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A MATHEMATICAL ANALYSIS ON BLOOD FLOW THROUGH AN ARTERY WITH A BRANCH CAPILLARY

By

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Abstract:

The paper is devoted to a theoretical study for the distribution of axial velocity for blood flow in a branch capillary emerging out of a parent artery at various locations of the branch. The results are computed for various values of r and the angle made by the parent artery and the branch capillary. Also due attention is given to the variation of n (fluid index). The output is compared with the results in the previous similar investigations. A theoretical estimate for the velocity of blood for various non negative values of the fluid index parameter and yield stress in different locations of the branch capillary is presented.

Keyword : *branch capillary, artery, fluid index, yield stress.*

¢hj§aÑ pj¡l *(Bengali version of the Abstract)*

j§m djZ£ -b-L Eá¨a ¢h¢iæ ÙÛ¡¢eL n¡M¡u n¡M¡ -L±¢nL e¡m£-a lš² fËh¡-ql SeÉ Ar£u N¢a-hN h¾V-el ašÄNa Ae¤på¡-e HC fœ¢V HL¡¿¹ i¡-h ¢e-u¡¢Saz j§m djZ£ Hhw n¡M¡ -L±¢nL e¡m£ à¡l¡ L«a ¢h¢iæ -L¡Z r-Hl j¡-el SeÉ NZe¡ Ll¡ q-u-Rz n- fËh¡q£ fc¡bÑ p§Q-Ll f¢lhaÑen£ma¡u J kb¡kb NZe¡ Ll¡ q-u-Rz f§hÑae pcªn Ae¤på¡-e ¢e¢ZÑa g-ml p-‰ fË¡ç g-ml a¥me¡ Ll¡ q-u-Rz fËh¡q£ fc¡-bÑl p§QL fËQm (Parameter) *Hhw pqe-f£s-el* (Yield Stress) *AGZ¡aÈL j¡-el SeÉ n¡M¡ -L±¢nL e¡m£l ¢h¢iæ ÙÛ¡¢eL -r-œ l-š²l N¢a-h-Nl ašÄNa pñ¡hÉ j¡e-L f¢l-hne Ll¡ q-u-Rz*

1. Introduction :

A systematic study of the rheologic and fluid dynamic properties of blood and blood flow could play a significant role in the basic understanding, diagnosis and treatment of many cardio-vascular, cerebra-vascular and arterial diseases. Therefore, the rheological complexities involved in the blood flow in the cardiovascular system have attracted serious attention from many researchers. Physiologically blood is an aquous liquid (plasma) having some suspended particles like white blood cells, erythrocytes, platelets and others. Rheologically blood behaves like a homogeneous Newtonian fluid in large blood vessels while it behaves non-Newtonian in narrow blood vessels e.g. capillaries. The flow behavior is further complicated due to the fact that at low shear rate certain chemical reactions occur that may cause significant changes in the flow behavior of blood. Since harmful experiments cannot be carried out on living human beings, model studies related to blood flow through human artery with a branch capillary have been carried out by many theoretical researchers. The complications in describing the flow of blood in the arterial system leads to develop a constitutive mathematical model that can explain its non-Newtonian behavior. Misra and Chakravarty[1] developed a mathematical model to study unsteady flow of blood through arteries treating blood as a Newtonian viscous incompressible fluid paying due attention to the orthotropic material behavior of the wall tissues. In an another theoretical study Misra et al ([4],[5],[6],[7]) presented a mathematical analysis in which the blood was treated as a non-Newtonian fluid and the artery as non linearly viscoelastic. A good number of analytical as well as experimental studies on the flow of blood through the arterial segments having stenosis or multiple stenoses were carried out by Young,Shukla et al,Chaturani[3] and Sami. They assumed that blood behaves like a Newtonian fluid. On the basis of the experimental observations, Han,Barnett and Whitemore[10] suggested that blood behaves like a non Newtonian fluid under certain conditions. At a low shear rate(

about 0.1/sec) blood behaves like a Casson model fluid. Misra and Kar [8] developed a mathematical analysis of branching when blood enters from a feeding artery into a right-angled branch capillary. On the basis of the above assumptions the present analysis deals with the distribution of the axial velocity (u) at various locations of the branch capillary for different non-negative values of n (fluid index) and θ (yield stress). In the present investigation, the following relevant assumptions are made:

i) A two layered structure of blood flow through narrow artery is studied. The two layers are (a) the peripheral layer and (b) the core layer

ii) The length of the main (feeding) artery is quite large in comparison to its diameter

iii) The blood is treated as a Herschel-Bulkley fluid

iv) The branch capillary makes an arbitrary angle α with the feeding artery

v) R^* and R_1^* ($<R^*$) respectively denote the radii of the feeding artery and the branch capillary

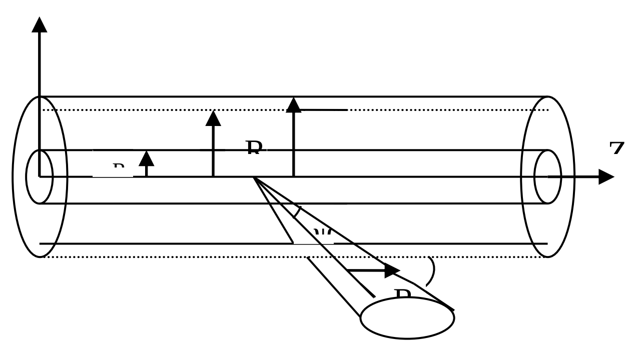

Fig1. Schematic diagram of an arterial segment with branch capillary

In case of steady flow, the multiphase flow of blood through a narrow artery is studied for the microcirculatory system. The angle made by the arteriole emerging from the feeding artery (α) is taken arbitrary. The model presented here is relevant to blood flow in micro vessels having diameter less than 200 μm and a shear rate below $10 sec^{-1}$

2. **Formulation of the problem and the method of solution :**

In terms of cylindrical polar coordinates(r, φ, z) with z-axis along the vessel axis, the governing equation of motion (for a low Reynolds number) may be taken as

$$\rho \frac{\delta u^*}{\delta t^*} = -\frac{\delta p^*}{\delta z^*} - \frac{1}{r^*}\frac{\delta (r^* \tau^*)}{\delta r^*} \qquad (2.1)$$

In conformity to the rheological properties of blood, the blood is considered non-Newtonian and of Herschel-Bulkey type. The constitutive equation for Herschel-Bulkley fluid may be taken as

$$-\frac{\delta u^*}{\delta r^*} = \frac{1}{\mu}(\tau^* - \tau_y)^n, \tau^* > \tau_y \qquad (2.2)$$

$$-\frac{\delta u^*}{\delta r^*} = 0, \tau^* \le \tau_y \qquad (2.3)$$

The multiphase flow of blood may be taken to be governed by the following system of equations:

$$-\frac{\delta u^*}{\delta r^*} = \frac{1}{\mu}\tau^{*n}, R_o^* \le r^* \le R^* \qquad (2.4)$$

$$-\frac{\delta u^*}{\delta r^*} = \frac{1}{\mu_1}(\tau^* - \tau_y)^n, R_p^* \le r^* < R_0^* \qquad (2.5)$$

$$\tau^* = \tau_{y,} r^* < R_p^* \tag{2.6}$$

where $\mu_1 = \mu(1 + k - kr^{*^n})$

Let a be a characteristics radius, p(t) is a non- dimensional pressure gradient along the axis of the tube which is taken to be a periodic function of t.

We non-dimensionlise the variables as

$$\tau = \frac{2\tau^*}{p_0 a}, r = \frac{r^*}{a}, u = \frac{u^*}{p_0 a^2 / 2\mu_p}, t = t^* w,$$

$$\mu_p = \mu\left(\frac{2}{ap_0}\right)^{n-1}, R = \frac{R^*}{a}, R_0 = \frac{R_0^*}{a}, R_P = \frac{R_P^*}{a}, R_1 = \frac{R_1^*}{a}$$

$$\frac{\delta p^*}{\delta z^*} = -p_0 p(t) \tag{2.7}$$

Using non-dimensional variables the equation(2.1) takes the form

$$\alpha^{*2} \frac{\delta u}{\delta t} = 2p(t) - \frac{1}{r}\frac{\delta(r\tau)}{\delta r}, 0 \le r \le R \tag{2.8}$$

where $\alpha^{*2} = \in = \frac{a^2 w}{\mu_p / \rho}$ is the Womersley parameter.

Equations (2.4)- (2.6) become

$$-\frac{\delta u}{\delta r} = \tau^n, R_0 \le r \le R \tag{2.9}$$

$$-\frac{\delta u}{\delta r} = \frac{1}{(1 + k - ka^m r^m)}(\tau - \theta)^n, R_p \le r < R_0 \tag{2.10}$$

$$-\frac{\delta u}{\delta r} = 0, 0 \le r < R_p \tag{2.11}$$

where $\theta = \frac{2\tau_y}{p_0 a}$

3. **Boundary Conditions :**

Mathematically the boundary conditions for the present problem are

$$u = 0 \quad \text{at} \quad r = R \tag{3.1}$$

$$\tau \text{ is finite at } r = 0 \tag{3.2}$$

4. **Method of Solution for Steady Flow :**

For steady flow of blood, equation (2.8) can be written as

$$\frac{1}{r}\frac{\delta(r\tau)}{\delta r} = 2p_s, 0 \le r \le R \tag{4.1}$$

p_s being the steady pressure gradient.

Integrating Eq.(4.1) and using boundary condition (3.2), we get

$$\tau = p_s r \tag{4.2}$$

Integrating Eqs. (2.9) –(2.11) and using boundary condition (3.1) and Eq(4.2) we get

$$u = \frac{p_s^n}{n+1}(R^{n+1} - r^{n+1}), R_0 \le r \le R \tag{4.3}$$

$$u=\frac{p_s^n}{n+1}(R^{n+1}-R_0^{\,n+1})+\frac{p_s^n}{1+k}\{[\frac{R_0^{n+1}}{1+n}-\frac{\theta R_0^n}{p_s}+\frac{n\theta^2 R_0^{n-1}}{2p_s^2}]+\frac{ka^m}{1+k}[\frac{R_0^{m+n+1}}{m+n+1}-\frac{n\theta R_0^{m+n}}{p_s(m+n)}+\frac{n(n-1)\theta^2 R_0^{m+n+1}}{2p_s^2(m+n-1)}]+$$
$$\frac{k^2a^{2m}}{(1+k)^2}[\frac{R_0^{2m+n+1}}{2m+n+1}-\frac{n\theta R_0^{2m+n}}{p_s(2m+n)}+\frac{n(n-1)\theta^2 R_0^{2m+n-1}}{2p_s^2(2m+n-1)}]\}-$$
$$\frac{p_s^n}{1+k}\{[\frac{r^{n+1}}{1+n}-\frac{\theta r^n}{p_s}+\frac{n\theta^2 r^{n-1}}{2p_s^2}]+\frac{ka^m}{1+k}[\frac{r^{m+n+1}}{m+n+1}-\frac{n\theta r^{m+n}}{p_s(m+n)}+\frac{n(n-1)\theta^2 r^{m+n-1}}{2p_s^2(m+n-1)}]+$$
$$\frac{k^2a^{2m}}{(1+k)^2}[\frac{r^{2m+n+1}}{2m+n+1}-\frac{n\theta r^{2m+n}}{p_s(2m+n)}+\frac{n(n-1)\theta^2 r^{2m+n-1}}{2p_s^2(2m+n-1)}]\},R_p\le r<R_0 \qquad (4.4)$$

By neglecting powers of θ higher than the second,

$$u=\frac{p_s^n}{n+1}(R^{n+1}-R_0^{\,n+1})+\frac{p_s^n}{1+k}\{[\frac{R_0^{n+1}}{1+n}-\frac{\theta R_0^n}{p_s}+\frac{n\theta^2 R_0^{n-1}}{2p_s^2}]+\frac{ka^m}{1+k}[\frac{R_0^{m+n+1}}{m+n+1}-\frac{n\theta R_0^{m+n}}{p_s(m+n)}+\frac{n(n-1)\theta^2 R_0^{m+n+1}}{2p_s^2(m+n-1)}]+$$
$$\frac{k^2a^{2m}}{(1+k)^2}[\frac{R_0^{2m+n+1}}{2m+n+1}-\frac{n\theta R_0^{2m+n}}{p_s(2m+n)}+\frac{n(n-1)\theta^2 R_0^{2m+n-1}}{2p_s^2(2m+n-1)}]\}-$$
$$\frac{p_s^n}{1+k}\{[\frac{R_p^{n+1}}{1+n}-\frac{\theta R_p^n}{p_s}+\frac{n\theta^2 R_p^{n-1}}{2p_s^2}]+\frac{ka^m}{1+k}[\frac{R_p^{m+n+1}}{m+n+1}-\frac{n\theta R_p^{m+n}}{p_s(m+n)}+\frac{n(n-1)\theta^2 R_p^{m+n+1}}{2p_s^2(m+n-1)}]+$$
$$\frac{k^2a^{2m}}{(1+k)^2}[\frac{R_p^{2m+n+1}}{2m+n+1}-\frac{n\theta R_p^{2m+n}}{p_s(2m+n)}+\frac{n(n-1)\theta^2 R_p^{2m+n-1}}{2p_s^2(2m+n-1)}]\},0\le r<R_P \qquad (4.5)$$

5. **Numerical Results and Discussion :**

For the purpose of the computational work, the following values of the material constants, and the rheological other parameters have been taken:

a = 0.0075 cm R = 1.0 m = 0.05 k = 0.2 R_p = 0.1 R_0 = 0.9 R_1 = 0.15

p_s = 1.0 A_1 = 0.1 (cf Misra, Adhikery and Shit [2])

The experimental work of Merrill [11] led to report that the yield stress θ lies within the range 0.0 to 0.3. The distribution of axial velocity is presented in the figures 2 – 4 along with our predictions on the basis of the present analysis. Fig 2 exhibits the distribution of axial velocity with varying fluid index parameter n from 0.75 to 2 and taking θ = 0.0. Our results presented in fig.2 shows that the axial velocity decreases with the increase in value of n. Within the whole range of the values of n , the axial velocity is zero for r = 1 and increases steadily as r is diminished. In fig.3, n is taken as 0.75 and θ is varied from θ = 0 to θ = 0.2. The axial velocity is found to decrease with the increase in the value of θ. In fig.4, the Newtonian model is considered taking θ = 0 and n = 1. The axial velocity becomes zero at r = 1 and increases consistently as we decrease the value of r from r = 0 to r = 1.

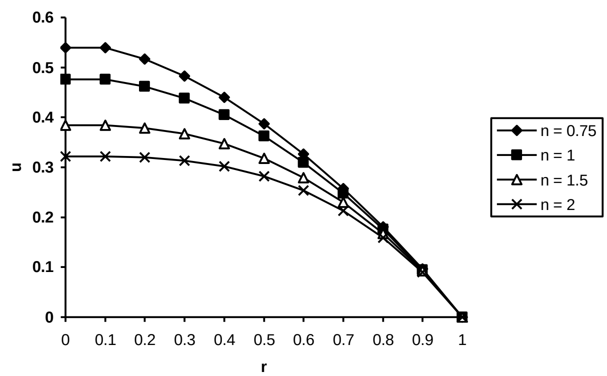

Fig.2 Distribution of axial velocity(u) for different values of n at θ = 0.0

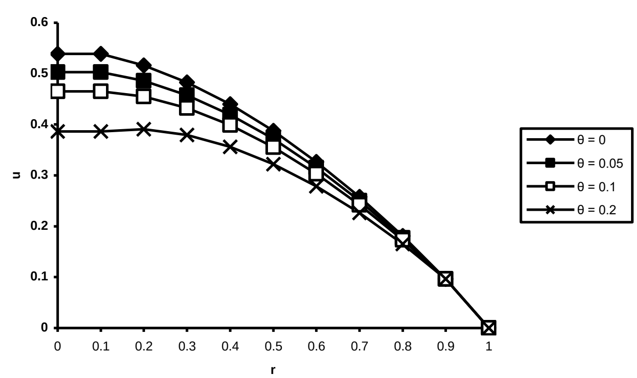

Fig.3 Distribution of axial velocity(u) for different values of θ at n = 0.75

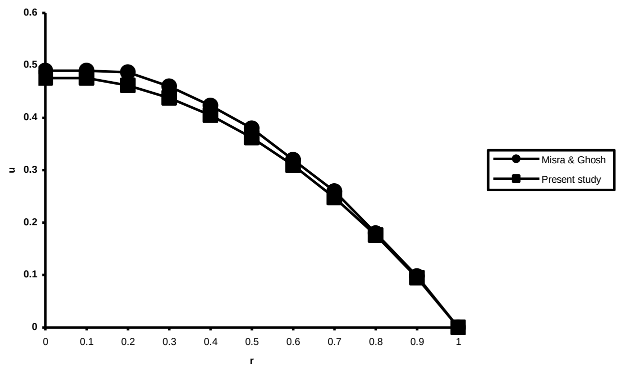

Fig.4 Comparison of axial velocity for Newtonian model(n = 1, θ= 0) with Misra and Ghosh in the steady case

The above results show close conformity to the corresponding results obtained by Misra et. al.[9]

6. Conclusion :

It follows from the above discussion that the fluid index parameter and the yield stress play an important role in the study of steady flow of blood in the arteries with branching. Also the consideration of the arbitrary α, the angle between the feeding artery and the branch artery gives the generalized form of the axial velocity.

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