b, c \rangle$. Then $r \wedge a \leq c$ and $s \wedge b \leq c$. Now

$\langle r \rangle_n \cap \langle a \rangle_n = [r \wedge n, r \vee n] \cap [n, a] = [n, (r \vee n) \wedge a] = [n, (r \wedge a) \vee n] \subseteq [n, c]$ $= \langle c \rangle_n$. Hence, $r \in \langle \langle a \rangle_n, \langle c \rangle_n \rangle$.

Similarly, $s \in \langle \langle b \rangle_n, \langle c \rangle_n \rangle$. Thus $x \in \langle \langle a \rangle_n, \langle c \rangle_n \rangle \vee \langle \langle b \rangle_n, \langle c \rangle_n \rangle$ and so $\langle \langle m(a, n, b) \rangle_n, \langle c \rangle_n \rangle \subseteq \langle \langle a \rangle_n, \langle c \rangle_n \rangle \vee \langle \langle b \rangle_n, \langle c \rangle_n \rangle$.

Since $\langle \langle a \rangle_n, \langle c \rangle_n \rangle \vee \langle \langle b \rangle_n, \langle c \rangle_n \rangle \subseteq \langle \langle m(a, n, b) \rangle_n, \langle c \rangle_n \rangle$ is obvious, so $\langle \langle m(a, n, b) \rangle_n, \langle c \rangle_n \rangle = \langle \langle a \rangle_n, \langle c \rangle_n \rangle \vee \langle \langle b \rangle_n, \langle c \rangle_n \rangle$.

A dual calculation of above proof proves (ii).

Following result on Stone lattices is well known due to [3, Thcorem-3, Page-161] and [2, Theorem-2.4].

Theorem 1.5: *Let L be a pseudocomplemented distributive lattice. Then the following conditions are equivalent:*

(i) L *is Stone;*

(ii) *For each* $x, y \in L,\ (x \wedge y)^* = x^* \vee y^*$

(iii) *I f* $x \wedge y = 0,\ x, y \in L,$ *then* $x^* \vee y^* = 1$.

Similarly we can easily prove the following result which is dual to above theorem.

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Theorem 1.6: *Let* L *be a dual pseudocomplemented distributive lattice. Then the following conditions are equivalent:*

(i) L *is dual Stone ;*

(ii) *For each* $x, y \in L,\ (x \vee y)^{*d} = x^{*d} \wedge y^{*d}$;

(iii) *if* $x \vee y = 1,\ x, y \in L,$ *then* $x^{*d} \wedge y^{*d} = 0$,

where x^{*d} *denotes the dual pseudcomplement of* x.

[2, Theorern-3.7] (also see [5]) has provided a nice characterization of relatively Stone lattices in terms of relative annihilators. By a similar technique we can easily prove its dual result.

Theorem 1.7: *Let* L *be a relatively dual pseudocomp1emented distributive lattice. Let* $a, b, c \in L$ *be arbitrary elements and* A, B *are arbitrary filters. Then the following are equivalent;*

(i) L *is relatively dual Stone;*

(ii) $< a, b >_d \vee < b, a >_d = L$;

(iii) $< c, a \wedge b >_d = < c, a >_d \vee < c, b >_d$;

(iv) $< [c), A \vee B >_d = < [c), A >_d \vee < [c), B >_d$;

(v) $< a \vee b, c >_d = < a, c >_d \vee < b, c >_d$.

[2, Theorem- 3.7] and [5, Theorern-5] provided some characterizations of relatively Stone lattices. Here generalize those results in terms of n-ideals.

Theorem 1.8: *Let* $F_n(L)$ *be a relatively pseudocomplemented distributive lattice and* A *and* B *be two n-ideals of* L. *Then for all* $a, b, c \in L$, *the following conditions are equivalent:*

(i) $F_n(L)$ *is relatively Stone ;*

(ii) $<< a >_n, < b >_n > \vee << b >_n, < a >_n >= L$;

(iii) $<< m(a, n, b) >_n, < c >_n >=<< a >_n, < c >_n > \vee << b >_n, < c >_n >$.

Proof: (i) => (ii). Let $z \in L$, consider the interval $I = [< a >_n \cap < b >_n \cap < z >_n, < z >_n]$ in $F_n(L)$. Then $< a >_n \cap < b >_n \cap < z >_n$ is the smallest element of the interval I. By (i), I is Stone. Then by Theorem 1.5, there exist finitely generated n-ideals $[p, q], \ [r, s] \in I$ such that,

$< a >_n \cap < z >_n \cap [p, q] = < a >_n \cap < b >_n \cap < z >_n =< b >_n \cap < z >_n \cap [r, s]$ and

$< z >_n = [p, q] \vee [r, s]$.

Now,

$< a >_n \cap [p, q] =< a >_n \cap [p, q] \cap < z >_n = < a >_n \cap < b >_n \cap < z >_n \subseteq < b >_n$

implies $[p, q] \subseteq << a >_n, < b >_n >$. Also $< b >_n \cap [r, s] = < b >_n \cap < z >_n \cap [r, s]$ $= < a >_n \cap < b >_n \cap < z >_n \subseteq < a >_n$ implies $[r, s] \subseteq << b >_n, < a >_n >$. Thus, $< z >_n \subseteq << a >_n, < b >_n > \vee << b >_n, < a >_n >$

and so $z \in << a >_n, < b >_n > \vee << b >_n, < a >_n >$.

Hence $<< a >_n, < b >_n > \vee << b >_n, < a >_n > = L$.

(ii) => (iii). In (iii) R.H.S $\subseteq$ L.H.S is obvious. Let $z \in$ L.H.S. Then $z \in << m(a, n, b) >_n, < c >_n >$, which implies $z \vee n \in << m(a, n, b) >_n, < c >_n >$. By (ii) $z \vee n \in << a >_n, < b >_n > \vee << b >_n, < a >_n >$. Then by Proposition 1.2, $z \vee n = x \vee y$, for some $x \in << a >_n, < b >_n >$ and $y \in << b >_n, < a >_n >$ and $x, y \geq n$. Thus, $< x >_n \cap < a >_n \subseteq < b >_n$, and so $< x >_n \cap < a >_n = < x >_n \cap < a >_n \cap < b >_n$ $\subseteq < z \vee n >_n \cap < a >_n \cap < b >_n = < z \vee n >_n \cap < m(a,n,b) >_n \subseteq < c >_n$. This implies $x \in << a >_n, < c >_n >$. Similarly $y \in << b >_n, < c >_n >$, and so $z \vee n \in << a >_n, < c >_n > \vee << b >_n, < c >_n >$. Similarly, a dual calculation of above shows that $z \wedge n \in << a >_n, < c >_n > \vee << b >_n, < c >_n >$. Thus by convexity, $z \in << a >_n, < c >_n > \vee << b >_n, < c >_n >$ and so (iii) holds.

(iii) => (i). Suppose (iii) holds. Let $a,b,c \geq n$. By (iii), $<< m(a,n,b) >_n, < c >_n > = << a >_n, < c >_n > \vee << b >_n, < c >_n >$. But by Lemma 1.4(i), this is equivalent to $< a \wedge b, c > = < a, c > \vee < b, c >$. Then by [2, Theorem-3.7], this shows that $[n)$ is a relatively Stone Lattice. Similarly, for $a,b,c \leq n$, using the Lemma 1.4(ii) and Theorem 1.7, we find that $(n]$

is relatively dual Stone. Therefore $F_n(L)$ is relatively Stone by Proposition 1.1.

We conclude the paper by generalizing another well known characterization of relatively Stone lattices. To prove this we need the following lemma which is due to [2, Lemma-3.4].

Lemma 1.9: *If* L_1 *is a sublattice of* L *and* P_1 *is a prime ideal in* L_1 *then there exists a prime ideal* P *in* L *such that* $P_1 = L_1 \cap P$.

Theorem 1.10: *Let* $F_n(L)$ *be a relatively pseudocomplemented distributive lattice. Then the following conditions are equivalent:*

(i) $F_n(L)$ *is relatively Stone*

(ii) *Any two incomparable prime n-ideals* P *and* Q *are comaximal, that is* $P \vee Q = L$.

Proof: Suppose (i) holds. Let P , Q be two incomparable prime n-deals of L . Then there exist $a, b \in L$ such that $a \in P - Q$ and $b \in Q - P$. Then $<a>_n \subseteq P - Q, <b>_n \subseteq Q - P$. Since $F_n(L)$ is relatively Stone, so by Theorem 1.8, $<<a>_n, <b>_n> \vee <<b>_n, <a>_n> = L$. But as P , Q are prime, so it is easy to see that, $<<a>_n, <b>_n> \subseteq Q$ and $<<b>_n, <a>_n> \subseteq P$. Therefore $L \subseteq P \vee Q$ and so $P \vee Q = L$. That is, (ii) holds.

Conversely, suppose (ii) holds. Let P_1 and Q_1 be two incomparable prime ideals of $[n)$. Then by Lemma 1.9, there exist incomparable prime ideals P and Q of L such that $P_1 = P \cap [n)$ and $Q_1 = Q \cap [n)$. Since $n \in P_1$

and $n \in Q_1$, so P, Q are in fact two incomparable prime n-ideals of L. Then by (ii), $P \vee Q = L$. Therefore, $P_1 \vee Q_1 = (P \vee Q) \cap [n) = [n)$. Thus by [2, Theorern-3.5], $[n)$ is relatively Stone. Similarly, considering two prime filters of $(n]$ and proceeding as above and using the dual result of [2, Theorern-3.5] we find that $(n]$ is relatively dual Stone. Therefore by Proposition 1.1, $F_n(L)$ is relatively Stone.

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